17. Economic Approach

A. Introduction

A.1 Setting the Stage

17.1 The *family of PPIs* provides price indices to deflate parts of the system of national accounts. As is well known,[1] there are three distinct approaches to measuring GDP:

- The production approach,
- The expenditure or final demand approach, and
- The income approach.

The production approach[2] to calculating nominal GDP involves calculating the value of outputs produced by an industry and subtracting the value of intermediate inputs (or intermediate consumption, to use the national accounting term) used in the industry. This difference in value is called the industry's *value added*. Summing these industry estimates of value added leads to an estimate of national GDP. PPIs are used to separately deflate both industry outputs and industry intermediate inputs.[3] A PPI also is used to deflate an industry's nominal value added into value added at constant prices.

17.2 The economic approach to the PPI begins not at the industry level, but at the *establishment* level. An establishment is the PPI counterpart to a *household* in the theory of the CPI. An establishment is an economic entity that undertakes *production* or *productive activity* at a specific geographic location in the country and is capable of providing basic accounting information on the prices and quantities of the outputs it produces and on the inputs it uses during an accounting period. This chapter focuses on establishments that undertake production under a *for-profit* motivation. In Chapter 14, it was shown that the *1993 SNA* output in the production account is broken down into market output (P.11), output for own final use (P.12), and other nonmarket output (P.13). The latter includes output of government and nonprofit institutions serving households distributed free or sold at prices not economically significant. The PPI covers all types of domestically produced or processed goods and services that are valued at market prices and thus excludes P.13.

17.3 *Production* is an activity that transforms or combines material inputs into other material outputs (for example, agricultural, mining, manufacturing, or construction activities) or transports materials from one location to another. Production also includes storage activities, which in effect transport materials in the same location from one time period to another. Finally, production also includes the use and creation of services of all types.[4]

17.4 There are two major problems with the above definition of an establishment. The first is that many production units at specific geographic locations do not have the capability of providing basic accounting information on their inputs used and outputs produced. These production units may simply be a division or single plant of a large firm, and detailed accounting information on prices may be available only at the head office (or not at all). If this is the case, the definition of an establishment is modified to include production units at a number of specific geographic locations in the country instead of just one location. The important aspect of the definition of an establishment is that it be able to provide accounting information on prices and quantities.[5] A second problem is that

[1]See Eurostat and others (1993) or Bloem, Dippelsman, and Maehle (2001, p. 17).

[2]Early contributors to this approach include Bowley (1922, p. 2), Rowe (1927, p. 173), Burns (1930, pp. 247–50), and Copeland (1932, pp. 3–5).

[3]Additional material relating national accounting aggregates to PPIs may be found in Chapter 14.

[4]See Peter Hill (1999) for a taxonomy for services.

[5]In this modified definition of an establishment, it is generally a smaller collection of production units than a *firm*, since a firm may be multinational. Thus, another way of de-
(continued)

while the establishment may be able to report accurate quantity information, its price information may be based on *transfer prices* set by a head office. These transfer prices are *imputed prices* and may not be very closely related to market prices.[6]

17.5 Thus the problems involved in obtaining the correct commodity prices for establishments are generally more difficult than the corresponding problems associated with obtaining market prices for households. However, in this chapter, these problems will be ignored, and it will be assumed that representative market prices are available for each output produced by an establishment and for each intermediate input used by the same establishment for at least two accounting periods.[7]

17.6 The economic approach to PPIs requires that establishment output prices *exclude* any indirect taxes that various layers of government might levy on outputs produced by the establishment. These indirect taxes are excluded because firms do not get to keep these tax revenues, even though they may collect them for governments. Thus, these taxes are not part of establishment revenue streams. On the other hand, the economic approach to PPIs requires that establishment intermediate input prices *include* any indirect taxes that governments might levy on these inputs used by the establishment. The reason for including these taxes is that they are actual costs paid by the establishment. These conventions on the treatment of indirect taxes on production are consistent with those specified in Section B.1 of Chapter 2.

17.7 For the first sections of this chapter, an *output price index*, an *intermediate input price index*, and a *value-added deflator*[8] will be defined for a *single establishment* from the economic perspective. In subsequent sections, aggregation will take place over establishments to define national counterparts to these establishment price indices.

17.8 Some notation is required. Consider the case of an establishment that produces N commodities during two periods, periods 0 and 1. Denote the period *t output price vector* by $p_y^t \equiv [p_{y1}^t,...,p_{yN}^t]$ and the corresponding period *t output quantity vector* by $y^t \equiv [y_1^t,...,y_N^t]$, for $t = 0,1$. Assume that the establishment uses M commodities as intermediate inputs during periods 0 and 1. An *intermediate input* is an input produced by another establishment in the country or an imported (nondurable) commodity.[9] The period *t intermediate*

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fining an establishment for our purposes is as follows: an establishment is the smallest aggregate of national production units able to provide accounting information on its inputs and outputs for the time period under consideration.

[6]For many highly specialized intermediate inputs in a multistage production process using proprietary technologies, market prices may simply not exist. Furthermore, several alternative concepts could be used to define transfer prices; see Diewert (1985) and Eden (1998). The *1993 SNA* (paragraph 6.82) notes that for deliveries between establishments belonging to the same enterprise

> Goods and services that one establishment provides to a different establishment belonging to the same enterprise are counted as part of the output of the producing establishment. Such goods and services may be used for intermediate consumption by the receiving establishment, but they also could be used for gross fixed capital formation. The goods and services should be valued by the producing establishment at current basic prices; the receiving establishment should value them at the same prices plus any additional transportation costs paid to third parties. The use of artificial transfer prices employed for internal accounting purposes within the enterprise should be avoided, if possible.

The difficulties in ascertaining such prices are recognized however:

> From an accounting point of view it can be difficult to partition a vertically integrated enterprise into establishments because values have to be imputed for the outputs from the earlier stages of production which are not actually sold on the market and which become intermediate inputs into later stages. Some of these enterprises may record the intra-enterprise deliveries at prices that reflect market values, but others may not. Even if adequate data are available on the costs incurred at each stage of production, it may be difficult to decide what is the appropriate way in which to allocate the operating surplus of the enterprise among the various stages. One possibility is that a uniform rate of profit could be applied to the costs incurred at each stage (*1993 SNA*, paragraph 5.33).

[7]These pricing problems are pursued in Chapter 6, where the concept of a market price for each product produced by an establishment during the accounting period under consideration is the value of production for that product divided by the quantity produced during that period; that is, the price is the average price for that product.

[8]While the value-added price index is just like any other price index in its definition, it is commonly referred to as the "value-added deflator," and the *Manual* will observe this common terminology.

[9]However, capital inputs or durable inputs are excluded from the list of intermediate inputs. A durable input is an input whose contribution to production lasts more than one accounting period. This makes the definition of a durable

(continued)

input price vector is denoted by $p_x^t \equiv [p_{x1}^t,...,p_{xM}^t]$ and the corresponding period *t intermediate input quantity vector* by $x^t \equiv [x_1^t,...,x_M^t]$ for $t = 0,1$. Finally, it is assumed that that the establishment uses the services of *K primary inputs* during periods 0 and 1. The period *t primary input vector* used by the establishment is denoted by $z^t \equiv [z_1^t,...,z_K^t]$ for $t = 0,1$.

17.9 Note it is assumed that the list of commodities produced by the establishment and the list of inputs used by the establishment *remains the same* over the two periods for which a price comparison is wanted. In real life, the list of commodities used and produced by an establishment does not remain constant over time. New commodities appear and old commodities disappear. The reasons for this churning of commodities include the following:

(i) Producers substitute new processes for older ones in response to changes in relative prices, and some of these new processes use new inputs.
(ii) Technical progress creates new processes or products, and the new processes use inputs not used in previous periods.
(iii) Seasonal fluctuations in the demand (or supply) of commodities cause some commodities to be unavailable in certain periods of the year.

The introduction of new commodities is dealt with in Chapter 21 and the problems associated with seasonal commodities in Chapter 22. In the present chapter, these complications are ignored, and it is assumed that the list of commodities remains the *same* over the two periods under consideration. It also will be assumed that all establishments are present in both periods under consideration; that is, there are no new or disappearing establishments.[10]

17.10 When convenient, the above notation will be simplified to match the notation used in Chapters 15 and 16. Thus, when studying the output price index, $p_y^t \equiv [p_{y1}^t,...,p_{yN}^t]$ and $y^t \equiv [y_1^t,...,y_N^t]$ will be replaced by $p^t \equiv [p_1^t,...,p_N^t]$ and $q^t \equiv [q_1^t,...,q_N^t]$; when studying the input price index, $p_x^t \equiv [p_{x1}^t,...,p_{xM}^t]$ and $x^t \equiv [x_1^t,...,x_M^t]$ will be replaced by $p^t \equiv [p_1^t,...,p_M^t]$ and $q^t \equiv [q_1^t,...,q_M^t]$; and when studying the value-added deflator, the composite vector of output and input prices $[p_y^t,p_x^t]$, will be replaced by $p^t \equiv [p_1^t,...,p_N^t]$; and the vector of net outputs $[y^t,-x^t]$, by $q^t \equiv [q_1^t,...,q_N^t]$ for $t = 0,1$ in each case. Thus, the appropriate definition for p^t and q^t depends on the context.

17.11 To most practitioners in the field, our basic framework, which assumes that detailed price and quantity data are available for each of the possibly millions of establishments in the economy, will seem to be utterly unrealistic. However, two answers can be directed at this very valid criticism:

- The spread of the computer and the ease of storing transaction data have made the assumption that the statistical agency has access to detailed price and quantity data less unrealistic. With the cooperation of businesses, it is now possible to calculate price and quantity indices of the type studied in Chapters 15 and

input dependent on the length of the accounting period. However, by convention, an input is classified as being durable if it lasts longer than two or three years. Thus, an intermediate input is a nondurable input that is also not a primary input. Durable inputs are classified as primary inputs even if they are produced by other establishments. Other primary inputs include labor, land, and natural resource inputs.

[10]Rowe (1927, pp. 174–75) was one of the first economists to appreciate the difficulties statisticians faced when attempting to construct price or quantity indices of production: "In the construction of an index of production there are three inherent difficulties which, inasmuch as they are almost insurmountable, impose on the accuracy of the index, limitations, which under certain circumstances may be somewhat serious. The first is that many of the products of industry are not capable of quantitative measurement. This difficulty appears in its most serious form in the case of the engineering industry. ... The second inherent difficulty is that the output of an industry, even when quantitatively measurable, may over a series of years change qualitatively as well as quantitatively. Thus during the last twenty years there has almost certainly been a tendency towards an improvement in the average quality of the yarn and cloth produced by the cotton industry .... The third inherent difficulty lies in the inclusion of new industries which develop importance as the years go on." These three difficulties still exist today: think of the difficulties involved in measuring the outputs of the insurance and gambling industries; an increasing number of industries produce outputs that are one of a kind, and, hence, price and quantity comparisons are necessarily difficult if not impossible; and, finally, the huge increases in research and development expenditures by firms and governments have led to ever increasing numbers of new products and industries. Chapter 8 considers the issues for index compilation arising from new and disappearing goods and services, as well as establishments.

16 using very detailed data on prices and quantities.[11]

- Even if it is not realistic to expect to obtain detailed price and quantity data for every transaction made by every establishment in the economy on a monthly or quarterly basis, it is still necessary to accurately specify the *universe* of transactions in the economy. Once the target universe is known, sampling techniques can be applied in order to reduce data requirements.

