18. Aggregation Issues

A. Introduction

18.1 In Chapter 15 a basic *index number problem* was identified: how exactly should the microeconomic information involving possibly millions of prices and quantities be aggregated into a smaller number of price and quantity variables? The primary concern of Chapters 15 to 17 was in deciding on an appropriate index number formula so that a *value ratio* pertaining to two periods of time could be decomposed into a component that measures the overall *change in prices* between the two periods (this is the *price index*) times a term that measures the overall *change in quantities* between the two periods (this is the *quantity index*). The summation for these indices was over items. For the fixed-basket and Divisia approach of Chapter 15 and the axiomatic and stochastic approach of Chapter 16, no distinction was drawn between aggregation over items produced by a single establishment, industry, or the economy as a whole. Microeconomic theory regarding the behavior of the establishment in a market was introduced in Chapter 17, and index number formulas were derived that corresponded to specific theoretical assumptions. There was nothing explicit in the analysis to suggest that the same findings would not hold when aggregation took place for outputs, inputs, or the value added of *all* establishments in the economy. Section B of this chapter examines the extent to which the various conclusions reached in Chapter 17 remain valid at an aggregate, economy level. The aggregation of price indices for establishments into national price indices is considered in turn for the output price index, input price index, and value-added deflator.[1] The details of the analysis for the output price index are given in Section B.1, but since a similar methodology is used for the input price index and value-added deflator, only the conclusions are given in Sections B.2 and B.3, respectively.

18.2 Section C notes that, in practice, PPIs tend to be calculated in two stages: first, commodities within establishments are calculated, and second, the commodity and establishment results are used as inputs for aggregation across commodities and establishments to provide industry, product group, and overall PPI results. Section C addresses whether indices calculated this way are consistent in aggregation; that is, if they have the same values whether calculated in a single operation or in two stages.

18.3 Section D considers the relationship between the three PPIs and, in particular, that separate deflation of inputs by the input price index and outputs by the output price index provide the components for the double-deflated value-added index. Section D also outlines a number of equivalent methods that may be used to derive estimates of double-deflated value added for a particular production unit. These are based on the separate deflation by price indices of input and output values, the separate escalation of input and output reference period values by quantity indices, and the use of value-added price and quantity indices. In Section E, the use of value-added price and quantity indices is reconsidered for two-stage aggregation over *industries* (rather than over commodities in a single industry as in Section D) to see if it is consistent with aggregation in a single stage. Finally, Section F considers under what conditions *national value-added price and quantity indices* will be identical to the corresponding *final-demand price and quantity indices*. Note that the final-demand indices are calculated using just the components of final demand, whereas the national value-added indices are constructed by aggregating outputs and intermediate inputs over all industries.

[1]While the value-added price index is just like any other price index in its definition, it is commonly referred to as the "value-added deflator," and the *Manual* will observe this common terminology.

B. Aggregation over Establishments

B.1 National output price index

18.4 An analysis undertaken for aggregation over products at the establishment level for the output price index in Chapter 17, Section B, will now be extended to aggregation over establishments. Assume now that there are E establishments in the economy (or industry, if the goal is to obtain an industry aggregate). The goal in this section is to obtain a national output price index that compares output prices in period 1 to those in period 0 and aggregates over these establishments.

18.5 For $e = 1,2,...,E$, let $p^e \equiv (p_1{}^e,\ldots p_N{}^e)$ denote a positive vector of output prices that establishment e might face in period t, and let $v^e \equiv [x^e,z^e]$ be a nonnegative vector of inputs that establishment e might have available for use during period t. Denote the period t technology set for establishment e by S^{et}. As in Chapter 17, Section B.1, *the revenue function for establishment* e can be defined using the period t technology as follows:

$$(18.1)\ R^{et}(p^e,v^e) \equiv \max_q \left\{ \sum_{n=1}^{N} p_n^e q_n : (q) \text{ belongs to } S^{et}(v^e) \right\},$$

where $e = 1,...,E$ and $t = 0,1$.

Now define the *national revenue function*, $R^t(p^1,...,p^E,v^1,...,v^E)$, using period t technologies as the sum of the period t establishment revenue functions R^{et} defined by equation (18.2):

$$(18.2)\ R^t(p^1,...,p^E,v^1,...,v^E) \equiv \sum_{e=1}^{E} R^{et}\left(p^e,v^e\right).$$

Simplify the notation by defining the national price vector p as $p \equiv [p^1,...,p^E]$ and the national input vector v as $v \equiv [v^1,...,v^E]$. With this new notation, $R^t(p^1,...,p^E,v^1,...,v^E)$ can be written as $R^t(p,v)$. Thus, $R^t(p,v)$ is the maximum value of output, $\sum_{e=1}^{E}\sum_{n=1}^{N} p_n^e q_n^e$, that all establishments in the economy can produce, given that establishment e faces the vector of output prices p^e and given that the vector of inputs v^e is available for use by establishment e, using the period t technologies.

18.6 The period t national revenue function R^t can be used to define the *national output price index* using the period t technologies P^t between any two periods, say, period 0 and period 1, as follows:

$$(18.3)\ P^t(p^0,p^1,v) = R^t\left(p^1,v\right) / R^t\left(p^0,v\right),$$

where $p^0 \equiv [p^{10},p^{20},...,p^{E0}]$ and $p^1 \equiv [p^{11},p^{21},...,p^{E1}]$ are the national vectors of output prices that the various establishments face in periods 0 and 1, respectively, and $v \equiv [v^1,v^2,...,v^E]$ is a reference vector of intermediate and primary inputs for each establishment in the economy.[2] The numerator in equation (18.3) is the maximum revenue that the economy could attain (using inputs v) if establishments faced the output prices of period 1, p^1, while the denominator in equation (18.3) is the maximum revenue that establishments could attain (using inputs v) if they faced the output prices of period 0, p^0. Note that all the variables in the numerator and denominator functions are exactly the same, except that the output price vectors differ.

18.7 As was the case of a single establishment studied in Chapter 17, Section B.1, there are a wide variety of price indices of the form in equation (18.3), depending on which reference technology t and reference input vector v are chosen. Thus, there is not a single economic price index of the type defined by equation (18.3)—there is an entire *family* of indices.

18.8 As usual, interest lies in two special cases of the general definition of the output price index in equation (18.3): (i) $P^0(p^0,p^1,v^0)$, which uses the period 0 establishment technology sets and the input vector v^0 that was actually used in period 0, and (ii) $P^1(p^0,p^1,v^1)$, which uses the period 1 establishment technology sets and the input vector v^1 that was actually used in period 1. Let q^{e0} and q^{e1} be the observed output vectors for the establishments in periods 0 and 1, respectively, for $e = 1,...,E$. If there is revenue-maximizing behavior on the part of each establishment in periods 0 and 1, then the sum of observed establishment revenues in periods 0 and 1 should be equal to $R^0(p^0,v^0)$ and $R^1(p^1,v^1)$, respectively; that is, the following equalities should hold:

[2]This concept for an economywide producer output price index may be found in Diewert (2001).

(18.4) $R^0(p^0,v^0) = \sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e0} q_n^{e0}$

and $R^1(p^1,v^1) = \sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e1} q_n^{e1}$.

Under these revenue-maximizing assumptions, adapting the arguments of F.M. Fisher and Shell (1972, pp. 57–58) and Archibald (1977, p. 66), Diewert (2001) showed that the two theoretical indices, $P^0(p^0,p^1,v^0)$ and $P^1(p^0,p^1,v^1)$, described in (i) and (ii) above, satisfy the following inequalities of equations (18.5) and (18.6):

(18.5) $P^0(p^0,p^1,v^0) \equiv R^0\left(p^1,v^0\right)\Big/R^0\left(p^0,v^0\right)$

using equation (18.3)

$$= R^0\left(p^1,v^0\right)\Big/\sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e0} q_n^{e0}$$

using equation (18.4)

$$\geq \sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e1} q_n^{e0} \Big/ \sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e0} q_n^{e0} ,$$

since q^{e0} is feasible for the maximization problem that defines $R^{e0}(p^{e1},v^{e0})$, and so

$$R^{e0}\left(p^{e1},v^{e0}\right) \geq \sum_{n=1}^{N} p_n^{e1} q_n^{e0} \quad \text{for } e = 1,\ldots,E$$

$$\equiv P_L(p^0,p^1,q^0,q^1),$$

where P_L is the Laspeyres output price index, which treats each commodity produced by each establishment as a separate commodity. Similarly,

(18.6) $P^1(p^0,p^1,v^1) \equiv R^1\left(p^1,v^1\right)\Big/R^1\left(p^0,v^1\right)$

using equation (18.3)

$$= \sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e1} q_n^{e1} \Big/ R^1\left(p^0,v^1\right)$$

using equation (18.4)

$$\leq \sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e1} q_n^{e1} \Big/ \sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e0} q_n^{e1} ,$$

since q^{e1} is feasible for the maximization problem that defines $R^{e1}(p^{e0},v^{e1})$ and so

$$R^{e1}\left(p^{e0},v^{e1}\right) \geq \sum_{n=1}^{N} p_n^{e0} q_n^{e1} \quad \text{for } e = 1,\ldots,E$$

$$\equiv P_P(p^0,p^1,q^0,q^1),$$

where P_P is the Paasche output price index, which treats each commodity produced by each establishment as a separate commodity. Thus, equation (18.5) says that the observable Laspeyres index of output prices P_L is a *lower bound* to the theoretical national output price index $P^0(p^0,p^1,v^0)$, and equation (18.6) says that the observable Paasche index of output prices P_P is an *upper bound* to the theoretical national output price index $P^1(p^0,p^1,v^1)$.

18.9 It is possible to relate the Laspeyres-type *national* output price index $P^0(p^0,p^1,v^0)$ to the *individual establishment* Laspeyres-type output price indices $P^{e0}(p^{e0},p^{e1},v^{e0})$, defined as follows:

(18.7) $P^{e0}(p^{e0},p^{e1},v^{e0})$

$$\equiv R^{e0}\left(p^{e1},v^{e0}\right)\Big/R^{e0}\left(p^{e0},v^{e0}\right)$$

$$= R^{e0}\left(p^{e1},v^{e0}\right)\Big/\sum_{n=1}^{N} p_n^{e0} q_n^{e0} \geq$$

for $e = 1,\ldots,E$,

where the establishment period 0 technology revenue functions R^{e0} were defined above by equation (18.1), and assumptions in equation (18.4) were used to establish the second set of equalities; that is, the assumption that each establishment's observed period 0 revenues, $\sum_{n=1}^{N} p_n^{e0} q_n^{e0}$, are equal to the optimal revenues, $R^{e0}(p^{e0},v^{e0})$. Now define *the revenue share of establishment* e *in national revenue for period 0* as

(18.8) $S_e^{\,0} \equiv \sum_{n=1}^{N} p_n^{e0} q_n^{e0} \Big/ \sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e0} q_n^{e0}$; $e = 1,\ldots,E$.

Using the definition of the Laspeyres-type national output price index $P^0(p^0,p^1,v^0)$, equation (18.3), for $(t,v) = (0,v^0)$, and using also equation (18.2),

(18.9) $P^0(p^0,p^1,v^0)$

$$\equiv \sum_{e=1}^{E} R^{e0}\left(p^{e1},v^{e0}\right)\Big/\sum_{e=1}^{E} R^{e0}\left(p^{e0},v^{e0}\right)$$

$$= \sum_{e=1}^{E} R^{e0}\left(p^{e0},v^{e0}\right)\frac{\left(\dfrac{R^{e0}\left(p^{e1},v^{e0}\right)}{R^{e0}\left(p^{e0},v^{e0}\right)}\right)}{\sum_{e=1}^{E} R^{e0}\left(p^{e0},v^{e0}\right)}$$

$$= \sum_{e=1}^{E} S_e^0 \left[\frac{R^{e0}\left(p^{e1}, v^{e0}\right)}{R^{e0}\left(p^{e0}, v^{e0}\right)} \right]$$

using equation (18.8)

$$= \sum_{i=1}^{E} S_e^0 P^{e0}\left(p^{e0}, p^{e1}, v^{e0}\right)$$

using equation (18.7).