A.2 An overview of the chapter

17.12 In this subsection, a brief overview of the contents of this chapter will be given. In Section B, the economic theory of the *output price index* for an establishment is outlined. This theory is credited primarily to Fisher and Shell (1972) and Archibald (1977). Various bounds to the output price index are developed, along with some useful approximations to the theoretical output price index. Diewert's (1976) theory of *superlative indices* is outlined. A superlative index can be evaluated using observable price and quantity data, but under certain conditions it can give exactly the same answer as the theoretical output price index.

17.13 In the previous two chapters, the Fisher (1922) ideal price index and the Törnqvist (1936) price index emerged as being supported by the test and stochastic approaches to index number theory, respectively. These two indices also will emerge as very good choices from the economic perspective. However, a practical drawback to their use is that current-period information on quantities is required, information that the statistical agency will usually not have on a current-period basis. Hence, in Section E, recent suggestions for *approximating* these indices are looked at using only current information on prices; that is, it is assumed that current-period information on quantities is not available.

17.14 Finally, in Appendix 17.1, the relationship between the Divisia price index introduced in Chapter 15 and an economic output price index is considered.

B. Fisher-Shell Output Price Index: The Case of One Establishment

B.1 Fisher-Shell output price index and observable bounds

17.15 This subsection includes an outline of the theory of the output price index for a single establishment developed by Fisher and Shell (1972) and Archibald (1977). This theory is the producer theory counterpart to the theory of the cost-of-living index for a single consumer (or household) that was first developed by the Russian economist, Konüs (1924). These economic approaches to price indices rely on the assumption of (competitive) *optimizing behavior* on the part of economic agents (consumers or producers). Thus, in the case of the output price index, given a vector of output prices p^t that the agent faces in a given time period *t,* it is assumed that the corresponding hypothetical quantity vector q^t is the solution to a revenue maximization problem that involves the producer's production function f or production possibilities set. (Hereafter the terms *value of output* and *revenue* are used interchangeably, inventory changes being ignored.)

17.16 In contrast to the axiomatic approach to index number theory, the economic approach does *not* assume that the two quantity vectors $q^0 \equiv [q_1^0,...,q_N^0]$ and $q^1 \equiv [q_1^1,...,q_N^1]$ are independent of the two price vectors $p^0 \equiv [p_1^0,...,p_N^0]$ and $p^1 \equiv [p_1^1,...,p_N^1]$. In the economic approach, the period 0 quantity vector q^0 is determined by the producer's period 0 production function and the period 0 vector of prices p^0 that the producer faces, and the period 1 quantity vector q^1 is determined by the producer's period 1 production function f and the period 1 vector of prices p^1.

17.17 Before the output price index is defined for an establishment, it is necessary to describe the establishment's technology in period *t.* In the economics literature, it is traditional to describe the technology of a firm or industry in terms of a production function, which reveals the maximum amount of output that can be produced using a given vector of inputs. However, since most estab-

[11]An early study that computed Fisher ideal indices for a distribution firm in western Canada for seven quarters aggregating over 76,000 inventory items is found in Diewert and Smith (1994).

lishments produce more than one output, it is more convenient to describe the establishment's technology in period t by means of a *production possibilities set* S^t. The set S^t describes what output vectors q can be produced in period t if the establishment has at its disposal the vector of inputs $v \equiv [x,z]$, where x is a vector of intermediate inputs and z is a vector of primary inputs. Thus, if $[q,v]$ belongs to S^t, then the nonnegative output vector q can be produced by the establishment in period t if it can use the nonnegative vector v of inputs.

17.18 Let $p \equiv (p_1,\ldots p_N)$ denote a vector of positive output prices that the establishment might face in period t, and let $v \equiv [x,z]$ be a nonnegative vector of inputs that the establishment might have available for use during period t. Then the establishment's *revenue function* using period t technology is defined as the solution to the following revenue maximization problem:

(17.1) $R^t(p,v) \equiv \max_q \{\sum_{n=1}^{N} p_n q_n : q \text{ belongs to } S^t(v)\}$.

Thus, $R^t(p,v)$ is the maximum value of output, $\sum_{n=1}^{N} p_n q_n$, that the establishment can produce, given that it faces the vector of output prices p and the vector of inputs v is available for use, using the period t technology.[12]

17.19 The period t revenue function R^t can be used to define the establishment's *period* t *technology output price index* P^t between any two periods, say, period 0 and period 1, as follows:

(17.2) $P^t(p^0,p^1,v) = R^t(p^1,v) / R^t(p^0,v)$,

where p^0 and p^1 are the vectors of output prices that the establishment faces in periods 0 and 1, respectively, and v is a reference vector of intermediate and primary inputs.[13] If $N = 1$ so that the establishment produces only one output, then it can be shown that the output price index collapses to the single-output price relative between periods 0 and 1, p_1^1 / p_1^0. In the general case, note that the output price index defined by equation (17.2) is a ratio of hypothetical revenues that the establishment could realize, given that it has the period t technology and the vector of inputs v to work with. The numerator in equation (17.2) is the maximum revenue that the establishment could attain if it faced the output prices of period 1, p^1, while the denominator in equation (17.2) is the maximum revenue that the establishment could attain if it faced the output prices of period 0, p^0. Note that all of the variables in the numerator and denominator functions are exactly the same, except that the output price vectors differ. This is a defining characteristic of an economic price index: all environmental variables are held constant with the exception of the prices in the domain of definition of the price index.

17.20 Note that there are a wide variety of price indices of the form equation (17.2), depending on which reference technology t and reference input vector v is chosen. Thus, there is not a single economic price index of the type defined by equation (17.2): there is an entire *family* of indices.

17.21 Usually, interest lies in two special cases of the general definition of the output price index in equation (17.2): (i) $P^0(p^0,p^1,v^0)$, which uses the period 0 technology set and the input vector v^0 that was actually used in period 0, and (ii) $P^1(p^0,p^1,v^1)$, which uses the period 1 technology set and the input vector v^1 that was actually used in period 1. Let q^0 and q^1 be the observed output vectors for the establishment in periods 0 and 1, respectively. If

[12]The function R^t is known as the *GDP function* or the *national product function* in the international trade literature (see Kohli, 1978 and 1991; or Woodland, 1982). It was introduced into the economics literature by Samuelson (1953). Alternative terms for this function include (i) the *gross profit function*, see Gorman (1968); (ii) the *restricted profit function*, see Lau (1976) and McFadden (1978); and (iii) the *variable profit function*, see Diewert (1973 and 1974a). The mathematical properties of the revenue function are laid out in these references.

[13]This concept of the output price index (or a closely related variant) was defined by Fisher and Shell (1972, pp. 56–58), Samuelson and Swamy (1974, pp. 588–92), Archibald (1977, pp. 60–61), Diewert (1980, pp. 460–61; 1983a, p. 1055), and Balk (1998a, pp. 83–89). Readers who are familiar with the theory of the true cost-of-living index will note that the output price index defined by equation (17.2) is analogous to the *true cost-of-living index,* which is a ratio of cost functions, say, $C(u,p^1)/C(u,p^0)$, where u is a reference utility level: r replaces $C,$ and the reference utility level u is replaced by the vector of reference variables (t,v). For references to the theory of the true cost-of-living index, see Konüs (1924), Pollak (1983a), or the CPI counterpart to this *Manual*, ILO and others (2004).

there is revenue-maximizing behavior on the part of the establishment in periods 0 and 1, then observed revenue in periods 0 and 1 should be equal to $R^0(p^0,v^0)$ and $R^1(p^1,v^1)$, respectively; that is, the following equalities should hold:

$$(17.3)\ R^0(p^0,v^0) = \sum_{n=1}^{N} p_n^0 q_n^0 \ \text{ and } R^1(p^1,v^1) = \sum_{n=1}^{N} p_n^1 q_n^1 .$$

17.22 Under these revenue-maximizing assumptions, Fisher and Shell (1972, pp. 57–58) and Archibald (1977, p. 66) have shown that the two theoretical indices, $P^0(p^0,p^1,v^0)$ and $P^1(p^0,p^1,v^1)$ described in (i) and (ii) above, satisfy equations (17.4) and (17.5):

$$(17.4)\ P^0(p^0,p^1,v^0) \equiv R^0(p^1,v^0) / R^0(p^0,v^0),$$

using equation (17.2)

$$= R^0(p^1,v^0) \Big/ \sum_{n=1}^{N} p_n^0 q_n^0 ,$$

using equation (17.3)

$$\geq \sum_{n=1}^{N} p_n^1 q_n^0 \Big/ \sum_{n=1}^{N} p_n^0 q_n^0 ,$$

since q^0 is feasible for the maximization problem, which defines $R^0(p^1,v^0)$, and so

$$R^0(p^1,v^0) \geq \sum_{n=1}^{N} p_n^1 q_n^0$$
$$\equiv P_L(p^0,p^1,q^0,q^1),$$

where P_L is the Laspeyres (1871) price index. Similarly,

$$(17.5)\ P^1(p^0,p^1,v^1) \equiv R^1(p^1,v^1) / R^1(p^0,v^1),$$

using equation (17.2)

$$= \sum_{n=1}^{N} p_n^1 q_n^1 \Big/ R^1(p^0,v^1),$$

using equation (17.3)

$$\leq \sum_{n=1}^{N} p_n^1 q_n^1 \Big/ \sum_{n=1}^{N} p_n^0 q_n^1 ,$$

since q^1 is feasible for the maximization problem, which defines $R^1(p^0,v^1)$, and so

$$R^1(p^0,v^1) \geq \sum_{n=1}^{N} p_n^0 q_n^1$$
$$\equiv P_P(p^0,p^1,q^0,q^1),$$

where P_P is the Paasche (1874) price index. Thus, the inequality in equation (17.4) says that the observable Laspeyres index of output prices P_L is a *lower bound* to the theoretical output price index $P^0(p^0,p^1,v^0)$, and the inequality (17.5) says that the observable Paasche index of output prices P_P is an *upper bound* to the theoretical output price index $P^1(p^0,p^1,v^1)$. Note that these inequalities are in the *opposite direction* compared with their counterparts in the theory of the true cost-of-living index.[14]

17.23 It is possible to illustrate the two inequalities in equations (17.4) and (17.5) if there are only two commodities; see Figure 17.1, which is based on diagrams credited to Hicks (1940, p. 120) and Fisher and Shell (1972, p. 57).

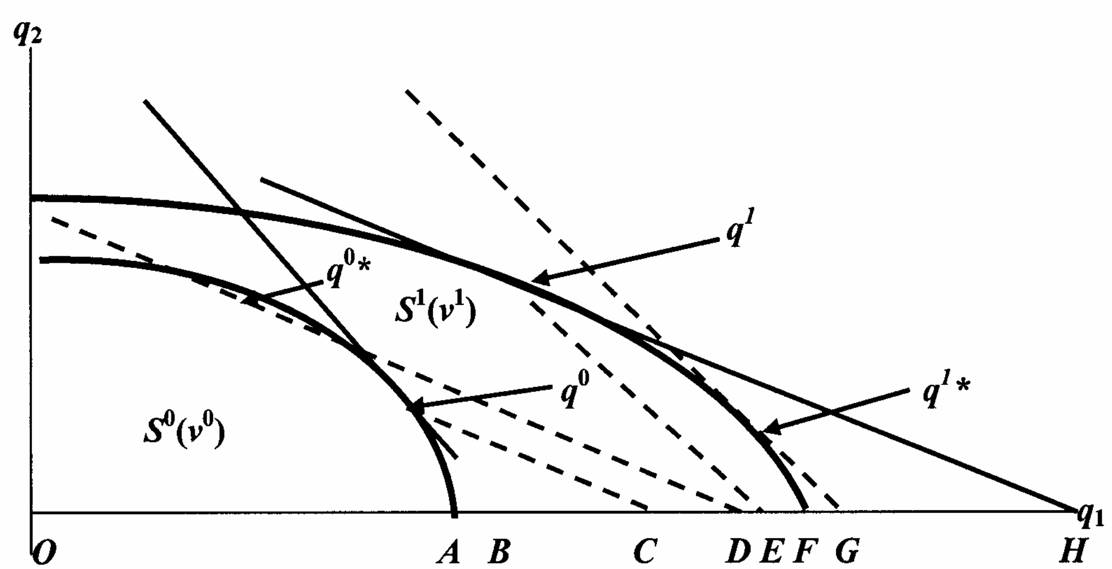
17.24 First the inequality in equation (17.4) is illustrated for the case of two outputs both produced in both periods. The solution to the period 0 revenue maximization problem is the vector q^0, and the straight line through B represents the revenue line that is just tangent to the period 0 output production possibilities set, $S^0(v^0) \equiv \{(q_1,q_2,v^0)$ belongs to $S^0\}$. The curved line through q^0 and A is the frontier to the producer's period 0 output production possibilities set $S^0(v^0)$. The solution to the period 1 revenue maximization problem is the vector q^1, and the straight line through H represents the revenue line that is just tangent to the period 1 output production possibilities set, $S^1(v^1) \equiv \{(q_1,q_2,v^1)$ belongs to $S^1\}$. The curved line through q^1 and F is the frontier to the producer's period 1 output production possibilities set $S^1(v^1)$. The point q^{0*} solves the hypothetical maximization problem of maximizing revenue when facing the period 1 price vector $p^1 = (p_1^1,p_2^1)$ but using the period 0 technology and input vector. This is given by $R^0(p^1,v^0) = p_1^1 q_1^{0*} + p_2^1 q_2^{0*}$, and the dashed line through D is the corresponding isorevenue line $p_1^1 q_1 + p_2^1 q_2 = R^0(p^1,v^0)$. Note that the hypothetical revenue line through D is parallel to the actual period 1 revenue line through H. From equation (17.4), the hypothetical Fisher-Shell output price index, $P^0(p^0,p^1,v^0)$, is $R^0(p^1,v^0) / [p_1^0 q_1^0 + p_2^0 q_2^0]$, while the ordinary Laspeyres output price index is $[p_1^1 q_1^0$

[14]This is because the optimization problem in the cost-of-living theory is a cost *minimization* problem as opposed to our present revenue *maximization* problem. The method of proof used to derive equations (17.4) and (17.5) dates back to Konüs (1924), Hicks (1940), and Samuelson (1950).

$+ p_2^1 q_2^0] / [p_1^0 q_1^0 + p_2^0 q_2^0]$. Since the denominators for these two indices are the same, the difference between the indices is due to the differences in their numerators. In Figure 17.1, this difference in the numerators is expressed by the fact that the revenue line through *C* lies *below* the parallel revenue line through *D*. Now, if the producer's period 0 output production possibilities set were block-shaped with a vertex at q^0, then the producer would not change production patterns in response to a change in the relative prices of the two commodities while using the period 0 technology and inputs. In this case, the hypothetical vector q^{0*} would coincide with q^0, the dashed line through *D* would coincide with the dashed line through *C*, and the true output price index, $P^0(p^0,p^1,v^0)$, would coincide with the ordinary Laspeyres price index. However, block-shaped production possibilities sets are generally not consistent with producer behavior; that is, when the price of a commodity increases, producers generally supply more of it. Thus, in the general case, there will be a gap between the points *C* and *D*. The magnitude of this gap represents the amount of *substitution bias* between the true index and the corresponding Laspeyres index; that is, the Laspeyres index generally will be *less* than the corresponding true output price index, $P^0(p^0,p^1,v^0)$.

17.25 Figure 17.1 also can be used to illustrate the inequality (17.5) for the two-output case. Note that technical progress or increases in input availability have caused the period 1 output production possibilities set $S^1(v^1) \equiv \{(q_1,q_2) : [q_1,q_2,v^1]$ belongs to $S^1\}$ to be much bigger than the corresponding period 0 output production possibilities set $S^0(v^0) \equiv$

Figure 17.1. Laspeyres and Paasche Bounds to the Output Price Index



$\{(q_1,q_2) : [q_1,q_2,v^0]$ belongs to $S^0\}$.[15] Note also that the dashed lines through E and G are parallel to the period 0 isorevenue line through B. The point q^{1*} solves the hypothetical problem of maximizing revenue using the period 1 technology and inputs when facing the period 0 price vector $p^0 = (p_1^0,p_2^0)$. This is given by $R^1(p^0,v^1) = p_1^0q_1^{1*} + p_2^0q_2^{1*}$, and the dashed line through G is the corresponding isorevenue line $p_1^1q_1 + p_2^1q_2 = R^1(p^0,v^1)$. From equation (17.5), the theoretical output price index using the period 1 technology and inputs is $[p_1^1q_1^1 + p_2^1q_2^1] / R^1(p^0,v^1)$, while the ordinary Paasche price index is $[p_1^1q_1^1 + p_2^1q_2^1] / [p_1^0q_1^1 + p_2^0q_2^1]$. Since the numerators for these two indices are the same, the difference between the indices is due to the differences in their denominators. In Figure 17.1, this difference in the denominators is expressed by the fact that the revenue line through E lies *below* the parallel cost line through G. The magnitude of this difference represents the amount of *substitution bias* between the true index and the corresponding Paasche index; that is, the Paasche index generally will be *greater* than the corresponding true output price index using current-period technology and inputs, $P^1(p^0,p^1,v^1)$. Note that this inequality goes in the opposite direction to the previous inequality (17.4). The reason for this change in direction is that one set of differences between the two indices takes place in the numerators of the two indices (the Laspeyres inequalities), while the other set takes place in the denominators of the two indices (the Paasche inequalities).

17.26 Equations (17.4) and (17.5) have two problems:

- *Two* equally valid economic price indices, $P^0(p^0,p^1,v^0)$ and $P^1(p^0, p^1 ,v^1)$, can be used to describe the amount of price change that took place between periods 0 and 1, whereas the public will demand that the statistical agency produce a *single* estimate of price change between the two periods.
- Only *one-sided* observable bounds to these two theoretical price indices[16] result from this analysis, and for most practical purposes, *two-sided* bounds are required.

The following subsection shows a possible solution to these two problems.

B.2 Fisher ideal index as an average of observable bounds

17.27 It is possible to define a theoretical output price index that falls *between* the observable Paasche and Laspeyres price indices. To do this, first define a hypothetical revenue function, $\pi(p,\alpha)$, that corresponds to the use of an α weighted average of the technology sets $S^0(v^0)$ and $S^1(v^1)$ for periods 0 and 1 as the reference technology:

$$(17.6)\ R(p,\alpha) \equiv \max_q \{\sum_{n=1}^{N} p_nq_n : q \text{ belongs to } (1-\alpha)S^0(v^0) + \alpha S^1(v^1)\}.$$

Thus, the revenue maximization problem in equation (17.6) corresponds to the use of a weighted average of the period 0 and period 1 technology sets, where the period 0 vector gets the weight $1-\alpha$ and the period 1 vector gets the weight α, where α is a number between 0 and 1.[17] The meaning of the weighted average technology set in equation (17.6) can be explained in terms of Figure 17.1 as follows. As α changes continuously from 0 to 1, the output production possibilities set changes in a continuous manner from the set $S^0(v^0)$ (whose frontier is the curve that ends in the point A) to the set $S^1(v^1)$ (whose frontier is the curve that ends in the point F). Thus, for any α between 0 and 1, a hypothetical establishment output production possibilities set is obtained that lies between the base-period set $S^0(v^0)$ and the current-period set $S^1(v^1)$. For each α, this hypothetical output production

[15]However, the validity of equation (17.5) does not depend on the relative position of the two output production possibilities sets. To obtain the strict inequality version of equation (17.5), two things are needed: (i) the frontier of the period 1 output production possibilities set to be "curved" and (ii) relative output prices to change going from period 0 to 1, so that the two price lines through G and H in Figure 17.1 are tangent to *different* points on the frontier of the period 1 output production possibilities set.

[16]The Laspeyres output price index is a lower bound to the theoretical index $P^0(p^0,p^1,v^0)$, while the Paasche output price index is an upper bound to the theoretical index $P^1(p^0,p^1,v^1)$.

[17]When $\alpha = 0$, $R(p,0) = R^0(p,v^0)$, and when $\alpha = 1$, $R(p,1) = R^1(p,v^1)$.

possibilities set can be used as the constraint set for a theoretical output price index.

17.28 The new revenue function in definition (17.6) is now used in order to define the following family (indexed by α) of theoretical net output price indices:

(17.7) $P(p^0,p^1,\alpha) \equiv R(p^1,\alpha) / R(p^0,\alpha)$.

The important advantage that theoretical output price indices of the form in equations (17.2) or (17.7) have over the traditional Laspeyres and Paasche output price indices P_L and P_P is that these theoretical indices deal adequately with *substitution effects*; that is, when an output price increases, the producer supply should increase, holding inputs and the technology constant.[18]

17.29 Diewert (1983a, pp. 1060–61) showed that, under certain conditions,[19] there exists an α between 0 and 1 such that the theoretical output price index defined by equation (17.7) lies between the observable (in principle) Paasche and Laspeyres output indices, P_P and P_L ; that is, there exists an α such that

(17.8) $P_L \le P(p^0,p^1,\alpha) \le P_P$ or
$P_P \le P(p^0,p^1,\alpha) \le P_L$.

17.30 The fact that the Paasche and Laspeyres output price indices provide upper and lower bounds to a "true" output price $P(p^0,p^1,\alpha)$ in equation (17.8) is a more useful and important result than the one-sided bounds on the "true" indices that were derived in equations (17.4) and (17.5). If the observable (in principle) Paasche and Laspeyres indices are not too far apart, then taking a symmetric average of these indices should provide a good approximation to an economic output price index, where the reference technology is somewhere between the base- and current-period technologies. The precise symmetric average of the Paasche and Laspeyres indices was determined in Section C.1 of Chapter 15 on axiomatic grounds and led to the geometric mean, the Fisher price index, P_F:

(17.9) $P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1) \times P_P(p^0,p^1,q^0,q^1)]^{1/2}$.

Thus, the Fisher ideal price index receives a fairly strong justification as a good approximation to an unobservable theoretical output price index.[20]

17.31 The bounds given by equations (17.4), (17.5), and (17.8) are the best that can be obtained on economic output price indices without making further assumptions. In the next subsection, further assumptions are made on the two technology sets S^0 and S^1 or, equivalently, on the two revenue functions $R^0(p,v)$ and $R^1(p,v)$. With these extra assumptions, it is possible to determine the geometric average of the two theoretical output price indices that are of primary interest, $P^0(p^0,p^1,v^0)$ and $P^1(p^0,p^1,v^1)$.