Thus, the Laspeyres-type national output price index $P^0(p^0,p^1,v^0)$ is equal to a base-period establishment revenue *share-weighted average* of the individual establishment Laspeyres-type output price indices $P^{e0}(p^{e0},p^{e1},v^{e0})$.

18.10 It is also possible to relate the Paasche-type *national* output price index $P^1(p^0,p^1,v^1)$ to the *individual establishment* Paasche-type output price indices $P^{e1}(p^{e0},p^{e1},v^{e1})$, defined as follows:

$$(18.10)\ P^{e1}(p^{e0},p^{e1},v^{e1}) \equiv \frac{R^{e1}\left(p^{e1}, v^{e1}\right)}{R^{e1}\left(p^{e0}, v^{e1}\right)} = \sum_{n=1}^{N} p_n^{e1} q_n^{e1} \Big/ R^{e1}\left(p^{e0}, v^{e1}\right);$$

$e = 1,...,E,$

where the establishment period 1 technology revenue functions R^{e1} were defined above by equation (18.1), and assumptions in equation (18.4) are used to establish the second set of equalities; that is, the assumption that each establishment's observed period 1 revenues, $\sum_{n=1}^{N} p_n^{e1} q_n^{e1}$, are equal to the optimal revenues, $R^{e1}(p^{e1},v^{e1})$. Now, define *the revenue share of establishment* e *in national revenue for period 1* as

$$(18.11)\ S_e^1 \equiv \sum_{n=1}^{N} p_n^{e1} q_n^{e1} \Big/ \sum_{i=1}^{E}\sum_{n=1}^{N} p_n^{i1} q_n^{i1}\ ;\ e = 1,...,E.$$

Using the definition of the Paasche-type national output price index $P^1(p^0,p^1,v^1)$, equation (18.3), for $(t,v) = (1,v^1)$, and using also equation (18.2),

$$(18.12)\ P^1(p^0,p^1,v^1) \equiv \sum_{e=1}^{E} R^{e1}\left(p^{e1}, v^{e1}\right) \Big/ \sum_{e=1}^{E} R^{e1}\left(p^{e0}, v^{e1}\right)$$

$$= \left\{ \left[\sum_{e=1}^{E} R^{e1}\left(p^{e0}, v^{e1}\right) \Big/ \sum_{e=1}^{E} R^{e1}\left(p^{e1}, v^{e1}\right) \right] \right\}^{-1}$$

$$= \left\{ \sum_{e=1}^{E} R^{e1}\left(p^{e1}, v^{e1}\right) \frac{\sum_{e=1}^{E} R^{e1}\left(p^{e0}, v^{e1}\right) \Big/ \sum_{e=1}^{E} R^{e1}\left(p^{e1}, v^{e1}\right)}{\sum_{e=1}^{E} R^{e1}\left(p^{e1}, v^{e1}\right)} \right\}^{-1}$$

$$= \left\{ \sum_{e=1}^{E} S_e^1 \left[\frac{\sum_{e=1}^{E} R^{e1}\left(p^{e1}, v^{e1}\right)}{\sum_{e=1}^{E} R^{e1}\left(p^{e0}, v^{e1}\right)} \right]^{-1} \right\}^{-1}$$

$$= \left\{ \sum_{i=1}^{E} S_e^1 \left[P^{e1}\left(p^{e0}, p^{e1}, v^{e1}\right) \right]^{-1} \right\}^{-1}$$

using equation (18.10).

Thus, the Paasche-type national output price index $P^1(p^0,p^1,v^1)$ is equal to *a period 1 establishment revenue share-weighted harmonic average* of the individual establishment Paasche-type output price indices $P^{e1}(p^{e0},p^{e1},v^{e1})$.

18.11 As was the case in Chapter 17, Section B.2, it is possible to define a national output price index that falls *between* the observable Paasche and Laspeyres national output price indices. To do this, first a *hypothetical revenue function*, $R^e(p^e,\alpha)$, is defined for each establishment that corresponds to the use of an α-weighted average of the technology sets $S^{e0}(v^0)$ and $S^{e1}(v^1)$ (with their associated input vectors) for periods 0 and 1 as the reference technologies and input vectors:

$$(18.13)\ R^e(p^e,\alpha) \equiv \max_q \left\{ \sum_{n=1}^{N} p_n^e q_n : q \text{ belongs to } (1-\alpha)S^{e0}(v^0) + \alpha S^{e1}(v^1) \right\};\ e = 1,...,E.$$

Once the establishment hypothetical revenue functions have been defined by equation (18.13), *the intermediate technology national revenue function* $R^t(p^1,...,p^E,v^1,...,v^E)$ can be defined as the sum of the period t intermediate technology establishment revenue functions R^e defined by equation (18.13):

(18.14) $R(p^1,...,p^E,\alpha) \equiv \sum_{e=1}^{E} R^e\left(p^e,\alpha\right)$.

Again, simplify the notation by defining the national price vector p as $p \equiv [p^1,...,p^E]$. With this new notation, $R(p^1,...,p^E,\alpha)$ can be written as $R(p,\alpha)$. Now, use the national revenue function defined by equation (18.14) in order to define the following family of *theoretical national output price indices*:

(18.15) $P(p^0,p^1,\alpha) \equiv R(p^1,\alpha) / R(p^0,\alpha)$.

18.12 As usual, the proof of Diewert (1983a, pp. 1060–61) can be adapted to show that there exists an α between 0 and 1 such that a theoretical national output price index defined by equation (18.15) lies between the observable (in principle) Paasche and Laspeyres national output price indices defined in equations (18.5) and (18.6), P_P and P_L ; that is, there exists an α such that

(18.16) $P_L \le P(p^0,p^1,\alpha) \le P_P$ or
$P_P \le P(p^0,p^1,\alpha) \le P_L$.

If the Paasche and Laspeyres indices are numerically close to each other, then equation (18.16) tells us that a true national output price index is fairly well determined and that a reasonably close approximation can be found to the true index by taking a symmetric average of P_L and P_P, such as the geometric average, which again leads to Irving Fisher's (1922) ideal price index, P_F, defined earlier by equation (17.9).

18.13 The above theory for the national output price indices is very general; in particular, no restrictive functional form or separability assumptions were made on the establishment technologies.

18.14 The translog technology assumptions used in Chapter 17, Section B.3 to justify the use of the Törnqvist-Theil output price index for a single establishment as an approximation to a theoretical output price index for a single establishment can be adapted to yield a justification for the use of a national Törnqvist-Theil output price index as an approximation to a theoretical national output price index.

18.15 Recall the definition of the national period t national revenue function, $R^t(p,v) \equiv R^t(p^1,...,p^E,v^1,...,v^E)$, defined earlier by equation (18.2) above. Assume that the period t national revenue function has the following *translog functional form* for $t = 0,1$:

(18.17) $\ln R^t(p,v)$

$$= \alpha_0^t + \sum_{n=1}^{NE} \alpha_n^t \ln p_n + \sum_{m=1}^{(N+K)E} \beta_m^t \ln v_m + \frac{1}{2}\sum_{n=1}^{NE}\sum_{j=1}^{NE} \alpha_{nj}^t \ln p_n \ln p_j + \sum_{n=1}^{NE}\sum_{m=1}^{(N+K)E} \beta_{nm}^t \ln p_n \ln v_m + \frac{1}{2}\sum_{m=1}^{(N+K)E}\sum_{k=1}^{(N+K)E} \gamma_{mk}^t \ln v_m \ln v_k ,$$

where the $\alpha_n{}^t$ coefficients satisfy the restrictions

(18.18) $\sum_{n=1}^{NE} \alpha_n^t = 1$ for $t = 0,1$,

and the $\alpha_{nj}{}^t$ coefficients satisfy the following restrictions:[3]

(18.19) $\sum_{n=1}^{NE} \alpha_{nj}^t = 0$ for $t = 0,1$ and n = 1,2,…,NE.

Note that the national output price vector p in equation (18.17) has dimension equal to NE, the number of outputs times the number of establishment—that is, $p \equiv [p_1,...,p_N;\ p_{N+1},...,p_{2N};\ ...;\ p_{(E-1)N+1},...,p_{NE}] = [p_1{}^1,...,p_N{}^1;\ p_1{}^2,...,p_N{}^2;\ ...;\ p_1{}^E,...,p_N{}^E]$. Similarly, the national input vector v in equation (18.17) has dimension equal to $(M + K)E$, the number of intermediate and primary inputs in the economy times the number of establishments.[4] The restrictions in equations (18.18) and (18.19)

[3]It is also assumed that the symmetry conditions $\alpha_{nj}{}^t = \alpha_{jn}{}^t$ for all n,j and for $t = 0,1$ and $\gamma_{mk}{}^t = \gamma_{km}{}^t$ for all m,k and for $t = 0,1$ are satisfied.

[4]It has also been implicitly assumed that each establishment can produce each of the N outputs in the economy and that each establishment uses all $M + K$ inputs in the economy. These restrictive assumptions can readily be relaxed, but only at the cost of notational complexity. All that is required is that each establishment produce the same set of outputs in each period.

are necessary to ensure that $R^t(p,v)$ is linearly homogeneous in the components of the output price vector p (which is a property that a revenue function must satisfy). Note that at this stage of the argument, the coefficients that characterize the technology in each period (the αs, βs, and γs) are allowed to be completely different in each period. Also note that the translog functional form is an example of a *flexible* functional form;[5] that is, it can approximate an arbitrary technology to the second order.

18.16 Define the national revenue share for establishment e and output n for period t as follows:

$$(18.20)\ s_n^{et} \equiv \sum_{n=1}^{N} p_n^{et} q_n^{et} \Big/ \sum_{i=1}^{E}\sum_{j=1}^{N} p_j^{it} q_j^{it}\ ;\ n = 1,\ldots,N\ ;$$
$$e = 1,\ldots,E\ ;\ t = 0,1.$$

Using the above establishment revenue shares and the establishment output price relatives, p_n^{e1}/ p_n^{e0}, define the logarithm of the *national Törnqvist-Theil output price index* P_T (Törnqvist, 1936; Törnqvist and Törnqvist, 1937; and Theil, 1967) as follows:

$$(18.21)\ \ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{e=1}^{E}\sum_{n=1}^{N} (½)\left(s_n^{e0} + s_n^{e1}\right)\ln\left(p_n^{e1}/p_n^{e0}\right).$$

18.17 Recall Theil's (1967) weighted stochastic approach to index number theory explained in Section D.2 of Chapter 16. In the present context, the discrete random variable R takes on the NE values for the logarithms of the establishment output price ratios between periods 0 and 1, $\ln\left(p_n^{e1}/p_n^{e0}\right)$, with probabilities $\left(\frac{1}{2}\right)\left(s_n^{e0} + s_n^{e1}\right)$. Thus, the right-hand side of equation (18.21) can also be interpreted as the *mean* of this distribution of economywide logarithmic output price relatives.

18.18 A result in Caves, Christensen, and Diewert (1982b, p. 1410) can be adapted to the present context: if the quadratic price coefficients in equation (18.17) are equal across the two periods where an index number comparison (that is, $\alpha_{ij}{}^0 = \alpha_{ij}{}^1$ for all i,j) is made, then the geometric mean of the national output price index that uses period 0 technology and the period 0 input vector v^0, $P^0(p^0,p^1,v^0)$, and the national output price index that uses period 1 technology and the period 1 input vector v^1, $P^1(p^0,p^1,v^1)$, is *exactly equal* to the Törnqvist output price index P_T defined by equation (18.21) above; that is,

$$(18.22)\ P_T(p^0,p^1,q^0,q^1) = \left[P^0\left(p^0,p^1,v^0\right)P^1\left(p^0,p^1,v^1\right)\right]^{1/2}.$$

As usual, the assumptions required for this result seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period, and our assumptions are consistent with technological progress occurring between the two periods being compared. Because the index number formula P_T is *exactly* equal to the geometric mean of two theoretical economic output price indices and this corresponds to a flexible functional form, the Törnqvist national output price index number formula is *superlative* following the terminology used by Diewert (1976).