B.3 Törnqvist index as an approximation to an economic output price index

17.32 An alternative to the Laspeyres and Paasche indices defined in equations (17.4) and

[18]This is a normal output substitution effect. However, empirically, it will often happen that observed period-to-period decreases in price are not accompanied by corresponding decreases in supply. However, these abnormal "substitution" effects can be rationalized as the effects of technological progress. For example, suppose the price of computer chips decreases substantially going from period 0 to 1. If the technology were constant over these two periods, one would expect domestic producers to decrease their supply of chips going from period 0 to 1. In actual fact, the opposite happens, because technological progress has led to a sharp reduction in the cost of producing chips, which is passed on to demanders of chips. Thus the effects of technological progress cannot be ignored in the theory of the output price index. The counterpart to technological change in the theory of the cost-of-living index is taste change, which is often ignored.

[19]Diewert adapted a method of proof credited originally to Konüs (1924) in the consumer context. Sufficient conditions on the period 0 and 1 technology sets for the result to hold are given in Diewert (1983a, p. 1105). The exposition of the material in Sections B.2, B.3, and C.1 also draws on Chapter 2 in Alterman, Diewert, and Feenstra (1999).

[20]Note that Irving Fisher (1922) constructed Laspeyres, Paasche, and Fisher output price indices for his U.S. data set. Fisher also adopted the view that the product of the price and quantity index should equal the value ratio between the two periods under consideration, an idea that he had already formulated (1911, p. 403). He did not consider explicitly the problem of deflating value added, but by 1930, his ideas on deflation and measuring quantity growth being essentially the same problem had spread to the problem of deflating nominal value added; see Burns (1930).

(17.5) or the Fisher index defined by equation (17.9) is to use the Törnqvist (1936) Theil (1967) price index P_T, whose natural logarithm is defined as follows:

$$(17.10)\ \ln P_T(p^0,p^1,q^0,q^1) = \sum_{n=1}^{N} (1/2)\left(s_n^0 + s_n^1\right)\ln\left(p_n^1/p_n^0\right),$$

where $s_n^t = p_n^t q_n^t \Big/ \sum_{n=1}^{N} p_n^t q_n^t$ is the revenue share of commodity n in the total value of sales in period t.

17.33 Recall the definition of the period t revenue function, $R^t(p,v)$, defined earlier by equation (17.1). Now assume that the period t revenue function has the following *translog functional form*[21] for $t = 0,1$:

$$(17.11)\ \ln R^t(p,v) = \alpha_0^t + \sum_{n=1}^{N}\alpha_n^t \ln p_n + \sum_{m=1}^{M+K}\beta_m^t \ln v_m + \frac{1}{2}\sum_{n=1}^{N}\sum_{j=1}^{N}\alpha_{nj}^t \ln p_n \ln p_j + \sum_{n=1}^{N}\sum_{m=1}^{M+K}\beta_{nm}^t \ln p_n \ln v_m + \frac{1}{2}\sum_{m=1}^{M+K}\sum_{k=1}^{M+K}\gamma_{mk}^t \ln v_m \ln v_k ,$$

where the α_n^t coefficients satisfy the restrictions:

$$(17.12)\ \sum_{n=1}^{N}\alpha_n^t = 1 \text{ for } t = 0,1,$$

and the α_{nj}^t coefficients satisfy the following restrictions:[22]

$$(17.13)\ \sum_{n=1}^{N}\alpha_{nj}^t = 0, \text{ for } t = 0,1 \text{ and } n = 1,2,\ldots,N.$$

The equations (17.12) and (17.13) are necessary to ensure that $R^t(p,v)$ is linearly homogeneous in the components of the output price vector p (which is a property that a revenue function must satisfy.)[23] Note that at this stage of our argument, the coefficients that characterize the technology in each period (the αs, βs and γs) are allowed to be completely different in each period. It also should be noted that the translog functional form is an example of a *flexible* functional form;[24] that is, it can approximate an arbitrary technology to the second order.

17.34 A result in Caves, Christensen, and Diewert (1982, p. 1410) now can be adapted to the present context: if the quadratic price coefficients in equation (17.11) are equal across the two periods of the index number comparison (that is, $\alpha_{nj}^0 = \alpha_{nj}^1$ for all n,j), then the geometric mean of the economic output price index that uses period 0 technology and the period 0 input vector v^0, $P^0(p^0,p^1,v^0)$, and the economic output price index that uses period 1 technology and the period 1 input vector v^1, $P^1(p^0,p^1,v^1)$, is *exactly equal* to the Törnqvist output price index P_T defined by equation (17.10) above; that is,

$$(17.14)\ P_T(p^0,p^1,q^0,q^1) = [P^0(p^0,p^1,v^0) \times P^1(p^0,p^1,v^1)]^{1/2}.$$

The assumptions required for this result seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period, and our assumptions are consistent with technological progress occurring between the two periods being compared. Because the index number formula P_T is *exactly* equal to the geometric mean of two theoretical economic output price indices, and it corresponds to a flexible functional form, the Törnqvist output price index number formula is said to be *superlative,* following the terminology used by Diewert (1976).

17.35 In the following section, additional superlative output price formulas are derived. However, this section concludes with a few words of caution on the applicability of the economic approach to PPIs.

[21]This functional form was introduced and named by Christensen, Jorgenson, and Lau (1971). It was adapted to the revenue function or profit function context by Diewert (1974a).

[22]It also is assumed that the symmetry conditions $\alpha_{nj}^t = \alpha_{jn}^t$ for all n,j and for $t = 0,1$, and $\gamma_{mk}^t = \gamma_{km}^t$ for all m,k, and for $t = 0,1$ are satisfied.

[23]See Diewert (1973 and 1974a) for the regularity conditions that a revenue or profit function must satisfy.

[24]The concept of flexible functional form was introduced by Diewert (1974a, p. 113).

17.36 The above economic approaches to the theory of output price indices have been based on the assumption that producers take the prices of their outputs as given fixed parameters that they cannot affect by their actions. However, a *monopolistic supplier* of a commodity will be well aware that the average price that can be obtained in the market for the commodity will depend on the number of units supplied during the period. Thus, under noncompetitive conditions when outputs are monopolistically supplied (or when intermediate inputs are monopsonistically demanded), the economic approach to PPIs breaks down. The problem of modeling noncompetitive behavior does not arise in the economic approach to CPIs because a single household usually does not have much control over the prices it faces in the marketplace.

17.37 The economic approach to producer output price indices can be modified to deal with certain monopolistic situations. The basic idea is credited to Frisch (1936, 14–15), and it involves linearizing the demand functions a producer faces in each period around the observed equilibrium points in each period and then calculating shadow prices that replace market prices. Alternatively, one can assume that the producer is a markup monopolist and simply adds a markup or premium to the relevant marginal cost of production.[25] However, to implement these techniques, econometric methods usually will have to be employed, and, hence, these methods are not really suitable for use by statistical agencies, except in very special circumstances when the problem of noncompetitive behavior is thought to be very significant and the agency has access to econometric resources.

B.4 Fisher ideal index revisited

17.38 In Section B.2, a justification was provided for the Fisher ideal index. It was argued, from the economic approach, that an appropriate index defined from economic theory should fall between Laspeyres and Paasche indices. On axiomatic grounds, the Fisher ideal index was then proposed as the best average of these two formulas. The justification for the Törnqvist index in Section B.3 was quite different. The theory of exact and superlative index numbers was used to justify its use. In the previous section, equation (17.14) showed that if the revenue function took a translog functional form, equation (17.11), then a theoretical price index based on this form would correspond exactly with the Törnqvist output price index, which is a price index number formula based on observable price and quantity data. Moreover, since the translog function is one form of a class of flexible functional forms, the Törnqvist output price index number formula was said to be *superlative,* following the terminology used by Diewert (1976). Flexible functional forms can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order, which is an attractive property of an index number. Bear in mind that Laspeyres and Paasche correspond to revenue functions that have restrictive Leontief forms, which allow no substitution, and the geometric Laspeyres and Paasche indices correspond to Cobb-Douglas forms, which restrict the elasticity of substitution to unity. The translog production technology is a form that allows for wider substitution possibilities and that can, to the second order, approximate a range of functional forms. The economic theory of index numbers provided a direct link between formulas used in practice and the implicit underlying economic behavior they represent. Diewert (1973) showed that if the functional form assumed is not flexible, it implicitly imposes restrictions on the elasticity of substitution. Index numbers that do not correspond to flexible functional forms, that is, are not superlative, are restrictive in this sense. In this section the findings for the Fisher ideal index are outlined. That is, the Fisher index, although justified on a mix of economic and axiomatic principles in Section B.2, is revisited here using the exact and superlative approach to economic index numbers. It will be shown that its derivation, while analogous to that of the Törnqvist index, requires more restrictive assumptions. In Section B.5 the findings on superlative indices are generalized.

17.39 The approach of the previous section is followed for the Fisher ideal index. However, first it is assumed that a linear homogeneous aggregator function exists for outputs. An additional (and considerably more restrictive) assumption is being invoked here than that required for the Törnqvist index: that outputs are said to be homogeneously weakly separable from the other commodities in the production function. The intuitive meaning of

[25]See Diewert (1993b, pp. 584–90) for a more detailed description of these techniques for modeling monopolistic behavior and for additional references to the literature.

the separability assumption defined by equation (17.15) is that an *output aggregate* $q \equiv f(q_1,...,q_N)$ *exists*; that is, a measure of the aggregate contribution to production of the amounts q_1 of the first output, q_2 of the second output, ..., and q_N of the Nth output is the number $q = f(q_1,q_2,...,q_N)$. Note that it is assumed that the linearly homogeneous output aggregator function f does not depend on t. These assumptions are quite restrictive from the viewpoint of empirical economics,[26] but strong assumptions are required to obtain the existence of output aggregates.

17.40 A *unit revenue function,*[27] r can be defined as follows:

$$(17.15)\ r(p) \equiv \max_q \{\sum_{n=1}^{N} p_n q_n : f(q) = 1\},$$

where $p \equiv [p_1,...,p_N]$ and $q \equiv [q_1,...,q_N]$. Thus $r(p)$ is the maximum revenue the establishment can make, given that it faces the vector of output prices p and is asked to produce a combination of outputs $[q_1,...,q_N] = q$ that will produce a unit level of aggregate output. Under the separability assumptions, the theoretical price index $r(p^1) / r(p^0)$ is a ratio of unit revenue functions.

17.41 Instead of starting with the a translog function for the revenue function of the Törnqvist index, the assumption of the Fisher ideal index is that the *unit* revenue function takes a homogeneous quadratic form given by

$$(17.16)\ r(p_1,\ldots,p_N) \equiv \left[\sum_{i=1}^{N}\sum_{k=1}^{N} b_{ik} p_i p_k\right]^{1/2},$$

where the parameters b_{ik} satisfy the following symmetry conditions:

$$(17.17)\ b_{ik} = b_{ki} \text{ for all } i \text{ and } k.$$

Differentiating $r(p)$ defined by equation (17.16) with respect to p_i yields the following equations:

$$(17.18)\ r_i(p) = \left(\frac{1}{2}\right)\left[\sum_{i=1}^{N}\sum_{k=1}^{N} b_{ik} p_i p_k\right]^{-1/2} 2\sum_{k=1}^{N} b_{ik} p_k\ ;$$

$i = 1,\ldots,N$ and using equation (17.16),

$$= \sum_{k=1}^{N} b_{ik} p_k \Big/ r(p),$$

where $r_i(p) \equiv \partial r(p^t) / \partial p_i$. To obtain the first equation in equation (17.18), it is necessary to use the symmetry conditions, equation (17.17). The second equation in equation (17.18) now is evaluated at the observed period t price vector $p^t \equiv (p_1^t,\ldots,p_N^t)$, and dividing both sides of the resulting equation by $r(p^t)$ yields

$$(17.19)\ \frac{r_i(p^t)}{r(p^t)} = \frac{\sum_{k=1}^{N} b_{ik} p_k^t}{\left[r(p^t)\right]^2},\ t = 0,1\ ;\ i = 1,\ldots,N.$$

The above equation defines a theoretical price index. It now is required to relate this theoretical price index, which comes from a particular functional form for the unit revenue function, that is, a homogeneous quadratic form, to an index number formula that can be used in practice. To do this, it is necessary to assume the establishment is maximizing revenue during the two periods, subject to the constraints of technology, and that the unit revenue function is differentiable, and to apply Hotelling's lemma: that the partial derivative of a unit revenue function with respect to an output price is proportional to the equilibrium output quantity.

$$(17.20)\ \frac{q_n^t}{\sum_{k=1}^{N} p_k^t q_k^t} = \frac{\left[\partial r(p^t) / \partial(p_n)\right]}{r(p^t)};\ n = 1,\ldots,N;$$

$$t = 0,1.$$

[26]Suppose that in period 0, the vector of inputs v^0 produces the vector of outputs q^0. Our separability assumptions imply that the same vector of inputs v^0 could produce *any* vector of outputs q such that $f(q) = f(q^0)$. In real life, as q varied, one would expect that the corresponding input requirements also would vary instead of remaining fixed.

[27]An alternative approach, which reaches the same conclusions, is to start with assuming the producer's aggregator function takes this quadratic form and, assuming outputs are homogeneously weakly separable from the other commodities in the production function, applies Wold's identity. It then can be shown that the Fisher ideal quantity index corresponds exactly to a homogeneous quadratic aggregator. Using the product rule, the unit revenue function can be derived to yield analogous results for the Fisher ideal price index.

In words, equation (17.20) says that the vector of period t establishment outputs q^t, divided by period t establishment revenues $\sum_{k=1}^{N} p_k^t q_k^t$, is equal to the vector of first-order partial derivatives of the establishment unit revenue function $\nabla r(p^t) \equiv [\partial r(p^t)/\partial p_1,...,\partial r(p^t)/\partial p_N]$, divided by the period t unit revenue function $r(p^t)$.

Now recall the definition of the Fisher ideal price index P_F defined by equations (15.12) or (17.9):

$$(17.21)\ P_F(p^0,p^1,q^0,q^1) = \left[\sum_{i=1}^{N} p_i^1 q_i^0 \Big/ \sum_{k=1}^{N} p_k^0 q_k^0\right]^{1/2} \left[\sum_{i=1}^{N} p_i^1 q_i^1 \Big/ \sum_{k=1}^{N} p_k^0 q_k^1\right]^{1/2}$$

substituting for $q_{nt} \Big/ \sum_{k=1}^{N} p_{kt} q_{kt}$ from equation (17.20) for $t = 0$

$$= \left[\sum_{i=1}^{N} p_i^1 r_i(p^0) \Big/ r(p^0)\right]^{1/2} \left[\sum_{i=1}^{N} p_i^1 q_i^1 \Big/ \sum_{k=1}^{N} p_k^0 q_k^1\right]^{1/2}$$

$$= \left[\sum_{i=1}^{N} p_i^1 r_i(p^0) \Big/ r(p^0)\right]^{1/2} \Big/ \left[\sum_{i=1}^{N} p_i^0 q_i^1 \Big/ \sum_{k=1}^{N} p_k^1 q_k^1\right]^{1/2}$$

and for $q_{nt} \Big/ \sum_{k=1}^{N} p_{kt} q_{kt}$ from equation (17.20) for $t = 1$

$$= \frac{\left[\sum_{i=1}^{N} p_i^1 r_i(p^0) \Big/ r(p^0)\right]^{1/2}}{\left[\sum_{i=1}^{N} p_i^0 r_i(p^1) \Big/ r(p^1)\right]^{1/2}}$$

and using equation (17.19)

$$= \frac{\left[\sum_{i=1}^{N}\sum_{k=1}^{N} b_{ik} p_k^0 p_i^1 \Big/ \left[r(p^0)\right]^2\right]^{1/2}}{\left[\sum_{i=1}^{N}\sum_{k=1}^{N} b_{ik} p_k^1 p_i^0 \Big/ \left[r(p^1)\right]^2\right]^{1/2}}$$

using equation (17.17) and canceling terms

$$= \left[1 \Big/ \left[r(p^0)\right]^2\right]^{1/2} \Big/ \left[1 \Big/ \left[r(p^1)\right]^2\right]^{1/2}$$

$$= r(p^1) \big/ r(p^0).$$

Thus, under the assumption that the producer engages in revenue-maximizing behavior during periods 0 and 1 and has technologies that satisfy the separability assumption, and that the unit revenue function is homogeneous quadratic, then the Fisher ideal price index P_F is *exactly* equal to the true price index, $r(p^1) / r(p^0)$.[28]

17.42 Since the homogeneous quadratic unit revenue function *r(p)* defined by equation (17.16) is also a flexible functional form, the fact that the Fisher ideal price index P_F exactly equals the true price index $r(p^1) / r(p^0)$ means that P_F is a *superlative index number formula*.[29]

B.5 Superlative output price indices

B.5.1 A general class of superlative output price indices

17.43 There are many other superlative index number formulas; that is, there exist many quantity indices $Q(p^0,p^1,q^0,q^1)$ that are exactly equal to $f(q^1) / f(q^0)$ and many price indices $P(p^0,p^1,q^0,q^1)$ that are exactly equal to $r(p^1) / r(p^0)$, where the aggregator function f or the unit revenue function r is a flexible functional form. Two families of superlative indices are defined below—quantity indices and price indices.

17.44 Suppose that the producer's output aggregator function[30] is the *following quadratic mean of order r aggregator function*:

$$(17.22)\ f^r(q_1,...,q_N) \equiv \left[\sum_{i=1}^{N}\sum_{k=1}^{N} a_{ik} q_i^{r/2} q_k^{r/2}\right]^{1/r},$$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k, and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976,

[28] This result was obtained by Diewert (1976, pp. 133–34) in the consumer context.

[29] Note that the Fisher index P_F is exact for the unit revenue function defined by equation (17.16). These two output aggregator functions do not coincide in general. However, if the N by N symmetric matrix **A** of the a_{ik} has an inverse, then it readily can be shown that the N by N matrix **B** of the b_{ik} will equal $\mathbf{A}^{-1}$.

[30] This terminology is credited to Diewert (1976, p. 129). This functional form was first defined by Denny (1974) as a unit cost function.

p. 130) showed that the aggregator function f^r defined by equation (17.22) is a flexible functional form; that is, it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when $r = 2$, f^r equals the homogeneous quadratic function defined by equation (17.16) above.

17.45 Define the *quadratic mean of order* r *quantity index* Q^r by

$$(17.23)\ Q^r\left(p^0,p^1,q^0,q^1\right) \equiv \left[\sum_{i=1}^{n} s_i^0 \left(\frac{q_i^1}{q_i^0}\right)^{r/2}\right]^{1/r} \left[\sum_{i=1}^{n} s_i^1 \left(\frac{q_i^1}{q_i^0}\right)^{-r/2}\right]^{1/-r},$$

where $s_i^t = p_i^t q_i^t \Big/ \sum_{i=1}^{N} p_i^t q_i^t$ is the period t revenue share for output i as usual. It can be verified that when $r = 2$, Q^r simplifies into Q_F, the Fisher ideal quantity index.

17.46 Using exactly the same techniques as were used in Section B.3, it can be shown that Q^r is exact for the aggregator function f^r defined by equation (17.22); that is,

$$(17.24)\ Q^r(p^0,p^1,q^0,q^1) = f^r\left(q^1\right)\Big/f^r\left(q^0\right).$$

Thus, under the assumption that the producer engages in revenue-maximizing behavior during periods 0 and 1 and has technologies that satisfy a linearly homogeneous aggregator function for outputs[31] where the output aggregator function $f(q)$ is defined by equation (17.22), then the quadratic mean of order r quantity index Q_F is *exactly* equal to the true quantity index, $f^r(q^1) / f^r(q^0)$.[32] Since Q^r is exact for f^r, and f^r is a flexible functional form, the quadratic mean of order r quantity index Q^r is a *superlative index* for each $r \neq 0$. Thus, there are an infinite number of superlative quantity indices.

17.47 For each quantity index Q^r, the product test in equation (15.3) can be used to define the corresponding *implicit quadratic mean of order* r *price index* P^r*:

$$(17.25)\ P^{r*}(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^{N} p_i^1 q_i^1 \Big/ \left[p_i^0 q_i^0\, Q^r\left(p^0,p^1,q^0,q^1\right)\right] = r^{r*}\left(p^1\right)\Big/r^{r*}\left(p^0\right),$$

where r^{r*} is the unit revenue function that corresponds to the aggregator function f^r defined by equation (17.22). For each $r \neq 0$, the implicit quadratic mean of order r price index P^{r*} is also a superlative index.

17.48 When $r = 2$, Q^r defined by equation (17.23) simplifies to Q_F, the Fisher ideal quantity index, and P^{r*} defined by equation (17.25) simplifies to P_F, the Fisher ideal price index. When $r = 1$, Q^r defined by equation (17.23) simplifies to

$$(17.26)\ Q^1(p^0,p^1,q^0,q^1) \equiv \left[\sum_{i=1}^{n} s_i^0\left(\frac{q_i^1}{q_i^0}\right)^{1/2}\right]^1 \Big/ \left[\sum_{i=1}^{n} s_i^1\left(\frac{q_i^1}{q_i^0}\right)^{-1/2}\right]^{-1}$$

$$= \frac{\sum_{i=1}^{N} p_i^1 q_i^1}{\sum_{i=1}^{N} p_i^0 q_i^0} \left[\sum_{i=1}^{N} p_i^0 q_i^0\left(\frac{q_i^1}{q_i^0}\right)^{1/2}\right]^1 \Big/ \left[\sum_{i=1}^{N} p_i^1 q_i^1\left(\frac{q_i^1}{q_i^0}\right)^{-1/2}\right]^{-1}$$

$$= \frac{\sum_{i=1}^{N} p_i^1 q_i^1}{\sum_{i=1}^{N} p_i^0 q_i^0} \left[\sum_{i=1}^{N} p_i^0\left(q_i^0 q_i^1\right)^{1/2}\right]^1 \Big/ \left[\sum_{i=1}^{N} p_i^1\left(q_i^0 q_i^1\right)^{-1/2}\right]^{-1}$$

$$= \frac{\sum_{i=1}^{N} p_i^1 q_i^1}{\sum_{i=1}^{N} p_i^0 q_i^0} \left[\sum_{i=1}^{N} p_i^1\left(q_i^0 q_i^1\right)^{1/2}\right] \Big/ \left[\sum_{i=1}^{N} p_i^0\left(q_i^0 q_i^1\right)^{1/2}\right]$$

$$= \frac{\sum_{i=1}^{N} p_i^1 q_i^1}{\sum_{i=1}^{N} p_i^0 q_i^0} \Big/ \left[P_W\left(p^0,p^1,q^0,q^1\right)\right],$$

where P_W is the *Walsh price index* defined previously by equation (15.19) in Chapter 15. Thus P^{1*} is equal to P_W, the *Walsh price index*, and hence it is also a superlative price index.