18.19 There are *four important results* in this section, which can be summarized as follows. Define the national Laspeyres output price index as follows:

$$(18.23)\ P_L(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e1} q_n^{e0}}{\sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e0} q_n^{e0}}.$$

Then, this national Laspeyres output price index is a *lower bound* to the economic output price index $P^0(p^0,p^1,v^0) \equiv R^0(p^1,v^0) / R^0(p^0,v^0)$, where the national revenue function $R^0(p,v^0)$ using the period 0 technology and input vector v^0 is defined by equations (18.1) and (18.2).

18.20 Define the national Paasche output price index as follows:

$$(18.24)\ P_P(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e1} q_n^{e1}}{\sum_{e=1}^{E}\sum_{n=1}^{N} p_n^{e0} q_n^{e1}}.$$

[5]In fact, the assumption that the period t national revenue function $R^t(p,v)$ has the translog functional form defined by equation (18.17) may be regarded as an approximation to the true technology, since equation (18.17) has not imposed any restrictions on the national technology, implied by the fact that the national revenue function is equal to the sum of the establishment revenue functions.

Then, this national Paasche output price index is an *upper bound* to the economic output price index $P^1(p^0,p^1,v^1) \equiv R^1(p^1,v^1) / R^1(p^0,v^1)$, where the national revenue function $R^1(p,v^1)$ using the period 1 technology and input vector v^1 is defined by equations (18.1) and (18.2).

18.21 Define the *national Fisher output price index* P_F as the square root of the product of the national Laspeyres and Paasche indices defined above:

$$(18.25)\ P_F(p^0,p^1,q^0,q^1) = \left[P_L\left(p^0,p^1,q^0,q^1\right)P_P\left(p^0,p^1,q^0,q^1\right)\right]^{1/2}.$$

Then, usually, the national Fisher output price index will be a good approximation to an economic output price index based on a revenue function that uses a technology set and an input vector that is intermediate to the period 0 and 1 technology sets and input vectors.

18.22 Under the assumption that the period 0 and 1 national revenue functions have translog functional forms, then the geometric mean of the national output price index that uses period 0 technology and the period 0 input vector v^0, $P^0(p^0,p^1,v^0)$, and the national output price index that uses period 1 technology and the period 1 input vector v^1, $P^1(p^0,p^1,v^1)$, is *exactly equal* to the Törnqvist output price index P_T defined by equation (18.21) above; that is, equation (18.22) holds true.

18.23 This section concludes with an observation. Economic justifications have been presented for the use of the national Fisher output price index, $P_F(p^0,p^1,q^0,q^1)$, defined by equation (18.25), and for the use of the national Törnqvist output price index, $P_T(p^0,p^1,q^0,q^1)$, defined by equation (18.21). The results in Chapter 17, Section B.5, indicate that for normal time-series data, these two indices will give virtually the same answer.

B.2 National intermediate input price index

18.24 The theory of the intermediate input price index for a single establishment that was developed in Chapter 17, Section C, can be extended to the case where there are E establishments in the economy. The techniques used for this extension are very similar to the techniques used in Section B.1 above, so it is not necessary to replicate this work here.

18.25 The observable national Laspeyres index of intermediate input prices P_L is found to be an *upper bound* to the theoretical national intermediate input price index using period 0 technology and inputs, and the observable national Paasche index of intermediate input prices P_P is a *lower bound* to the theoretical national intermediate input price index using period 1 technology and inputs.

18.26 As was the case in Section B.1, it is possible to define a theoretical national intermediate input price index that falls *between* the observable Paasche and Laspeyres national intermediate input price indices. The details are omitted, although they follow along the lines used in Section B.1. Usually, the *national Fisher intermediate input price index* P_F, defined as the square root of the product of the national Laspeyres and Paasche indices, will be a good approximation to this economic intermediate input price index. Such an index is based on a national cost function that uses establishment technology sets, target establishment output vectors, and establishment primary input vectors intermediate to the period 0 and 1 technology sets, observed output vectors, and observed primary input vectors.

18.27 The translog technology assumptions used in Section B.1 to justify the use of the Törnqvist-Theil intermediate input price index for a single establishment as an approximation to a theoretical intermediate input price index for a single establishment can be adapted to yield a justification for the use of a national Törnqvist-Theil intermediate input price index as an approximation to a theoretical national intermediate input price index.

B.3 National value-added deflator

18.28 In this section, it is the theory of the *value-added deflator for a single establishment* developed in Chapter 17, Section D, that is drawn on and extended to the case where there are E establishments in the economy. The techniques used for this extension are, again, very similar to the techniques used in Section B.1, except that an establishment net revenue functions π^{et} is used in place of establishment revenue functions R^{et}.

18.29 The observable Laspeyres index of net output prices is shown to be a *lower bound* to the theoretical national value-added deflator based on period 0 technology and inputs, and the observable Paasche index of net output prices is an *upper bound* to the theoretical national value-added deflator based on period 1 technology and inputs.

18.30 Constructing industry indices, such as Laspeyres and Paasche, from individual establishment indices, and national indices from individual industry indices requires weights. It should be noted that *establishment shares of national value added* are used for national value-added deflators, whereas *establishment shares of the national value of (gross) outputs produced* were used in Section B.1 for national output price indices. Results supporting the use of Fisher's ideal index and the Törnqvist index arise from arguments similar to those presented for the national output price index.

18.31 Recall Theil's (1967) weighted stochastic approach to index number theory that was explained in Section D.2 of Chapter 16. If his approach is adapted to the present context, then the discrete random variable R would take on the $(N + M)E$ values for the logarithms of the establishment net output price relatives between periods 0 and 1, $\ln\left(p_n^{e1}/p_n^{e0}\right)$, with probabilities $\left(\frac{1}{2}\right)\left(s_n^{e0}+s_n^{e1}\right)$. Thus, under this interpretation of the stochastic approach, it would appear that the right-hand side of the Törnqvist-Theil index could be interpreted as the *mean* of this distribution of economywide logarithmic output and intermediate input price relatives. However, in the present context, this stochastic interpretation for the Törnqvist-Theil net output price formula breaks down because the shares $\left(\frac{1}{2}\right)\left(s_n^{e0}+s_n^{e1}\right)$ are negative when n corresponds to an intermediate input.

C. Laspeyres, Paasche, Superlative Indices and Two-Stage Aggregation

18.32 The above analysis has been conducted as if the aggregation had been undertaken in a single stage. Most statistical agencies use the Laspeyres formula to aggregate prices in two stages. At the first stage of aggregation, the Laspeyres formula is used to aggregate components of the overall index (for example, agricultural output prices, other primary industry output prices, manufacturing prices, service output prices). Then, at the second stage of aggregation, these component subindices are further combined into the overall index. The following question then naturally arises: does the index computed in two stages coincide with the index computed in a single stage? This question is initially addressed in the context of the Laspeyres formula.[6]

18.33 Now, suppose that the price and quantity data for period t, p^t and q^t, can be written in terms of j subvectors as follows:

$$(18.26)\ p^t = (p^{t1}, p^{t2}, \ldots, p^{tJ})\ ;\quad q^t = (q^{t1}, q^{t2}, \ldots, q^{tJ})\ ;\ t = 0,1,$$

where the dimensionality of the subvectors p^{tj} and q^{tj} is N_j for $j = 1,2,\ldots,J$ with the sum of the dimensions N_j equal to N. These subvectors correspond to the price and quantity data for subcomponents of the producer output price index for period t. The analysis is undertaken for output price indices here, but similar conclusions hold for input price indices. Construct subindices for each of these components going from period 0 to 1. For the base period, the price for each of these subcomponents, say, P_j^0 for $j = 1,2,\ldots J$, is set equal to 1, and the corresponding base-period subcomponent quantities, say, Q_j^0 for $j = 1,2,\ldots,J$, are set equal to the base-period value of production for that subcomponent. For $j = 1,2,\ldots,J$; that is,

$$(18.27)\ P_j^0 \equiv 1\ ;\ Q_j^0 \equiv \sum_{i=1}^{Nj} p_i^{1j} q_i^{0j} \quad \text{for } j = 1,2,\ldots,J.$$

Now use the Laspeyres formula to construct a period 1 price for each subcomponent, say, P_j^1 for $j = 1,2,\ldots,J$, of the producer price index. Since the dimensionality of the subcomponent vectors, p^{tj} and q^{tj}, differ from the dimensionality of the complete period t vectors of prices and quantities, p^t and q^t, different symbols for these subcomponent Laspeyres indices will be used, say, P_L^j for $j = 1,2,\ldots J$. Thus, the period 1 subcomponent prices are defined as follows:

[6]Much of the initial material in this section is adapted from Diewert (1978) and Alterman, Diewert, and Feenstra (1999). See also Vartia (1976a; 1976b) and Balk (1996b) for a discussion of alternative definitions for the two-stage aggregation concept and references to the literature on this topic.

$$(18.28)\ P_j^1 \equiv P_L^j(p^{0j},p^{1j},q^{0j},q^{1j}) \equiv \frac{\sum_{i=1}^{Nj} p_i^{1j} q_i^{0j}}{\sum_{i=1}^{Nj} p_i^{0j} q_i^{0j}}$$

for $j = 1,2,\ldots,J$.

Once the period 1 prices for the j subindices have been defined by equation (18.28), then corresponding subcomponent period 1 quantities Q_j^1 for $j = 1,2,\ldots J$ can be defined by deflating the period 1 subcomponent values $\sum_{i=1}^{Nj} p_i^{1j} q_i^{1j}$ by the prices P_j^1 defined by equation (18.28); that is,

$$(18.29)\ Q_j^1 \equiv \sum_{i=1}^{Nj} p_i^{1j} q_i^{1j} \Big/ P_j^1 \text{ for } j = 1,2,\ldots,J.$$

Subcomponent price and quantity vectors for each period $t = 0,1$ can now be defined using equations (18.27) to (18.29). Thus, define the period 0 and 1 subcomponent price vectors P^0 and P^1 as follows:

$$(18.30)\ P^0 = (P_1^0, P_2^0,\ldots,P_J^0) \equiv 1_J\ ;\ P^1 = (P_1^1, P_2^1,\ldots,P_J^1),$$

where 1_J denotes a vector of ones of dimension j, and the components of P^1 are defined by equation (18.28). The period 0 and 1 subcomponent quantity vectors Q^0 and Q^1 are defined as follows:

$$(18.31)\ Q^0 = (Q_1^0, Q_2^0,\ldots,Q_J^0)\ ;\quad Q^1 = (Q_1^1, Q_2^1,\ldots,Q_J^1),$$

where the components of Q^0 are defined in equation (18.27) and the components of Q^1 are defined by equation (18.29). The price and quantity vectors in equations (18.30) and (18.31) represent the results of the first-stage aggregation. These vectors can now be used as inputs into the second-stage aggregation problem; that is, the Laspeyres price index formula can be applied using the information in equations (18.30) and (18.31) as inputs into the index number formula. Since the price and quantity vectors that are inputs into this second-stage aggregation problem have dimension j instead of the single-stage formula that used vectors of dimension N_j, a different symbol is needed for our new Laspeyres index, which is chosen to be P_L*. Thus, the Laspeyres price index computed in two stages can be denoted as $P_L*(P^0,P^1,Q^0,Q^1)$. It is now appropriate to ask whether this two-stage Laspeyres index equals the corresponding single-stage index P_L studied in the previous sections of this chapter; that is, whether

$$(18.32)\ P_L*(P^0,P^1,Q^0,Q^1) = P_L(p^0,p^1,q^0,q^1).$$

If the Laspeyres formula is used at each stage of each aggregation, the answer to the above question is yes: straightforward calculations show that the Laspeyres index calculated in two stages equals the Laspeyres index calculated in one stage. The answer is also yes if the Paasche formula is used at each stage of aggregation; that is, the Paasche formula is consistent in aggregation just like the Laspeyres formula.