[31]This method for justifying aggregation over commodities is due to Shephard (1953, pp. 61–71). It is assumed that $f(q)$ is an increasing, positive, and convex function of q for positive q. Samuelson and Swamy (1974) and Diewert (1980, pp. 438–42) also developed this approach to index number theory.

[32]See Diewert (1976, p. 130).

17.49 Suppose the producer's unit revenue function[33] is the following quadratic mean of order r unit revenue function:

$$(17.27)\ r^r(p_1,\ldots,p_n) \equiv \left[\sum_{i=1}^{N}\sum_{k=1}^{N} b_{ik}\, p_i^{r/2} p_k^{r/2}\right]^{1/r},$$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976, p. 130) showed that the unit revenue function r^r defined by equation (17.27) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note again that when $r = 2$, r^r equals the homogeneous quadratic function defined by equation (17.16) above.

17.50 Define the quadratic mean of order r price index P^r by:

$$(17.28)\ P^r(p^0,p^1,q^0,q^1) \equiv \left[\sum_{i=1}^{n} s_i^0 \left(\frac{p_i^1}{p_i^0}\right)^{r/2}\right]^{1/r} \left[\sum_{i=1}^{n} s_i^1 \left(\frac{p_i^1}{p_i^0}\right)^{-r/2}\right]^{-1/r}$$

where $s_i^t = p_i^t q_i^t \Big/ \sum_{i=1}^{N} p_i^t q_i^t$ is the period t revenue share for output i as usual. It can be verified that when $r = 2$, P^r simplifies into P_F, the Fisher ideal price index.

17.51 Using exactly the same techniques as were used in Section B.3, it can be shown that P^r is exact for the unit revenue function r^r defined by (17.27); that is,

$$(17.29)\ P^r(p^0,p^1,q^0,q^1) = r^r\left(p^t\right) \Big/ r^r\left(p^0\right).$$

Thus, under the assumption that the producer engages in revenue-maximizing behavior during periods 0 and 1 and has technologies that are homogeneously weakly separable where the output aggregator function $f(q)$ corresponds to the unit revenue function $r^r(p)$ defined by (17.27), then the quadratic mean of order r price index P^r is *exactly* equal to the true output price index, $r^r(p^1)/r^r(p^0)$.[34] Since P^r is exact for r^r and r^r is a flexible functional form, that the quadratic mean of order r price index P^r is a *superlative index* for each $r \neq 0$. Thus there are an infinite number of superlative price indices.

17.52 For each price index P^r, the product test (15.3) can be used in order to define the corresponding *implicit quadratic mean of order r quantity index* Q^{r*}:

$$(17.30)\ Q^{r*}(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^{N} p_i^1 q_i^1 \Big/ \left\{ p_i^0 q_i^0 P^r\left(p^0,p^1,q^0,q^1\right)\right\} = f^{r*}\left(p^1\right) \Big/ f^{r*}\left(p^0\right)$$

where f^{r*} is the aggregator function that corresponds to the unit cost function r^r defined by (17.27) above.[35] For each $r \neq 0$, the implicit quadratic mean of order r quantity index Q^{r*} is also a superlative index.

17.53 When $r = 2$, P^r defined by (17.28) simplifies to P_F, the Fisher ideal price index and Q^{r*} defined by (17.30) simplifies to Q_F, the Fisher ideal quantity index. When $r = 1$, P^r defined by (17.28) simplifies to:

$$(17.31)\ P^1(p^0,p^1,q^0,q^1) \equiv \left[\sum_{i=1}^{n} s_i^0 \left(\frac{p_i^1}{p_i^0}\right)^{1/2}\right]^1 \Bigg/ \left[\sum_{i=1}^{n} s_i^1 \left(\frac{p_i^1}{p_i^0}\right)^{-1/2}\right]^{-1}$$

$$= \frac{\sum_{i=1}^{N} p_i^1 q_i^1}{\sum_{i=1}^{N} p_i^0 q_i^0} \left[\sum_{i=1}^{N} p_i^0 q_i^0 \left(\frac{p_i^1}{p_i^0}\right)^{1/2}\right]^1 \Bigg/ \left[\sum_{i=1}^{N} p_i^1 q_i^1 \left(\frac{p_i^1}{p_i^0}\right)^{-1/2}\right]^{-1}$$

[33] Again, the approach here is by way of a unit revenue function. An alternative formulation is via a quadratic mean of order r superlative quantity index. Using the product rule, the quantity index defines an implicit quadratic mean of order r price index that also is a superlative index.

[34] See Diewert (1976, pp. 133–34).

[35] The function f^{r*} can be defined by using r^r as follows: $f^{r*}(q) \equiv \max_p \left\{\sum_{i=1}^{n} p_i q_i : r^r\left(p\right) = 1\right\}$.

$$= \frac{\sum_{i=1}^{N} p_i^1 q_i^1}{\sum_{i=1}^{N} p_i^0 q_i^0} \left[\sum_{i=1}^{N} q_i^0 \left(p_i^0 p_i^1 \right)^{1/2} \right]^1 \Big/ \left[\sum_{i=1}^{N} q_i^1 \left(p_i^0 p_i^1 \right)^{-1/2} \right]^{-1}$$

$$= \frac{\sum_{i=1}^{N} p_i^1 q_i^1}{\sum_{i=1}^{N} p_i^0 q_i^0} \left[\sum_{i=1}^{N} q_i^1 \left(p_i^0 p_i^1 \right)^{1/2} \right] \Big/ \left[\sum_{i=1}^{N} q_i^0 \left(p_i^0 p_i^1 \right)^{1/2} \right]$$

$$= \frac{\sum_{i=1}^{N} p_i^1 q_i^1}{\sum_{i=1}^{N} p_i^0 q_i^0} \Big/ \left[Q_W \left(p^0, p^1, q^0, q^1 \right) \right],$$

where Q_W is the *Walsh quantity index* defined previously by equation (16.34) in Chapter 16. Thus Q^{1*} is equal to Q_W, the Walsh (1901; 1921) quantity index, and hence it is also a superlative quantity index.

17.54 Essentially, the economic approach to index number theory provides reasonably strong justifications for the use of the Fisher price index P_F defined by equation (15.12) or equation (17.9), the Törnqvist-Theil price index P_T defined by equation (16.22) or equation (17.10), and the quadratic mean of order r price indices P^r defined by equation (17.28) (when $r = 1$, this index is the Walsh price index defined by equation [15.19] in Chapter 15). It is now necessary to ask if it matters which one of these formulas is chosen as best.

B.5.2 Approximation properties of superlative indices

17.55 The analysis in this chapter has led to three superlative index number formulas, the Fisher price index, the Törnqvist-Theil price index, and the Walsh price index, all of which appear to have good properties from the viewpoint of the economic approach to index number theory.

17.56 Two questions arise as a consequence of these results:

- Does it matter which formula is chosen?
- If it does matter, which formula should be chosen?

With respect to the first question, the justifications for the Törnqvist index presented in Section B.3 are stronger than the justifications for the other superlative indices presented in Section B.2, because the economic derivation did not rely on restrictive separability assumptions. The justification for the Fisher index, however, took a different form. Economic theory established that Laspeyres and Paasche bounded a true index, and axiomatic grounds were found for the Fisher being the best average of the two. However, Diewert (1978, p. 888) showed that the three superlative index number formulas listed approximate each other to the second order around any point where the two price vectors, p^0 and p^1, are equal and where the two quantity vectors, q^0 and q^1, are equal. He concluded that "all superlative indices closely approximate each other" (Diewert, 1978, p. 884).

17.57 However, the above conclusion requires a caveat. The problem is that the quadratic mean of order r price indices P^r is a (continuous) function of the parameter r. Hence, as r becomes very large in magnitude, the index P^r can differ substantially from, say, $P^2 = P_F$, the Fisher ideal index. In fact, using equation (17.28) and the limiting properties of means of order r,[36] R.J. Hill (2000, p. 7) showed that P^r has the following limit as r approaches plus or minus infinity:

$$(17.32)\ \lim_{r\to+\infty} P^r(p^0,p^1,q^0,q^1) = \lim_{r\to-\infty} P^r(p^0,p^1,q^0,q^1) = [\min_i\{p_i^1/p_i^0\}\max_i\{p_i^1/p_i^0\}]^{1/2}.$$

Thus for r large in magnitude, P^r can differ substantially from the Törnqvist-Theil price index, the Walsh price index, and the Fisher ideal index.[37]

17.58 Although R.J. Hill's theoretical and empirical results demonstrate conclusively that all superlative indices will not necessarily closely approximate each other, there is still the question of how well the more commonly used superlative indices will approximate each other. All of the commonly used superlative indices, P^r and P^{r*}, fall into the interval $0 \le r \le 2$. Diewert (1980,

[36]See Hardy, Littlewood, and Polyá (1934). Actually, Allen and Diewert (1981, p. 434) obtained the result (17.32) but they did not appreciate its significance.

[37]R.J. Hill (2000) documents this for two data sets. His time-series data consists of annual expenditure and quantity data for 64 components of U.S. GDP from 1977 to 1994. For this data set, Hill (2000, p. 16) found that "superlative indices can differ by more than a factor of two (i.e., by more than 100 percent), even though Fisher and Törnqvist never differ by more than 0.6 percent."

p. 451) showed that the Törnqvist index P_T is a limiting case of P^r as r tends to 0. R.J. Hill (2000, p. 16) summarized how far apart the Törnqvist and Fisher indices were making all possible bilateral comparisons between any two data points for his time-series data set as follows:

> The superlative spread $S(0,2)$ is also of interest since, in practice, Törnqvist (r = 0) and Fisher (r = 2) are by far the two most widely used superlative indices. In all 153 bilateral comparisons, $S(0,2)$ is less than the Paasche-Laspeyres spread and on average, the superlative spread is only 0.1 percent. It is because attention, until now, has focused almost exclusively on superlative indices in the range $0 \leq r \leq 2$ that a general misperception has persisted in the index number literature that all superlative indices approximate each other closely.

17.59 Thus for R.J. Hill's time-series data set covering 64 components of U.S. GDP from 1977 to 1994 and making all possible bilateral comparisons between any two years, the Fisher and Törnqvist price indices differed by only 0.1 percent on average. This close correspondence is consistent with the results of other empirical studies using annual time-series data.[38] Additional evidence on this topic may be found in Chapter 19.

17.60 A reasonably strong justification has been provided by the economic approach for a small group of index numbers: the *Fisher ideal index* $P_F = P^2 = P^2{*}$ defined by equation (15.12) or equation (17.9), the *Törnqvist-Theil index* P_T defined by equations (17.10) or (15.81), and the *Walsh index* P_W defined by equation (15.19) (which is equal to the implicit quadratic mean of order r price indices $P^r{*}$ defined by equation (17.25) when r = 1). They share the property of being *superlative* and approximate each other to the second order around any point. This indicates that for normal time-series data, these three indices will give virtually the same answer. The economic approach gave particular support to the Fisher and Törnqvist-Theil indices, albeit on different grounds. The Fisher index was advocated as the only symmetrically weighted average of Laspeyres and Paasche bounds that satisfied the time reversal test. Economic theory argued for the existence of Laspeyres and Paasche bounds on a suitable true theoretical index. The support for the Törnqvist index arose from its requiring less restrictive assumptions to show it was superlative than the Fisher and Walsh indices. The Törnqvist-Theil index seemed to be best from the stochastic viewpoint, and the Fisher ideal index was supported from the axiomatic viewpoint in that it best satisfied the quite reasonable tests presented. The Walsh index seemed to be best from the viewpoint of the pure price index. To determine precisely which one of these three alternative indices to use as a theoretical target or actual index, the statistical agency will have to decide which approach to bilateral index number theory is most consistent with its goals. It is reassuring that, as illustrated in Chapter 19, for normal time series data, these three indices give virtually the same answer.

C. Economic Approach to an Intermediate Input Price Index for an Establishment

17.61 Attention now is turned to the economic theory of the intermediate input price index for an establishment. This theory is analogous to the economic theory of the output price index explained in Section B but now uses the *joint cost function* or the *conditional cost function C* in place of the revenue function r that was used in Section B. Section E will continue the analysis in a similar vein for the value-added deflator. The approach in this section for the intermediate input price index is analogous to the Konüs (1924) theory for the true cost-of-living index in consumer theory.

17.62 Recall that the set $S^t(v^t)$ describes what output vectors y can be produced in period t if the establishment has at its disposal the vector of inputs $v \equiv [x,z]$, where x is a vector of intermediate inputs and z is a vector of primary inputs. Thus if $[y,x,z]$ belongs to S^t, then the nonnegative output vector y can be produced by the establishment in period t, if it can use the nonnegative vector x of intermediate inputs and the nonnegative vector z of primary inputs.

17.63 Let $p_x \equiv (p_{x1},\ldots p_{xM})$ denote a positive vector of intermediate input prices that the establishment might face in period t, let y be a nonnegative vector of output targets, and let z be a nonnegative vector of primary inputs that the establishment might have available for use during period t. Then the establishment's *conditional cost function* using

[38]See, for example, Diewert (1978, p. 894) or Fisher (1922), which is reproduced in Diewert (1976, p. 135).

period t technology is defined as the solution to the following intermediate input cost minimization problem:

(17.33) $C^t(p_x,y,z) \equiv \min_x \{\sum_{m=1}^{M} p_{xm}x_m : [y,x,z]$ belongs to $S^t\}$.

Thus $C^t(p_x,y,z)$ is the minimum intermediate input cost, $\sum_{m=1}^{M} p_{xm}x_m$, that the establishment must pay to produce the vector of outputs y, given that it faces the vector of intermediate input prices p_x and the vector of primary inputs z is available for use, using the period t technology.[39]

17.64 To make the notation for the intermediate input price index comparable to the notation used in Chapters 15 and 16 for price and quantity indices, in the remainder of this subsection the intermediate input price vector p_x is replaced by the vector p, and the vector of intermediate quantities x is replaced by the vector q. Thus $C^t(p_x,y,z)$ is rewritten as $C^t(p,y,z)$.

17.65 The period t conditional cost function C^t can be used to define the economy's *period* t *technology intermediate input price index* P^t between any two periods, say, period 0 and period 1, as follows:

(17.34) $P^t(p^0,p^1,y,z) = C^t(p^1,y,z) / C^t(p^0,y,z)$,

where p^0 and p^1 are the vectors of intermediate input prices that the establishment faces in periods 0 and 1, respectively; y is a reference vector of outputs that the establishment must produce, and z is a reference vector of primary inputs.[40] If $M = 1$, so that there is only one intermediate input that the establishment uses, then it can be shown that the intermediate input price index collapses to the single intermediate input price relative between periods 0 and 1, p_1^1/ p_1^0. In the general case, note that the intermediate input price index defined by equation (17.34) is a ratio of hypothetical intermediate input costs that the establishment must pay to produce the vector of outputs y, given that it has the period t technology and the vector of primary inputs v to work with. The numerator in equation (17.34) is the minimum intermediate input cost that the establishment could attain if it faced the intermediate input prices of period 1, p^1, while the denominator in equation (17.34) is the minimum intermediate input cost that the establishment could attain if it faced the output prices of period 0, p^0. Note that all variables in the numerator and denominator of equation (17.34) are held constant except the vectors of intermediate input prices.

17.66 As was the case with the theory of the output price index, there are a wide variety of price indices in equation (17.34) depending on which reference vector (t,y,z) is chosen (the reference technology is indexed by t, the reference output vector is indexed by y, and the reference primary input vector is indexed by z). As in the theory of the output price index, two special cases of the general definition of the intermediate input price index, equation (17.34), are of interest: (i) $P^0(p^0,p^1,y^0,z^0)$, which uses the period 0 technology set, the output vector y^0 produced in period 0, and the primary input vector z^0 used in period 0; and (ii) $P^1(p^0,p^1,y^1,z^1)$, which uses the period 1, technology set, the output vector y^1 produced in period 1, and the primary input vector z^1 used in period 1. Let I^0 and q^1 be the observed intermediate input vectors for the establishment in periods 0 and 1, respectively. If there is cost-minimizing behavior on the part of the producer in periods 0 and 1, then the observed intermediate input cost in periods 0 and 1 should equal $C^0(p^0,y^0,z^0)$ and $C^1(p^1,y^1,z^1)$, respectively; that is, the following equalities should hold:

(17.35) $C^0(p^0,y^0,z^0) = \sum_{m=1}^{M} p_m^0 q_m^0$ and

$$C^1(p^1,y^1,z^1) = \sum_{m=1}^{M} p_m^1 q_m^1 .$$

17.67 Under these cost-minimizing assumptions, adapt the arguments of Fisher and Shell (1972, pp. 57–58) and Archibald (1977, p. 66) to show that the two theoretical indices, $P^0(p^0,p^1,y^0,z^0)$ and

[39]See McFadden (1978) for the mathematical properties of a conditional cost function. Alternatively, note that $-C^t(p_x,y,z)$ has the same mathematical properties as the revenue function R^t defined earlier in this chapter.

[40]This concept of the intermediate input price index is analogous to the import price index defined in Alterman, Diewert, and Feenstra (1999). If the vector of primary inputs is omitted from equation (17.34), then the resulting intermediate input price index reduces to the physical production cost index defined by Court and Lewis (1942–43, p. 30).

$P^1(p^0,p^1,y^1,z^1)$ described in (i) and (ii) above, satisfy the inequalities of equations (17.36) and (17.37):

(17.36) $P^0(p^0,p^1,y^0,z^0) \equiv C^0(p^1,y^0,z^0) / C^0(p^0,y^0,z^0)$

using equation (17.34)

$$= C^0(p^1,y^0,z^0) \Big/ \sum_{m=1}^{M} p_m^0 q_m^0$$

using equation (17.35)

$$\leq \sum_{m=1}^{M} p_m^1 q_m^0 \Big/ \sum_{m=1}^{M} p_m^0 q_m^0 \ ,$$

since q^0 is feasible for the minimization problem that defines $C^0(p^1,y^0,z^0)$, and so

$$C^0(p^1,y^0,z^0) \leq \sum_{m=1}^{M} p_m^1 q_m^0 \equiv P_L(p^0,p^1,q^0,q^1),$$

where P_L is the Laspeyres intermediate input price index. Similarly,

(17.37) $P^1(p^0,p^1,y^1,z^1) \equiv C^1(p^1,y^1,z^1) / C^1(p^0,y^1,z^1)$

using equation (17.34)

$$= \sum_{m=1}^{M} p_m^1 q_m^1 \Big/ C^1\left(p^0, y^1, z^1\right)$$

using equation (17.35)

$$\geq \sum_{m=1}^{M} p_m^1 q_m^1 \Big/ \sum_{m=1}^{M} p_m^0 q_m^1 \ ,$$

since q^1 is feasible for the minimization problem that defines $C^1(p^0,y^1,z^1)$, and so

$$C^1(p^0,y^1,z^1) \leq \sum_{m=1}^{M} p_m^0 q_m^1 \equiv P_P(p^0,p^1,q^0,q^1),$$

where P_P is the Paasche price index. Thus, equation (17.36) says that the observable Laspeyres index of intermediate input prices, P_L , is an *upper bound* to the theoretical intermediate input price index, $P^0(p^0,p^1,y^0,z^0)$, and the equation (17.37) says that the observable Paasche index of intermediate input prices, P_P, is a *lower bound* to the theoretical intermediate input price index, $P^1(p^0,p^1,y^1,z^1)$. Note that these inequalities are the reverse of earlier equations (17.4) and (17.5) found for the output price index, but the new inequalities are analogous to their counterparts in the theory of the true cost-of-living index.

17.68 As was the case in Section B.2, it is possible to define a theoretical intermediate input price index that falls *between* the observable Paasche and Laspeyres intermediate input price indices. To do this, first define a *hypothetical intermediate input cost function*, $C(p,\alpha)$, that corresponds to the use of an α weighted average of the technology sets $S^0(y^0,z^0)$ and $S^1(y^1,z^1)$ for periods 0 and 1 as the reference technology and that uses an α-weighted average of the period 0 and period 1 output vectors y^0 and y^1 and primary input vectors z^0 and z^1 as the reference output and primary input vectors:

(17.38) $C(p,\alpha)$

$$\equiv \min q \ \{\sum_{m=1}^{M} p_m q_m : q \text{ belongs to } (1-\alpha)\, S^0(y^0,z^0) + \alpha\, S^1(y^1,z^1)\}.$$

Thus, the intermediate input cost minimization problem in equation (17.38) corresponds to the intermediate output target $(1-\alpha)y^0 + \alpha y^1$ and the use of an average of the period 0 and 1 primary input vectors z^0 and z^1 , where the period 0 vector gets the weight $1-\alpha$ and the period 1 vector gets the weight α. An average is used of the period 0 and period 1 technology sets, where the period 0 set gets the weight $1-\alpha$ and the period 1 set gets the weight α, where α is a number between 0 and 1. The new intermediate input cost function defined by equation (17.38) now can be used to define the following *family of theoretical intermediate input price indices*:

(17.39) $P(p^0,p^1,\alpha) \equiv C(p^1,\alpha) / C(p^0,\alpha)$.

17.69 Adapting the proof of Diewert (1983a, pp. 1060–61) shows that there exists an α between 0 and 1 such that the theoretical intermediate input price index defined by equation (17.39) lies between the observable (in principle) Paasche and Laspeyres intermediate input price indices, P_P and P_L ; that is, there exists an α such that

(17.40) $P_L \leq P(p^0,p^1,\alpha) \leq P_P$
or $P_P \leq P(p^0,p^1,\alpha) \leq P_L$.

17.70 If the Paasche and Laspeyres indices are numerically close to each other, then equation (17.40) tells us that a true economic intermediate input price index is fairly well determined, and a reasonably close approximation to the true index can be found by taking a symmetric average of

P_L and P_P such as the geometric average, which again leads to Irving Fisher's (1922) ideal price index, P_F, defined earlier by equation (17.40).

17.71 It is worth noting that the above theory of the economic intermediate input price indices was very general; in particular, no restrictive functional form or separability assumptions were made on the technology.