18.34 Now suppose the Fisher or Törnqvist formula is used at each stage of the aggregation; that is, in equation (18.28), suppose the Laspeyres formula $P_L^j(p^{0j},p^{1j},q^{0j},q^{1j})$ is replaced by the Fisher formula $P_F^j(p^{0j},p^{1j},q^{0j},q^{1j})$ or by the Törnqvist formula $P_T^j(p^{0j},p^{1j},q^{0j},q^{1j})$, and in equation (18.32), $P_L*(P^0,P^1,Q^0,Q^1)$ is replaced by P_F* (or by P_T*) and $P_L(p^0,p^1,q^0,q^1)$ replaced by P_F (or by P_T). Then, do counterparts to the two-stage aggregation result for the Laspeyres formula, equation (18.32)? The answer is no; it can be shown that, in general,

$$(18.33)\ P_F*(P^0,P^1,Q^0,Q^1) \neq P_F(p^0,p^1,q^0,q^1) \text{ and}$$
$$P_T*(P^0,P^1,Q^0,Q^1) \neq P_T(p^0,p^1,q^0,q^1).$$

Similarly, it can be shown that the quadratic mean of order r index number formula P^r defined by equation (17.28) and the implicit quadratic mean of order r index number formula P^r* defined by equation (17.25) are also not consistent in aggregation.

18.35 However, even though the Fisher and Törnqvist formulas are not *exactly* consistent in aggregation, it can be shown that these formulas are *approximately* consistent in aggregation. More specifically, it can be shown that the two-stage Fisher formula P_F* and the single-stage Fisher formula P_F in equation (18.33), both regarded as functions of the 4N variables in the vectors p^0,p^1,q^0,q^1, approximate each other to the second order around a point where the two price vectors are equal (so that $p^0 = p^1$) and where the two quantity vectors are equal (so that $q^0 = q^1$). A similar result holds for the two-stage and single-stage Törnqvist indices in equation (18.33).[7] As it was shown in the previous section, the single-stage Fisher and Törnqvist indices have a similar approximation property, and so all four indices in equation (18.33) approximate each other to the second order around an equal (or proportional) price and quantity point. Thus, for normal time-series data, single-stage and two-stage Fisher and Törnqvist indices usually will be numerically very close.[8] This result for an artificial data set is illustrated in Chapter 19.

18.36 Similar approximate consistency in aggregation results (to the results for the Fisher and Törnqvist formulas explained in the previous paragraph) can be derived for the *quadratic mean of order* r *indices*, P^r, and for the implicit quadratic mean of order r indices, P^{r*}; see Diewert (1978, p. 889). However, the results of R.J. Hill (2000) again imply that *the second-order approximation property of the single-stage quadratic mean of order* r *index* P^r *to its two-stage counterpart will break down as* r *approaches either plus or minus infinity*. To see this, consider a simple example where there are only four commodities in total. Let the first price relative p_1^1 / p_1^0 be equal to the positive number a, let the second two price relatives p_i^1 / p_i^0 equal b, and let the last price relative p_4^1 / p_4^0 equal c where it is assumed that $a < c$ and $a \leq b \leq c$. Using R.J. Hill's result in equation (17.32), the limiting value of the single-stage index is

$$(18.34)\ \lim_{r\to+\infty} P^r(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^r(p^0,p^1,q^0,q^1) = [\min_i\{p_i^1 / p_i^0\}\max_i\{p_i^1 / p_i^0\}]^{1/2} = [ac]^{1/2}.$$

If commodities 1 and 2 are aggregated into a subaggregate and commodities 3 and 4 into another subaggregate, using R.J. Hill's result in equation (17.32) again, it is found that the limiting price index for the first subaggregate is $[ab]^{1/2}$ and the limiting price index for the second subaggregate is $[bc]^{1/2}$. Now, apply the second stage of aggregation and use R.J. Hill's result once again to conclude that the limiting value of the two-stage aggregation using P^r as our index number formula is $[ab^2c]^{1/4}$. Thus, the limiting value as r tends to plus or minus infinity of the single-stage aggregate over the two-stage aggregate is $[ac]^{1/2} / [ab^2c]^{1/4} = [ac/b^2]^{1/4}$. Now b can take on any value between a and c, and the ratio of the single-stage limiting P^r to its two-stage counterpart can take on any value between $[c/a]^{1/4}$ and $[a/c]^{1/4}$. Since c/a is less than 1 and a/c is greater than 1, it can be seen that the ratio of the single-stage to the two-stage index can be arbitrarily far from 1 as r becomes large in magnitude with an appropriate choice of the numbers a, b, and c.

18.37 The results in the previous paragraph show that caution is required in assuming that *all* superlative indices will be approximately consistent in aggregation. However, for the three most commonly used superlative indices (the Fisher ideal P_F, the Törnqvist-Theil P_T, and the Walsh P_W), the available empirical evidence indicates that these indices satisfy the consistency-in-aggregation property to a sufficiently high enough degree of approximation that users will not be unduly troubled by any inconsistencies.[9]

18.38 A similar analysis could be undertaken for *input price indices*, and similar conclusions would hold. The value-added deflator is considered in the next subsection.

D. Value-Added Deflators—Relationships Between Producer Price Indices

D.1 Output price, intermediate input price, and deflation of value added

18.39 Let the vectors of output price, output quantity, intermediate input price, and intermediate input price vectors for an establishment[10] in period t be denoted by p_y^t, y^t, p_x^t, and x^t, respectively, for $t = 0,1$. Suppose a bilateral index number formula P is used to construct an establishment output price

[7] See Diewert (1978, p. 889), who used some results credited to Vartia (1976a; 1976b).

[8] For an empirical comparison of the four indices, see Diewert (1978, pp. 894–95). For the Canadian consumer data considered there, the chained two-stage Fisher in 1971 was 2.3228 and the corresponding chained two-stage Törnqvist was 2.3230, the same values as for the corresponding single-stage indices.

[9] See Chapter 19 for additional evidence on this topic.

[10] Instead of "establishment," one could substitute the words "industry" or "national economy."

index, $P(p_y^0,p_y^1,y^0,y^1)$, an establishment intermediate input price index, $P(p_x^0,p_x^1,x^0,x^1)$, and an establishment *value-added deflator*, $P(p^0,p^1,q^0,q^1)$ where, as usual, $p^t \equiv [p_y^t,p_x^t]$ and $q^t \equiv [y^t,-x^t]$ for $t = 0,1$. Two related questions arise:

- How is the *value-added deflator* related to the output price index and the intermediate input price index?
- How can the output price index and the intermediate input price index be combined to obtain a *value-added deflator*?

Answers to the above questions can be obtained using the two-stage aggregation procedure explained in Section C.

18.40 In the present application of the two-stage aggregation procedure explained in Section C, let $j = 2$. The price and quantity vectors p^{tj} and q^{tj} that appeared in equation (18.26) are now defined as follows:

(18.35) $p^{t1} \equiv p_y^t$; $p^{t2} \equiv p_x^t$; $q^{t1} \equiv y^t$; $q^{t2} \equiv -x^t$; $t = 0,1$.

Thus, the first group of commodities aggregated in the first stage of aggregation are the outputs y^t of the establishment, and the second group of commodities aggregated in the first stage of aggregation are minus the intermediate inputs $-x^t$ of the establishment.

18.41 The base-period first-stage aggregate prices and quantities, P_j^0 and Q_j^0, that appeared in equation (18.27) are now defined as follows:

(18.36) $$P_1^0 = P_2^0 \equiv 1 ;\quad Q_1^0 \equiv \sum_{n=1}^{N} p_{yn}^0 y_n^0 ;\quad Q_2^0 \equiv -\sum_{m=1}^{M} p_{xm}^0 x_m^0 .$$

Note that Q_1^0 is the base-period value of outputs produced by the establishment, and Q_2^0 is minus the value of intermediate inputs used by the establishment in period 0.

18.42 Now, use a chosen index number formula to construct an output price index, $P(p_y^0,p_y^1,y^0,y^1)$, and an intermediate input price index, $P(p_x^0,p_x^1,x^0,x^1)$. These two numbers are set equal to the aggregate price of establishment output P_1^1 and the aggregate price of intermediate input P_2^1 in period 1; that is, the bilateral index number formula P is used to form the following counterparts to equation (18.28) in Section C:

(18.37) $P_1^1 \equiv P(p_y^0,p_y^1,y^0,y^1)$; $P_2^1 \equiv P(p_x^0,p_x^1,x^0,x^1)$.

18.43 Finally, the following counterparts to equation (18.29) generate the period 1 output quantity aggregate Q_1^1 and minus the period 1 input aggregate Q_2^1:

(18.38) $$Q_1^1 \equiv \sum_{n=1}^{N} p_{yn}^1 y_n^1 \Big/ P_1^1 = \sum_{n=1}^{N} p_{yn}^1 y_n^1 \Big/ P\left(p_y^0, p_y^1, y^0, y^1\right);$$
$$Q_2^1 \equiv -\sum_{m=1}^{M} p_{xm}^1 x_m^1 \Big/ P_2^1 = -\sum_{m=1}^{M} p_{xm}^1 x_m^1 \Big/ P\left(p_x^0, p_x^1, x^0, x^1\right).$$

Thus, the period 1 output aggregate, Q_1^1, is equal to the value of period 1 production, $\sum_{n=1}^{N} p_{yn}^1 y_n^1$, divided by the output price index, $P(p_y^0,p_y^1,y^0,y^1)$. The period 1 intermediate input aggregate, Q_2^1, is equal to minus the period 1 cost of intermediate inputs, $\sum_{m=1}^{M} p_{xm}^1 x_m^1$, divided by the intermediate input price index, $P(p_x^0,p_x^1,x^0,x^1)$. Thus, the period 1 output and intermediate input quantity aggregates are constructed by deflating period 1 value aggregates by an appropriate price index, which may be considered to be a type of double-deflation procedure.

18.44 Following equation (18.30), the period 0 and 1 subcomponent price vectors P^0 and P^1 and the period 0 and 1 subcomponent quantity vectors Q^0 and Q^1 are defined as follows:

(18.39) $$P^0 \equiv [P_1^0,P_2^0] ;\ P^1 \equiv [P_1^1,P_2^1] ;\quad Q^0 \equiv [Q_1^0,Q_2^0] ;\ Q^1 \equiv [Q_1^1,Q_2^1].$$

Finally, given the aggregate prices and quantity vectors defined in equation (18.39), again make use of the chosen bilateral index number formula P, and calculate the *two-stage value-added deflator* for the establishment, $P(P^0,P^1,Q^0,Q^1)$. The construction of this two-stage value-added deflator provides an answer to the second question asked above; that is, how can the output price index and the intermediate input price index be combined to obtain a value-added deflator?

18.45 It is now necessary to ask whether the two-stage value-added deflator that was just constructed, $P(P^0,P^1,Q^0,Q^1)$, using the bilateral index number formula P in both stages of aggregation, is equal to the value-added deflator that was constructed in a single-stage aggregation, $P(p^0,p^1,q^0,q^1)$, using the same index number formula P. That is, ask whether

$$(18.40)\ P(P^0,P^1,Q^0,Q^1) = P(p^0,p^1,q^0,q^1).$$

The answer to this question is yes, if the Laspeyres or Paasche price index is used at each stage of aggregation; that is, if $P = P_L$ or if $P = P_P$. The answer is no if a superlative price index is used at each stage of aggregation; that is, if $P = P_F$ or if $P = P_T$. However, using the results explained in Section C, the difference between the right-hand and left-hand sides of equation (18.40) will be very small if the Fisher or Törnqvist-Theil formulas, P_F or P_T, are used consistently at each stage of aggregation. Thus, using a superlative index number formula to construct output price, intermediate input price, and value-added deflators comes at the cost of small inconsistencies as prices are aggregated up in two or more stages of aggregation, whereas the Laspeyres and Paasche formulas are exactly consistent in aggregation. However, the use of the Laspeyres or Paasche formulas also comes at a cost: these indices will have an indeterminate amount of substitution bias compared with their theoretical counterparts,[11] whereas superlative indices will be largely free of substitution bias.

D.2 Laspeyres and Paasche value-added deflators

18.46 Given the importance of Paasche and Laspeyres price indices in statistical agency practice, it is worth writing out explicitly the value-added deflator using the two-stage aggregation procedure explained above when these two indices are used as the basic index number formula. If the Laspeyres formula is used, the two sides of equation (18.40) become

$$(18.41)\ P_L(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{n=1}^{N} p_{yn}^1 y_n^0 - \sum_{m=1}^{M} p_{xm}^1 x_m^0}{\sum_{n=1}^{N} p_{yn}^0 y_n^0 - \sum_{m=1}^{M} p_{xm}^0 x_m^0}$$

$$= s_y^0 \left(\frac{\sum_{n=1}^{N} p_{yn}^1 y_n^0}{\sum_{n=1}^{N} p_{yn}^0 y_n^0} \right) + s_x^0 \left(\frac{\sum_{m=1}^{M} p_{xm}^1 x_m^0}{\sum_{m=1}^{M} p_{xm}^0 x_m^0} \right)$$

$$= s_y^0\, P_L(p_y^0,p_y^1,y^0,y^1) + s_x^0\, P_L(p_x^0,p_x^1,x^0,x^1),$$

where the period 0 output share s_y^0 and the period 0 intermediate input share s_x^0 are defined as follows:

$$(18.42)\ s_y^0 = \frac{\sum_{n=1}^{N} p_{yn}^1 y_n^0}{\sum_{n=1}^{N} p_{yn}^1 y_n^0 - \sum_{m=1}^{M} p_{xm}^1 x_m^0} = \frac{P_1^0 Q_1^0}{\left(P_1^0 Q_1^0 + P_2^0 Q_2^0\right)};$$

$$s_x^0 \equiv \frac{-\sum_{m=1}^{M} p_{xm}^0 x_m^0}{\sum_{n=1}^{N} p_{yn}^0 y_n^0 - \sum_{m=1}^{M} p_{xm}^0 x_m^0} = \frac{P_2^0 Q_2^0}{\left(P_1^0 Q_1^0 + P_2^0 Q_2^0\right)}.$$

Note that s_y^0 will be greater than 1, and s_x^0 will be negative. Thus, equation (18.41) says that the Laspeyres value-added deflator can be written as a weighted average of the Laspeyres output price index, $P_L(p_y^0,p_y^1,y^0,y^1)$, and the Laspeyres intermediate input price index, $P_L(p_x^0,p_x^1,x^0,x^1)$. Although the weights sum to 1, s_x^0 is negative and s_y^0 is greater than 1, so these weights are rather unusual.

18.47 There is an analogous two-stage decomposition for the Paasche value-added deflator:

$$(18.43)\ P_P(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{n=1}^{N} p_{yn}^1 y_n^1 - \sum_{m=1}^{M} p_{xm}^1 x_m^1}{\sum_{n=1}^{N} p_{yn}^0 y_n^1 - \sum_{m=1}^{M} p_{xm}^0 x_m^1}$$

[11]Recall Figure 17.1, which illustrated substitution biases for the Laspeyres and Paasche output price indices.

$$= 1 \Bigg/ \left(\frac{\sum_{n=1}^{N} p_{yn}^0 y_n^1 - \sum_{m=1}^{M} p_{xm}^0 x_m^1}{\sum_{n=1}^{N} p_{yn}^1 y_n^1 - \sum_{m=1}^{M} p_{xm}^1 x_m^1} \right)$$

$$= 1 \Bigg/ \left[s_y^1 \left(\frac{\sum_{n=1}^{N} p_{yn}^0 y_n^1}{\sum_{n=1}^{N} p_{yn}^1 y_n^1} \right) + s_x^1 \left(\frac{\sum_{m=1}^{M} p_{xm}^0 x_m^1}{\sum_{m=1}^{M} p_{xm}^1 x_m^1} \right) \right]$$

$$= \{s_y^1[P_P(p_y^0,p_y^1,y^0,y^1)]^{-1} + s_x^1[P_P(p_x^0,p_x^1,x^0,x^1)]^{-1}\}^{-1}$$

where the period 1 output share s_y^1 and the period 1 intermediate input share s_x^1 are defined as follows:

$$(18.44)\ s_y^1 \equiv \frac{\sum_{n=1}^{N} p_{yn}^1 y_n^1}{\sum_{n=1}^{N} p_{yn}^1 y_n^1 - \sum_{m=1}^{M} p_{xm}^1 x_m^1} = \frac{P_1^1 Q_1^1}{\left(P_1^1 Q_1^1 + P_2^1 Q_2^1\right)};$$

$$s_x^1 \equiv \frac{-\sum_{m=1}^{M} p_{xm}^1 x_m^1}{\sum_{n=1}^{N} p_{yn}^1 y_n^1 - \sum_{m=1}^{M} p_{xm}^1 x_m^1} = \frac{P_2^1 Q_2^1}{\left(P_1^1 Q_1^1 + P_2^1 Q_2^1\right)}.$$

Note that s_y^1 will be greater than 1, and s_x^1 will be negative. Thus, equation (18.43) says that the Paasche value-added deflator can be written as a weighted *harmonic* average of the Paasche output price index, $P_P(p_y^0,p_y^1,y^0,y^1)$, and the Paasche intermediate input price index, $P_P(p_x^0,p_x^1,x^0,x^1)$.

18.48 The analysis presented in this section on the relationships between the output price, the intermediate input price, and the value-added deflator for an establishment can be extended to the industry or national levels.

D.3 Value-added deflators and double-deflation method for constructing real value added

18.49 In the previous section, it was shown how the Paasche and Laspeyres value-added deflators for an establishment were related to the Paasche and Laspeyres output and intermediate input price indices for an establishment. In this section, this analysis will be extended to look at the problems involved in using these indices to deflate nominal values into real values. Having defined a value-added deflator, $P(p^0,p^1,q^0,q^1)$, using some index number formula, equation (15.4) in Chapter 15 can be used to define a corresponding quantity index, $Q(p^0,p^1,q^0,q^1)$, which can be interpreted as the growth rate for real value added from period 0 to 1; that is, given P, Q can be defined as follows:

$$(18.45)\ Q(p^0,p^1,q^0,q^1) \equiv \left[\frac{V^1}{V^0}\right] P\left(p^0,p^1,q^0,q^1\right),$$

where V^t is the nominal establishment value added for period $t = 0,1$.

18.50 When the Laspeyres value-added deflator, $P_L(p^0,p^1,q^0,q^1)$, is used as the price index in equation (18.45), the resulting quantity index Q is the Paasche value-added quantity index Q_P defined as follows:

$$(18.46)\ Q_P(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{n=1}^{N} p_{yn}^1 y_n^1 - \sum_{m=1}^{M} p_{xm}^1 x_m^1}{\sum_{n=1}^{N} p_{yn}^1 y_n^0 - \sum_{m=1}^{M} p_{xm}^1 x_m^0}.$$

When the Paasche value-added deflator, $P_P(p^0,p^1,q^0,q^1)$, is used as the price index in equation (18.45), the resulting quantity index Q is the Laspeyres value-added quantity index Q_L defined as follows:

$$(18.47)\ Q_L(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{n=1}^{N} p_{yn}^0 y_n^1 - \sum_{m=1}^{M} p_{xm}^0 x_m^1}{\sum_{n=1}^{N} p_{yn}^0 y_n^0 - \sum_{m=1}^{M} p_{xm}^0 x_m^0}.$$

18.51 Given a generic value-added quantity index, $Q(p^0,p^1,q^0,q^1)$, real value added in period 1 at the prices of period 0, rva^1, can be defined as the period 0 nominal value added of the establishment escalated by the value-added quantity index Q; that is,

$$(18.48)\ rva^1 \equiv V^0\, Q(p^0,p^1,q^0,q^1)$$

$$= \sum_{n=1}^{N} p_{yn}^0 y_n^0 - \sum_{m=1}^{M} p_{xm}^0 x_m^0 \quad Q\left(p^0, p^1, q^0, q^1\right).$$

18.52 If the Laspeyres value-added quantity index $Q_L(p^0,p^1,q^0,q^1)$ defined by equation (18.47) is used as the escalator of nominal value added in equation (18.48), the following rather interesting decomposition for the resulting period 1 real value added at period 0 prices is obtained:

(18.49) rva^1

$$\equiv \sum_{n=1}^{N} p_{yn}^0 y_n^0 - \sum_{m=1}^{M} p_{xm}^0 x_m^0 \quad Q_L\left(p^0, p^1, q^0, q^1\right)$$

$$= \sum_{n=1}^{N} p_{yn}^0 y_n^1 - \sum_{m=1}^{M} p_{xm}^0 x_m^1$$

using equation (18.47)

$$= \sum_{n=1}^{N} p_{yn}^0 y_n^0 \left(\frac{\sum_{n=1}^{N} p_{yn}^0 y_n^1}{\sum_{n=1}^{N} p_{yn}^0 y_n^0} \right) - \sum_{m=1}^{M} p_{xm}^0 x_m^0 \left(\frac{\sum_{m=1}^{M} p_{xm}^0 x_m^1}{\sum_{m=1}^{M} p_{xm}^0 x_m^0} \right)$$

$$\equiv \sum_{n=1}^{N} p_{yn}^0 y_n^0 \, Q_L\left(p_y^0, p_y^1, q_y^0, q_y^1\right) - \sum_{m=1}^{M} p_{xm}^0 x_m^0 \, Q_L\left(p_x^0, p_x^1, q_x^0, q_x^1\right).$$

Thus, period 1 real value added at period 0 prices, rva^1, is defined as period 0 nominal value added, $\sum_{n=1}^{N} p_{yn}^0 y_n^0 - \sum_{m=1}^{M} p_{xm}^0 x_m^0$, escalated by the Laspeyres value-added quantity index, $Q_L(p^0,p^1,q^0,q^1)$, defined by equation (18.47). But the last line of equation (18.49) shows that rva^1 is also equal to the period 0 value of production, $\sum_{n=1}^{N} p_{yn}^0 y_n^0$, escalated by the Laspeyres output quantity index,[12] $Q_L\left(p_y^0, p_y^1, q_y^0, q_y^1\right)$, minus the period 0 intermediate input cost, $\sum_{m=1}^{M} p_{xm}^0 x_m^0$, escalated by the Laspeyres intermediate input quantity index, $Q_L\left(p_x^0, p_x^1, q_x^0, q_x^1\right)$.

[12]The use of the Laspeyres output quantity index can be traced back to Bowley (1921, p. 203).

18.53 Using equation (18.45) yields the following formula for the Laspeyres value-added quantity index, Q_L, in terms of the Paasche value-added deflator, P_P:

(18.50) $Q_L(p^0,p^1,q^0,q^1) = \left[\frac{V^1}{V^0}\right] \Big/ P_P\left(p^0, p^1, q^0, q^1\right).$

Now, substitute equation (18.50) into the first line of equation (18.49) to obtain the following alternative decomposition for the period 1 real value added at period 0 prices, rva^1:

(18.51) $$rva^1 \equiv \frac{\sum_{n=1}^{N} p_{yn}^1 y_n^1 - \sum_{m=1}^{M} p_{xm}^1 x_m^1}{P_P\left(p^0, p^1, q^0, q^1\right)}$$

$$= \sum_{n=1}^{N} p_{yn}^0 y_n^1 - \sum_{m=1}^{M} p_{xm}^0 x_m^1$$

using equation (18.43)

$$= \sum_{n=1}^{N} p_{yn}^1 y_n^1 \Bigg/ \left(\frac{\sum_{n=1}^{N} p_{yn}^1 y_n^1}{\sum_{n=1}^{N} p_{yn}^0 y_n^1} \right) - \sum_{m=1}^{M} p_{xm}^1 x_m^1 \Bigg/ \left(\frac{\sum_{m=1}^{M} p_{xm}^1 x_m^1}{\sum_{m=1}^{M} p_{xm}^0 x_m^1} \right)$$

$$\equiv \frac{\sum_{n=1}^{N} p_{yn}^1 y_n^1}{P_P\left(p_y^0, p_y^1, q_y^0, q_y^1\right)} - \frac{\sum_{m=1}^{M} p_{xm}^1 x_m^1}{P_P\left(p_x^0, p_x^1, q_x^0, q_x^1\right)}.$$

Thus, period 1 real value added at period 0 prices, rva^1, is equal to period 1 nominal value added, $\sum_{n=1}^{N} p_{yn}^1 y_n^1 - \sum_{m=1}^{M} p_{xm}^1 x_m^1$, deflated by the Paasche *value-added deflator*, $P_P\left(p^0, p^1, q^0, q^1\right)$, defined by equation (18.43). But the last line of equation (18.51) shows that rva^1 is also equal to the period 1 value of production, $\sum_{n=1}^{N} p_{yn}^1 y_n^1$, deflated by the Paasche output price index, $P_P\left(p_y^0, p_y^1, q_y^0, q_y^1\right)$, minus the period 1 intermediate input cost, $\sum_{m=1}^{M} p_{xm}^{1} x_{m}^{1}$, deflated by the Paasche intermediate input price index, $P_P\left(p_x^0, p_x^1, q_x^0, q_x^1\right)$. Thus, the use of the Paasche value-added deflator leads to a measure of period 1 real value added at period 0 prices, rva^1, that is equal to period 1 deflated output minus period 1 deflated intermediate input. Hence, this method for constructing a real value-added measure is called the *double-deflation method*.[13] The method of double deflation has been subject to some criticism. Peter Hill (1996) has shown that errors in measurement in the individual components, reflected in a higher variance of price changes, can lead to even larger errors in the double-deflated value added, since the subtraction of the two variances compounds the overall error.

18.54 There is a less well-known method of double deflation that reverses the above roles of the Paasche and Laspeyres indices. Instead of expressing real value added in period 1 at the prices of period 0, it is also possible to define real value added in period 0 at the prices of period 1, rva^0. Using this methodology, given a generic value-added quantity index, the counterpart to equation (18.48) is

$$(18.52)\quad rva^0 \equiv V^1 \big/ Q\left(p^0, p^1, q^0, q^1\right) = \frac{\sum_{n=1}^{N} p_{yn}^{1} y_{n}^{1} - \sum_{m=1}^{M} p_{xm}^{1} x_{m}^{1}}{Q\left(p^0, p^1, q^0, q^1\right)}.$$

Thus, to obtain period 0 real value added at the prices of period 1, rva^0, take the nominal period 1 value added, V^1, and deflate it by the value-added quantity index, $Q(p^0,p^1,q^0,q^1)$.

18.55 If the Paasche value-added quantity index, $Q_P(p^0,p^1,q^0,q^1)$, defined by equation (18.46) above is used as the deflator of nominal value added in (18.52), the following interesting decomposition for the resulting period 0 real value added at period 1 prices is obtained:

$$(18.53)\quad rva^0 \equiv \left[\sum_{n=1}^{N} p_{yn}^{1} y_{n}^{1} - \sum_{m=1}^{M} p_{xm}^{1} x_{m}^{1}\right] \Big/ Q_P\left(p^0, p^1, q^0, q^1\right)$$

$$= \left[\sum_{n=1}^{N} p_{yn}^{1} y_{n}^{0} - \sum_{m=1}^{M} p_{xm}^{1} x_{m}^{0}\right]$$

using equation (18.46)

$$= \left[\sum_{n=1}^{N} p_{yn}^{1} y_{n}^{1}\right] \Bigg/ \left(\frac{\sum_{n=1}^{N} p_{yn}^{1} y_{n}^{1}}{\sum_{n=1}^{N} p_{yn}^{1} y_{n}^{0}}\right) - \sum_{m=1}^{M} p_{xm}^{1} x_{m}^{1} \Bigg/ \left(\frac{\sum_{m=1}^{M} p_{xm}^{1} x_{m}^{1}}{\sum_{m=1}^{M} p_{xm}^{1} x_{m}^{0}}\right)$$

$$\equiv \frac{\sum_{n=1}^{N} p_{yn}^{1} y_{n}^{1}}{Q_P\left(p_y^0, p_y^1, y^0, y^1\right)} - \frac{\sum_{m=1}^{M} p_{xm}^{1} x_{m}^{1}}{Q_P\left(p_x^0, p_x^1, x^0, x^1\right)}.$$

Thus, period 0 real value added at period 1 prices, rva^0, is defined as period 1 nominal value added, $\sum_{n=1}^{N} p_{yn}^{1} y_{n}^{1} - \sum_{m=1}^{M} p_{xm}^{1} x_{m}^{1}$, deflated by the Paasche value-added quantity index, $Q_P(p^0,p^1,q^0,q^1)$, defined by equation (18.46). But the last line of equation (18.53) shows that rva^0 is also equal to the period 1 value of production, $\sum_{n=1}^{N} p_{yn}^{1} y_{n}^{1}$, deflated by the Paasche output quantity index, $Q_P(p_y^0,p_y^1,y^0,y^1)$, minus the period 1 intermediate input cost, $\sum_{m=1}^{M} p_{xm}^{1} x_{m}^{1}$, deflated by the Paasche intermediate input quantity index, $Q_P(p_x^0,p_x^1,x^0,x^1)$.

18.56 Using equation (18.45) yields the following formula for the Paasche value-added quantity index, Q_P, in terms of the Laspeyres value-added deflator, P_L:

$$(18.54)\quad Q_P(p^0,p^1,q^0,q^1) = \left[\frac{V^1}{V^0}\right] \Big/ P_L\left(p^0, p^1, q^0, q^1\right).$$

Now, substitute equation (18.54) into the first line of equation (18.53) to obtain the following alternative decomposition for the period 0 real value added at period 1 prices, rva^0:

[13] See Schreyer (2001, p. 32). A great deal of useful material in this book will be of interest to price statisticians.

(18.55) rva^0

$$\equiv \left[\sum_{n=1}^{N} p_{yn}^0 y_n^0 - \sum_{m=1}^{M} p_{xm}^0 x_m^0\right] P_L\left(p^0, p^1, q^0, q^1\right)$$

$$= \left[\sum_{n=1}^{N} p_{yn}^1 y_n^0 - \sum_{m=1}^{M} p_{xm}^1 x_m^0\right]$$

using equation (18.41)

$$= \left[\sum_{n=1}^{N} p_{yn}^0 y_n^0\right]\left(\frac{\sum_{n=1}^{N} p_{yn}^1 y_n^0}{\sum_{n=1}^{N} p_{yn}^0 y_n^0}\right) - \sum_{m=1}^{M} p_{xm}^0 x_m^0 \left(\frac{\sum_{m=1}^{M} p_{xm}^1 x_m^0}{\sum_{m=1}^{M} p_{xm}^0 x_m^0}\right)$$

$$= \sum_{n=1}^{N} p_{yn}^0 y_n^0 \; P_L\left(p_y^0, p_y^1, y^0, y^1\right) - \sum_{m=1}^{M} p_{xm}^0 x_m^0 \; P_L\left(p_x^0, p_x^1, x^0, x^1\right).$$

Thus, period 0 real value added at period 1 prices, rva^0, is equal to period 0 nominal value added, $\sum_{n=1}^{N} p_{yn}^0 y_n^0 - \sum_{m=1}^{M} p_{xm}^0 x_m^0$, escalated by the Laspeyres *value-added deflator*, $P_L(p^0,p^1,q^0,q^1)$, defined by equation (18.41). But the last line of equation (18.55) shows that rva^0 is also equal to the period 0 value of production, $\sum_{n=1}^{N} p_{yn}^0 y_n^0$, escalated by the Laspeyres output price index, $P_L(p_y^0,p_y^1,y^0,y^1)$, minus the period 0 intermediate input cost, $\sum_{m=1}^{M} p_{xm}^0 x_m^0$, escalated by the Laspeyres intermediate input price index, $P_L(p_x^0,p_x^1,x^0,x^1)$.[14]

E. Aggregation of Establishment Deflators into a National Value-Added Deflator

18.57 Once establishment value-added deflators have been constructed for each establishment, there remains the problem of aggregating up these deflators into an industry or regional or national value-added deflator. Only the national aggregation problem is considered in this section, but the same logic will apply to the regional and industry aggregation problems.[15]

18.58 Let the vectors of output price, output quantity, intermediate input price, and intermediate input price vectors for an establishment e in period t be denoted by p_y^{et}, y^{et}, p_x^{et}, and x^{et}, respectively, for t = 0,1 and e = 1,...,E. As usual, the net price and net quantity vectors for establishment e in period t are defined as $p^{et} \equiv [p_y^{et},p_x^{et}]$ and $q^{et} \equiv [y^{et},-x^{et}]$ for t = 0,1 and e = 1,...,E. Suppose that a bilateral index number formula P is used to construct a value-added deflator, $P(p^{e0},p^{e1},q^{e0},q^{e1})$, for establishment e where e = 1,...,E. Our problem is somehow to aggregate up these establishment indices into a national value-added deflator.

18.59 The two-stage aggregation procedure explained in Section C above is used to do this aggregation. The first stage of the aggregation of price and quantity vectors is for the establishment net output price vectors, p^{et}, and the establishment net output quantity vectors, q^{et}. These establishment price and quantity vectors are combined into national price and quantity vectors, p^t and q^t, as follows:[16]

(18.56) $p^t = (p^{1t}, p^{2t}, \ldots ,p^{Et})$; $q^t = (q^{1t}, q^{1t}, \ldots ,q^{Et})$; t = 0,1.

For each establishment e, its aggregate price of value added P_e^0 in the base period is set equal to 1, and the corresponding establishment e base-period quantity of value added Q_e^0 is defined as the establishment's period 0 value added; that is,

(18.57) $P_e^0 \equiv 1$; $Q_e^0 \equiv \sum_{i=1}^{N+M} p_i^{e0} q_i^{e0}$ for e = 1,2,…,E.

Now, the chosen price index formula P is used to construct a period 1 price for the price of value added for each establishment e, say, P_e^1 for e = 1,2,…,E:

[14] This method for constructing real value-added measures was used by Phillips (1961, p. 320).

[15] The algebra developed in Section E can also be applied to the problem of aggregating establishment or industry output or intermediate input price indices into national output or intermediate input price indices.

[16] Equation (18.56) is the counterpart to equation (18.26) in Section C. Equations (18.57)–(18.61) are counterparts to equations (18.27)–(18.38) in Sections C and D.

(18.58) $P_e^1 \equiv P(p^{e0},p^{e1},q^{e0},q^{e1})$ for $e = 1,2,\ldots E$.

Once the period 1 prices for the E establishments have been defined by equation (18.58), then corresponding establishment e period 1 quantities Q_e^1 can be defined by deflating the period 1 establishment values $\sum_{i=1}^{N+M} p_i^{e1}q_i^{e1}$ by the prices P_e^1 defined by equation (18.58); that is,

(18.59) $Q_e^1 \equiv \sum_{i=1}^{N+M} p_i^{e1}q_i^{e1} \Big/ P_e^1$ for $e = 1,2,\ldots,E$.

The aggregate establishment price and quantity vectors for each period $t = 0,1$ can be defined using equations (18.57) to (18.59). Thus, the period 0 and 1 establishment value-added price vectors P^0 and P^1 are defined as follows:

(18.60) $P^0 = (P_1^0, P_2^0,\ldots,P_E^0) \equiv 1_E$;
$P^1 = (P_1^1, P_2^1,\ldots,P_E^1)$,

where 1_E denotes a vector of ones of dimension E, and the components of P^1 are defined by equation (18.58). The period 0 and 1 establishment value-added quantity vectors Q^0 and Q^1 are defined as

(18.61) $Q^0 = (Q_1^0, Q_2^0,\ldots,Q_E^0)$;
$Q^1 = (Q_1^1, Q_2^1,\ldots,Q_E^1)$,

where the components of Q^0 are defined in equation (18.57) and the components of Q^1 are defined in equation (18.59). The price and quantity vectors in equations (18.60) and (18.61) represent the results of the first-stage aggregation (over commodities within an establishment). These vectors can now be inputs into the second-stage aggregation problem (which aggregates over establishments); that is, our chosen price index formula can be applied using the information in equations (18.60) and (18.61) as inputs into the index number formula. The resulting two-stage aggregation national value-added deflator is $P(P^0,P^1,Q^0,Q^1)$. It should be asked whether this two-stage index equals the corresponding single-stage index $P(p^0,p^1,q^0,q^1)$ that treats each output or intermediate input produced or used by each establishment as a separate commodity, using the same index number formula P. That is, it is asked whether

(18.62) $P(P^0,P^1,Q^0,Q^1) = P(p^0,p^1,q^0,q^1)$.

18.60 If the Laspeyres or Paasche formula is used at each stage of each aggregation, the answer to the above question is yes. Thus, in particular, the national Laspeyres value-added deflator that is constructed in a single stage of aggregation, $P_L(p^0,p^1,q^0,q^1)$, is equal to the two-stage Laspeyres value-added deflator, $P_L(P^0,P^1,Q^0,Q^1)$, where the Laspeyres formula is used in equation (18.58) to construct establishment value-added deflators in the first stage of aggregation. If a superlative formula is used at each stage of aggregation, the answer to the above consistency-in-aggregation question is no: equation (18.62) using a superlative P will hold only approximately. However, if the Fisher, Walsh, or Törnqvist price index formulas are used at each stage of aggregation, the differences between the right- and left-hand sides of equation (18.62) will be very small using normal time series data.

F. National Value-Added Deflator versus Final-Demand Deflator

18.61 In this section, we ask whether there are any relationships between the *national value-added deflator* defined in the preceding sections of this chapter and the *national deflator for final-demand expenditures*. In particular, we look for conditions that will imply that the two deflators are exactly equal.

18.62 Assume that the commodity classification for intermediate inputs is exactly the same as the commodity classification for outputs, so that, in particular, N, the number of outputs, is equal to M, the number of intermediate inputs. This assumption is not restrictive, since if N is chosen to be large enough, all produced intermediate inputs can be accommodated in the expanded output classification.[17] With this change in assumptions, the

[17]It is not necessary to assume that each establishment or sector of the economy produces all outputs and uses all intermediate inputs in each of the two periods being compared. All that is required is that if an output is not produced in one period by establishment e, then that output is also not produced in the other period. Similarly, it is required that if an establishment does not use a particular intermediate input in one period, then it also does not use it in the other period.

same notation can be used as was used in the previous section. Thus, let the vectors of output price, output quantity, intermediate input price, and intermediate input price vectors for an establishment e in period t be denoted by p_y^{et}, y^{et}, p_x^{et}, and x^{et}, respectively, for $t = 0,1$ and $e = 1,...,E$. As usual, the net price and net quantity vectors for establishment e in period t are defined as $p^{et} \equiv [p_y^{et}, p_x^{et}]$ and $q^{et} \equiv [y^{et}, -x^{et}]$ for $t = 0,1$ and $e = 1,...,E$. Again, define the national price and quantity vectors, p^t and q^t, as

$$\begin{aligned} p^t &\equiv (p^{1t}; p^{2t}; \dots ; p^{Et}) \\ &= (p_y^{1t}, p_x^{1t}; p_y^{2t}, p_x^{2t}; \dots ; p_y^{Et}, p_x^{Et}) \text{ and} \\ q^t &\equiv (q^{1t}, q^{1t}, \dots, q^{Et}) \\ &= (y^{1t}, -x^{1t}, y^{2t}, -x^{2t}; \dots ; y^{Et}, -x^{Et}) \text{ for } t = 0,1. \end{aligned}$$

As in the previous section, an index number formula P is chosen and the national value-added deflator denoted as $P(p^0, p^1, q^0, q^1)$.

18.63 Using the above notation, the period $t$$N$ by E *make matrix* for the economy, Y^t, and the period $t$$N$ by E *use matrix*, X^t, are defined as follows:

$$(18.63)\quad Y^t \equiv [y^{1t}, y^{2t}, \dots, y^{Et}]\ ;\ X^t \equiv [x^{1t}, x^{2t}, \dots, x^{Et}]\ ;\ t = 0,1.$$

The *period* t *final-demand vector* for the economy, f^t, can be defined by summing up all the establishment output vectors y^{et} in the period t make matrix and subtracting all the establishment intermediate input-demand vectors x^{et} in the period t use matrix; that is, define f^t by[18]

$$(18.64)\quad f^t \equiv \sum_{e=1}^{E} y^{et} - \sum_{e=1}^{E} x^{et}\ ;\ t = 0,1.$$

18.64 Final-demand prices are required to match up with the components of the period t final-demand quantity vector $f^t = [f_1^t, \dots, f_N^t]$. The net value of production for commodity n in period t divided by the net deliveries of this commodity to final demand f_n^t is *the period* t *final-demand unit value for commodity* n, p_{fn}^t:

$$(18.65)\quad p_{fn}{}^t \equiv \frac{\sum_{e=1}^{E} p_{yn}^{et} y_n^{et} - \sum_{e=1}^{E} p_{xn}^{et} x_n^{et}}{f_n^t}\ ;\ n = 1,\dots,N\ ;\ t = 0,1.$$

If equation (18.64) is to hold so that production minus intermediate input use equals deliveries to final demand for each commodity in period t, and if the value of production minus the value of intermediate demands is to equal the value of final demand for each commodity in period t, then the value-added prices defined by equation (18.65) must be used as final-demand prices.

18.65 Define the vector of period t final-demand prices as $p_f^t \equiv [p_{f1}^t, p_{f2}^t, \dots, p_{fN}^t]$ for $t = 0,1$, where the components p_{fn}^t are defined by equation (18.65). The corresponding final-demand quantity vector f^t has already been defined by equation (18.64). Hence, a generic price index number formula P can be taken to form the *final-demand deflator*, $P(p_f^0, p_f^1, f^0, f^1)$. It is now asked whether this final-demand deflator is equal to the national value-added deflator $P(p^0, p^1, q^0, q^1)$ defined in Section B.3; that is, whether

$$(18.66)\quad P(p_f^0, p_f^1, f^0, f^1) = P(p^0, p^1, q^0, q^1).$$

Note that the dimensionality of each price and quantity vector that occurs in the left-hand side of equation (18.66) is N (the number of commodities in our output classification), while the dimensionality of each price and quantity vector that occurs in the right-hand side of equation (18.66) is $2NE$, where E is the number of establishments (or industries or sectors that have separate price and quantity vectors for both outputs and intermediate inputs) that are aggregating over.

18.66 The answer to the question asked in the previous paragraph is no; in general, it will *not* be the case that the final-demand deflator is equal to the national value-added deflator.

18.67 However, under certain conditions, equation (18.66) will hold as an equality. A set of conditions is now developed. The first assumption is that all establishments face the same vector of prices p^t in period t for both the outputs that they

[18] Components of f^t can be negative if the corresponding commodity is being imported into the economy during period t, or if the component corresponds to the change in an inventory item.

produce and for the intermediate inputs that they use. That is, it is assumed[19]

(18.67) $p_y^{et} = p_x^{et} = p^t$; $e = 1,...,E$; $t = 0,1$.

If assumptions in equation (18.67) hold, then it is easy to verify that the vector of period t final-demand prices p_f^t defined above by equation (18.65) is also equal to the vector of period t basic prices p^t.

18.68 If assumptions in equation (18.67) hold and the price index formula used in both sides of equation (18.66) is the *Laspeyres formula*, then it can be verified that equation (18.66) will hold as an equality; that is, the Laspeyres final-demand deflator will be equal to the national Laspeyres value-added deflator. To see why this is so, use the Laspeyres formula in equation (18.66) and, for the left-hand side of the index, collect all the quantity terms both in the numerator and denominator of the index that correspond to the common establishment price for the nth commodity, $p_n^t = p_{yn}^{et} = p_{xn}^{et}$, for $e = 1,...,E$. Using equation (18.64) for $t = 0$, the resulting sum of collected quantity terms will sum to f_n^0. Since this is true for $n = 1,...,N$, it can be seen that the left-hand side of the Laspeyres index is equal to the right-hand side of the Laspeyres index.

18.69 If assumptions in equation (18.67) hold and the price index formula used in both sides of equation (18.66) is the *Paasche formula*, then it can be verified that equation (18.66) will also hold as an equality; that is, the Paasche final-demand deflator will equal the national Paasche value-added deflator. To see why this is so, use the Paasche formula in equation (18.66) and, for the left-hand side of the index, collect all the quantity terms both in the numerator and denominator of the index that correspond to the common establishment price for the nth commodity, $p_n^t = p_{yn}^{et} = p_{xn}^{et}$, for $e = 1,...,E$. Using equation (18.64) for $t =1$, the resulting sum of collected quantity terms will sum to f_n^1. Since this is true for $n = 1,...,N$, it can be seen that the left-hand side of the Paasche index is equal to the right-hand side of the Paasche index.

18.70 The results in the previous two paragraphs imply that the national value-added deflator will equal the final-demand deflator provided that Paasche or Laspeyres indices are used and provided that assumptions in equation (18.67) hold. But these two results immediately imply that if equation (18.67) holds and Fisher ideal price indices are used, then an important equality is obtained—that the Fisher national value-added deflator is equal to the Fisher final-demand deflator.

18.71 Recall equation (18.21) of the national Törnqvist-Theil output price index P_T in Section B.1 above. The corresponding national Törnqvist-Theil value-added deflator P_T was defined in Section B.3. Make assumptions in equation (18.67), start with the national Törnqvist-Theil value-added deflator, and collect all the exponents that correspond to the common price relative for commodity n, p_n^1 / p_n^0. Using equation (18.65), the sum of these exponents will equal the exponent for the nth price term, $p_{fn}^1 / p_{fn}^0 = p_n^1 / p_n^0$, in the Törnqvist-Theil final-demand deflator. Since this equality holds for all $n = 1,...,N$, the equality of the national value-added deflator to the final-demand deflator is also obtained if the Törnqvist formula P_T is used on both sides of equation (18.66).

18.72 Summarizing the above results, it has been shown that the national value-added deflator is equal to the final-demand deflator, provided that all establishments face the same vector of prices in each period for both the outputs that they produce and for the intermediate inputs that they use, and provided that either the Laspeyres, Paasche, Fisher, or Törnqvist price index formula is used for both deflators.[20] However, these results were established ignoring the existence of indirect taxes and subsidies that may be applied to the outputs and intermediate inputs of each establishment. It is necessary to extend the initial results to deal with situations where there are indirect taxes on deliveries to final demand and indirect taxes on the use of intermediate inputs.

18.73 Again, it is assumed that all establishments face the same prices for their inputs and outputs, but it is now assumed that their deliveries

[19]Under these hypotheses, the vector of producer prices p^t can be interpreted as the vector of *basic producer prices* that appears in the *1993 SNA*.

[20]This result does not carry over if we use the Walsh price index formula.

to the final-demand sector are *taxed*.[21] Let τ_n^t be the period t ad valorem commodity tax rate on deliveries to final demand of commodity n for $t = 0,1$ and $n = 1,...,N$.[22] Thus, the period t final-demand price for commodity n is now

$$(18.68)\quad p_{fn}^t = p_n^t(1 + \tau_n^t)\ ;\ n = 1,...,N\ ;\ t = 0,1.$$

These tax-adjusted final-demand prices defined by equation (18.68) can be used to form new vectors of final-demand price vectors, $p_f^t \equiv [p_{f1}^t,...,p_{fN}^t]$ for $t = 0,1$. The corresponding final-demand quantity vectors, f^0 and f^1, are still defined by the commodity balance equation (18.64). Now, pick an index number formula P and form the *final-demand deflator* $P(p_f^0,p_f^1,f^0,f^1)$ using the new tax-adjusted prices, p_f^0,p_f^1. If the commodity tax rates τ_n^t are substantial, the new final-demand deflator $P(p_f^0,p_f^1,f^0,f^1)$ can be substantially different from the national value-added deflator $P(p^0,p^1,q^0,q^1)$ defined earlier in this section (because all the commodity tax terms are missing from the national value-added deflator).

[21]Hicks (1940, p. 106) appears to have been the first to note that the treatment of indirect taxes in national income accounting depends on the purpose for which the calculation is to be used. Thus, for measuring productivity, Hicks (1940, p. 124) advocated using prices that best represented marginal costs and benefits from the perspective of producers—that is, basic prices should be used. On the other hand, if the measurement of economic welfare is required, Hicks (1940, pp. 123–24) advocated the use of prices that best represent marginal utilities of consumers—that is, final-demand prices should be used. Bowley (1922, p. 8) advocated the use of final-demand prices, but he implicitly took a welfare point of view: "To the purchaser of whisky, tobacco and entertainment tickets, the goods bought are worth what he pays; it is indifferent to him whether the State or the producer gets the money."

[22]If commodity n is subsidized during period t, then τ_n^t can be set equal to minus the subsidy rate. In most countries, the commodity tax regime is much more complex than we have modeled it above, in that some sectors of final demand are taxed differently than other sectors; for example, exported commodities are generally not taxed or are taxed more lightly than other final-demand sectors. To deal with these complications, it would be necessary to decompose the single final-demand sector into a number of sectors (for example, the familiar $C + I + G + X - M$ decomposition) where the tax treatment in each sector is uniform. In this disaggregated framework, tariffs on imported goods and services can readily be accommodated. There are additional complications owing to the existence of commodity taxes that fall on intermediate inputs. To deal adequately with all these complications would require a rather extended discussion. The purpose here is to indicate to the reader that the national value-added deflator is closely connected to the final-demand deflator.

18.74 However, it is possible to adjust our national value-added deflator in an attempt to make it more comparable to the final-demand deflator. Recall that the price and quantity vectors, p^t and q^t, that appear in the national value-added deflator are defined as follows:[23]

$$(18.69)\quad p^t \equiv [p_y^{1t},p_x^{1t};p_y^{2t},p_x^{2t};...;p_y^{Et},p_x^{Et}]\ ;\ t = 0,1;$$
$$q^t \equiv [y^{1t},-x^{1t};y^{2t},-x^{2t};...;y^{Et},-x^{Et}]\ ;\ t = 0,1;$$

where p_y^{et} is the vector of output prices that establishment e faces in period t, p_x^{et} is the vector of input prices that establishment e faces in period t, y^{et} is the production vector for establishment e in period t, and x^{et} is the vector of intermediate inputs used by establishment e during period t. The adjustment made to the national value-added deflator is that an additional N artificial commodities are added to the list of outputs and inputs that the national value-added deflator aggregates over. Define the price and quantity of the nth extra *artificial commodity* as follows:

$$(18.70)\quad p_n^{At} \equiv p_n^t\tau_n^t\ ;\ q_n^{At} \equiv f_n^t\ ;\ n = 1,...,N\ ;\ t = 0,1.$$

Thus, the period t price of the nth artificial commodity is just the product of the nth basic price, p_n^t, times the nth commodity tax rate in period t, τ_n^t. The period t quantity for the nth artificial commodity is simply equal to period t final demand for commodity n, f_n^t. Note that the period t value of all N artificial commodities is just equal to period t commodity tax revenue. Define the period t price and quantity vectors for the artificial commodities in the usual way; that is, $p^{At} \equiv [p_1^{At},...,p_N^{At}]$ and $q^{At} \equiv [q_1^{At},...,q_N^{At}] = f^t$, $t = 0,1$. Now, add the extra price vector p^{At} to the initial period t price vector p^t that was used in the national value-added deflator, and add the extra quantity vector q^{At} to the initial period t quantity vector q^t that was used in the national value-added deflator. That is, define the *augmented national price and quantity vectors*, p^{t*} and q^{t*} as follows:

$$(18.71)\quad p^{t*} \equiv [p^t,p^{At}]\ ;\ q^{t*} \equiv [q^t,q^{At}]\ ;\ t = 0,1.$$

[23]Under assumptions in equation (18.67), the definition of p^t simplifies dramatically.

Using the augmented price and quantity vectors defined above, calculate a *new tax-adjusted national value-added deflator* using the chosen index number formula, $P(p^0*,p^1*,q^0*,q^1*)$, and ask whether it will equal the final-demand deflator, $P(p_f^{\,0},p_f^{\,1},f^{\,0},f^{\,1})$ using the new tax-adjusted prices, $p_f^{\,0},p_f^{\,1}$, defined by equation (18.68). That is, ask whether the following equality holds:

(18.72) $P(p^0*,p^1*,q^0*,q^1*) = P(p_f^{\,0},p_f^{\,1},f^{\,0},f^{\,1})$.

18.75 Choose P to be P_L, the Laspeyres formula, and evaluate the left-hand side of equation (18.72). Using assumptions in equation (18.67), collect all terms in the numerator of the Laspeyres national value-added deflator, $P_L(p^0*,p^1*,q^0*,q^1*)$, that correspond to the nth commodity price $p_n^{\,1}$. Using equation (18.64) for $t = 0$, it is found that the sum of these terms involving $p_n^{\,1}$ is $p_n^{\,1}(1 + \tau_n^{\,1})f_n^{\,0}$, which is equal to the nth term in the numerator of the final-demand deflator, $P_L(p_f^{\,0},p_f^{\,1},f^{\,0},f^{\,1})$. In a similar fashion, collect all terms in the denominator of the Laspeyres national value-added deflator, $P_L(p^0*,p^1*,q^0*,q^1*)$, that correspond to the nth commodity price $p_n^{\,0}$. Using equation (18.64) for $t = 0$, it is found that the sum of these terms involving $p_n^{\,0}$ is $p_n^{\,0}(1 + \tau_n^{\,1})f_n^{\,0}$, which is equal to the nth term in the denominator of the final-demand deflator, $P_L(p_f^{\,0},p_f^{\,1},f^{\,0},f^{\,1})$. Thus, equation (18.72) does hold as an exact equality under the above assumptions if the Laspeyres price index is used for each of the deflators.

18.76 Now choose P to be P_P, the Paasche formula, and evaluate the left-hand side of equation (18.72). Using assumptions in equation (18.67), collect all terms in the numerator of the Paasche national value-added deflator, $P_P(p^0*,p^1*,q^0*,q^1*)$, that correspond to the nth commodity price $p_n^{\,1}$. Using equation (18.64) for $t = 1$, it is found that the sum of these terms involving $p_n^{\,1}$ is $p_n^{\,1}(1 + \tau_n^{\,1})f_n^{\,1}$, which is equal to the nth term in the numerator of the final-demand deflator, $P_P(p_f^{\,0},p_f^{\,1},f^{\,0},f^{\,1})$. In a similar fashion, collect all terms in the denominator of the Paasche national value-added deflator, $P_P(p^0*,p^1*,q^0*,q^1*)$, that correspond to the nth commodity price $p_n^{\,0}$. Using equation (18.64) for $t = 1$, it is found that the sum of these terms involving $p_n^{\,0}$ is $p_n^{\,0}(1 + \tau_n^{\,1})f_n^{\,1}$, which is equal to the nth term in the denominator of the final-demand deflator, $P_P(p_f^{\,0},p_f^{\,1},f^{\,0},f^{\,1})$. Thus, equation (18.72) does hold as an exact equality under the above assumptions if the Paasche price index is used for each of the deflators. Putting this result together with the result in the previous paragraph, we see that under the above assumptions, equation (18.72) also holds as an exact equality if the Fisher index is used for both the final-demand deflator and tax-adjusted national value-added deflator, which is built up using industry information.

18.77 Finally, choose P to be P_T, the Törnqvist-Theil formula for a price index, and evaluate both sides of equation (18.79). In general, this time an exact equality is *not* obtained between the national Törnqvist-Theil tax-adjusted value-added deflator $P_T(p^0*,p^1*,q^0*,q^1*)$ and the Törnqvist-Theil final-demand deflator $P_T(p_f^{\,0},p_f^{\,1},f^{\,0},f^{\,1})$.

18.78 However, if the extra assumption—in addition to equation (18.67), the assumption of equal basic prices across industries—is made that the commodity tax rates are equal in periods 0 and 1 so that

(18.73) $\tau_n^{\,0} = \tau_n^{\,1}$ for $n = 1,...,N$,

then it can be shown that the national Törnqvist-Theil tax-adjusted value-added deflator $P_T(p^0*,p^1*,q^0*,q^1*)$ and the Törnqvist-Theil final-demand deflator $P_T(p_f^{\,0},p_f^{\,1},f^{\,0},f^{\,1})$ are *exactly equal*.

The last few results can be modified to work in reverse: that is, start with the final-demand deflator and make some adjustments to it using artificial commodities. Then the resulting tax-adjusted final-demand deflator can equal the original unadjusted national value-added deflator. To implement this reverse procedure, it is necessary to add an additional N artificial commodities to the list of outputs and inputs that the final-demand deflator aggregates over. Define the price and quantity of the nth extra *artificial commodity* as follows:

(18.74) $p_n^{At} \equiv p_n^{\,t}\tau_n^{\,t}$; $q_n^{At} \equiv -f_n^{\,t}$; $n = 1,...,N$; $t = 0,1$.

Thus, the period t price of the nth artificial commodity is just the product of the nth basic price, $p_n^{\,t}$, times the nth commodity tax rate in period t, $\tau_n^{\,t}$. The period t quantity for the nth artificial commodity is simply equal to minus period t final demand for commodity n, $-f_n^{\,t}$. Note that the period t value of all N artificial commodities is just equal to *minus* period t commodity tax revenue. Define the period t price and quantity vectors for the artificial commodities in the usual way; that is, $p^{At} \equiv [p_1^{At},...,p_N^{At}]$ and $q^{At} \equiv [q_1^{At},...,q_N^{At}] = f^{\,t}$, $t = 0,1$. The extra price vector p^{At} is now added to the old period t price vector p_f^t that was used in the final-demand deflator, and the extra quantity vector q^{At} is added to the initial period t quantity vector $f^{\,t}$ that was used in the final-demand deflator. That is, define the *augmented final-demand price and quantity vectors*, p^t* and $f^{\,t}*$, as follows:

$$(18.75)\quad p_f^{\,t}* \equiv [p_f^{\,t}, p^{At}]\ ;\ f^{\,t}* \equiv [f^{\,t}, q^{At}]\ ;\ t = 0,1.$$

Using the augmented price and quantity vectors defined above, a *new tax-adjusted final-demand deflator* is calculated using the chosen index number formula, $P(p_f^{\,0}*, p_f^{\,1}*, f^{\,0}*, f^{\,1}*)$, and the question asked is whether it will equal our initial *national value-added deflator* (that did not make any tax adjustments for commodity taxes on final demands), $P(p^0,p^1,q^0,q^1)$; that is, ask whether the following equality holds:

$$(18.76)\ P(p_f^{\,0}*, p_f^{\,1}*, f^{\,0}*, f^{\,1}*) = P(p^0,p^1,q^0,q^1).$$

18.79 Under the assumption that all establishments face the same prices, it can be shown *that the tax-adjusted final-demand deflator will exactly equal the national value-added deflator*, provided that the index number formula in equation (18.76) is chosen to be the Laspeyres, Paasche, or Fisher formulas, P_L, P_P, or P_F. In general, equation (18.76) will not hold as an exact equality if the Törnqvist-Theil formula, P_T, is used. However, if the commodity tax rates are equal in periods 0 and 1, so that assumptions in equation (18.73) hold in addition to assumptions in equation (18.67), then it can be shown that equation (18.76) will hold as an exact equality when P is set equal to P_T, the Törnqvist-Theil formula. These results are of some practical importance for the following reason. Most countries do not have adequate surveys that will support a complete system of value-added price indices for each sector of the economy.[24] Adequate information is generally available that will enable the statistical agency to calculate the final-demand deflator. However, for measuring the productivity of the economy using the economic approach to index number theory, the national value-added deflator is the preferred deflator.[25] The results that have just been stated show how the final-demand deflator can be modified to give a close approximation to the national value-added deflator under certain conditions.

18.80 It has always been a bit of a mystery how tax payments should be decomposed into price and quantity components in national accounting theory. The results presented in this section may be helpful in suggesting reasonable decompositions under certain conditions.

[24]In particular, information on the prices and quantities of intermediate inputs used by sector are generally lacking. These data deficiencies were noted by Fabricant (1938, pp. 566–70) many years ago, and he indicated some useful methods that are still used today in attempts to overcome these data deficiencies.

[25]See Schreyer (2001) for more explanation.