17.72 The translog technology assumptions used in Section B.3 to justify the use of the Törnqvist-Theil output price index as an approximation to a theoretical output price index can be adapted to yield a justification for the use of the Törnqvist-Theil intermediate input price index as an approximation to a theoretical intermediate input price index. Recall the definition of the period t conditional intermediate input cost function, $C^t(p_x,y,z)$, defined by equation (17.33). Replace the vector of intermediate input prices p_x by the vector p, and define the $N + K$ vector u as $u \equiv [y,z]$. Now assume that the period t conditional cost function has the following *translog functional form* : for t = 0,1 :

(17.41) $\ln C^t(p,u)$

$$= \alpha_0^t + \sum_{m=1}^{M} \alpha_m^t \ln p_m + \sum_{j=1}^{N+K} \beta_j^t \ln u_j + \frac{1}{2}\sum_{m=1}^{M}\sum_{j=1}^{M} \alpha_{mj}^t \ln p_m \ln p_j + \sum_{m=1}^{M}\sum_{n=1}^{N+K} \beta_{mn}^t \ln p_m \ln u_n + \frac{1}{2}\sum_{n=1}^{N+K}\sum_{k=1}^{N+K} \gamma_{nk}^t \ln u_n \ln u_k \ ,$$

where the $\alpha_n{}^t$ and the $\gamma_n{}^t$ coefficients satisfy the following restrictions:

(17.42) $\alpha_{mj}{}^t = \alpha_{jm}{}^t$ for all m,j and for $t = 0,1$;

(17.43) $\gamma_{nk}{}^t = \gamma_{kn}{}^t$ for all k,n and for $t = 0,1$;

(17.44) $\sum_{m=1}^{M} \alpha_m^t = 1$ for $t = 0,1$; and

(17.45) $\sum_{m=1}^{M} \alpha_m^t = 0$ for $t = 0,1$ and $m = 1,2,\ldots,M$.

The restrictions in equations (17.44) and (17.45) are necessary to ensure that $C^t(p,u)$ is linearly homogeneous in the components of the intermediate input price vector p (which is a property that a conditional cost function must satisfy). Note that at this stage of our argument the coefficients that characterize the technology in each period (the αs, βs, and γs) are allowed to be completely different in each period.

17.73 Adapting the result in Caves, Christensen, and Diewert (1982b, p. 1410) to the present context;[41] if the quadratic price coefficients in equation (17.41) are equal across the two periods where an index number comparison (that is, $\alpha_{mj}{}^0 = \alpha_{mj}{}^1$ for all m,j) is being made, then the geometric mean of the economic intermediate input price index that uses period 0 technology, the period 0 output vector y^0, and the period 0 vector of primary inputs z^0, $P^0(p^0,p^1,y^0,z^0)$, and the economic intermediate input price index that uses period 1 technology, the period 1 output vector y^1, and the period 1 primary input vector z^1, $P^1(p^0,p^1,y^1,z^1)$, is *exactly* equal to the Törnqvist intermediate input price index P_T defined by equation (17.10);[42] that is,

(17.46) $P_T(p^0,p^1,q^0,q^1)$
$$= [P^0(p^0,p^1,y^0,z^0)\, P^1(p^0,p^1,y^1,z^1)]^{1/2} \ .$$

17.74 As was the case with our previous result in equation (17.40), the assumptions required for the result (17.46) seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period, and our assumptions are consistent with comparing technological progress occurring between the two periods. Because the index number formula P_T is *exactly* equal to the geometric mean of two theoretical economic intermediate input price index, and this corresponds to a flexible functional form, the Törnqvist intermediate input index number formula is said to be *superlative*.

[41]The Caves, Christensen, and Diewert translog exactness result is slightly more general than a similar translog exactness result obtained earlier by Diewert and Morrison (1986, p. 668); Diewert and Morrison assumed that all of the quadratic terms in equation (17.41) were equal during the two periods under consideration, whereas Caves, Christensen, and Diewert assumed only that $\alpha_{mj}{}^0 = \alpha_{mj}{}^1$ for all m,j.

[42]In the present context, output prices are replaced by intermediate input prices, and the number of terms in the summation of terms defined by equation (17.10) is changed from N to M.

17.75 It is possible to adapt the analysis of the output price index that was developed in Sections C.3 and C.4 to the intermediate input price index, and show that the two families of superlative output price indices, P^{r*} defined by equation (17.25) and P^{r} defined by equation (17.23), also are superlative intermediate input price indices. However, the details are omitted here since to derive these results, rather restrictive separability restrictions are required on the technology of the establishment.[43]

17.76 In the following section, the analysis presented in this section is modified to provide an economic approach to the value-added deflator.

D. Economic Approach to the Value-Added Deflator for an Establishment

17.77 Attention now is turned to the economic theory of the value-added deflator for an establishment. This theory is analogous to the economic theory of the output price index explained in Section B, but now the *profit function* π is used in place of the revenue function r used in Section B.

17.78 Recall that the set S^t describes which output vectors y can be produced in period t if the establishment has at its disposal the vector of inputs $[x,z]$, where x is a vector of intermediate inputs and z is a vector of primary inputs. Thus, if $[y,x,z]$ belongs to S^t, then the nonnegative output vector y can be produced by the establishment in period t, if it can use the nonnegative vector x of intermediate inputs and the nonnegative vector z of primary inputs.

17.79 Let $p_y \equiv (p_{y1},\ldots p_{yN})$ and $p_x \equiv (p_{x1},\ldots p_{xM})$ denote positive vectors of output and intermediate input prices that the establishment might face in period t, and let z be a nonnegative vector of primary inputs that the establishment might have available for use during period t. Then the establishment's *(gross) profit function* or *net revenue function* using period t technology is defined as the solution to the following net revenue maximization problem:

(17.47) $\pi^t(p_y,p_x,z) \equiv \max_{y,x}$

$$\left\{\sum_{n=1}^{N} p_{yn}y_n - \sum_{m=1}^{M} p_{xm}x_m \quad : (y,x) \text{ belongs to } S^t(z)\right\},$$

where, as usual, $y \equiv [y_1,\ldots,y_N]$ is an output vector and $x \equiv [x_1,\ldots,x_M]$ is an intermediate input vector. Thus, $\pi^t(p_y,p_x,z)$ is the maximum output revenue, $\sum_{n=1}^{N} p_{yn}y_n$, less intermediate input cost, $\sum_{m=1}^{M} p_{xm}x_m$, that the establishment could generate, given that it faces the vector of output prices p_y and the vector of intermediate input prices p_x , and given that the vector of primary inputs z is available for use, using the period t technology.[44]

17.80 To make the notation for the value-added deflator comparable to the notation used in Chapters 15 and 16 for price and quantity indices, in the remainder of this subsection, the *net output price vector p* is defined as $p \equiv [p_y,p_x]$, and the *net output quantity vector q* is defined as $q \equiv [y,-x]$. Thus, all output and intermediate input prices are positive, output quantities are positive, but intermediate inputs are indexed with a minus sign. With these definitions, $\pi^t(p_y,p_x,z)$ can be rewritten as $\pi^t(p,z)$.

17.81 The period t profit function π^t can be used to define the economy's *period* t *technology value added deflator* P^t between any two periods, say, period 0 and period 1, as follows:[45]

(17.48) $P^t(p^0,p^1,z) = \pi^t(p^1,z) / \pi^t(p^0,z)$,

where p^0 and p^1 are the $N + M$ dimensional vectors of net output prices that the establishment faces in periods 0 and 1, and z is a reference vector of primary inputs. Note that all variables in the numerator and denominator of equation (17.48) are held

[43]The counterpart to our earlier separability assumption in equation (17.15) is now $z_1 = F^t(y,x,z_2,\ldots,z_K) = G^t(y,f(x),z_2,\ldots,z_K)$ for $t = 0,1$, where the intermediate input aggregator function f is linearly homogeneous and independent of t.

[44] The profit function π^t has the same mathematical properties as the revenue function R^t.

[45]If there are no intermediate inputs, this concept reduces to Archibald's (1977) fixed-input quantity output price index. In the case where there is no technical progress between the two periods, this concept reduces to Diewert's (1980, pp. 455–61) (net) output price deflator. Diewert (1983a) considered the general concept, which allows for technical progress between periods.

constant, except the vectors of net output (output and intermediate input) prices.

17.82 As was the case with the theory of output price index, there are various price indices of the form of equation (17.48), depending on which reference vector (t,z) is chosen. The analysis follows that of the output price index in Section B. As in the theory of the output price index, interest lies in two special cases of the general definition of the intermediate input price index of the form of equation (17.48): a theoretical index that uses the period 0 technology set and the primary input vector z^0 used in period 0, and one that uses the period 1 technology set and the primary input vector z^1 used in period 1. The observable Laspeyres index of output and intermediate input prices P_L is shown to be a *lower bound* to the former theoretical value-added deflator, and the observable Paasche index of output and intermediate input prices P_P is an *upper bound* to the latter theoretical value-added deflator.[46] These inequalities go in the same direction as the earlier inequalities of equations (17.4) and (17.5) obtained for the output price index.

17.83 As was the case in Section B.2, it is possible to define a value-added deflator that falls *between* the observable Paasche and Laspeyres value-added deflators. To do this, a *hypothetical net revenue function*, $\pi(p,\alpha)$, is defined to correspond to an α-weighted average of the period 0 and 1 technology sets, and an α-weighted average of the primary input vectors z^0 and z^1 is used as the reference primary input vector.

17.84 Following the arguments made for the output price index if the Paasche and Laspeyres indices are numerically close, then a true economic value-added deflator is fairly well determined. A reasonably close approximation to the true index is a symmetric average of P_L and P_P , such as the geometric average, which again leads to Irving Fisher's ideal price index.[47]

17.85 The translog technology assumptions used in Section B.3 to justify the use of the Törnqvist-Theil output price index as an approximation to a theoretical output price index can be adapted to yield a justification for the use of the Törnqvist-Theil value-added price index as an approximation to a theoretical value-added deflator. Recall the definition of the period t net revenue function, $\pi^t(p_y,p_x,z)$, defined by equation (17.47). Replace the vectors of output prices p_y and the vector of intermediate input prices p_x by the vector $p \equiv [p_y,p_x]$, and assume that the period t net revenue function has the *translog functional form*. Following the argument for the output price index, if the quadratic price coefficients are equal across the two periods, Törnqvist value-added deflator is exactly equal to this form of the theoretical index. Because the index number formula is *exactly* equal to an underlying *flexible* functional form, the Törnqvist value-added deflator formula is *superlative*. As was the case with the output price index, the assumptions required for this finding seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period, and the assumptions are consistent with technological progress occurring between the two periods being compared.

17.86 It is possible to adapt the analysis of the output price index developed in Sections B.4 and B.5 to the value-added deflator and show that the family of superlative output price indices, P^r defined by equation (17.28), also are superlative

[46]To derive this inequality, the hypothetical value added $\sum_{n=1}^{N+M} p_n^0 q_n^1 \equiv \sum_{n=1}^{N} p_{yn}^0 y_n^1 - \sum_{m=1}^{M} p_{xm}^0 x_m^1$ must be positive to establish the inequality in (17.4). If the periods 0 and 1 are quite distant in time, or if there are dramatic changes in output or intermediate input prices between the two periods, this hypothetical value added can be negative. In this case, one can try to use the chain principle to break up the large price and quantity changes that occurred between periods 0 and 1 into a series of smaller changes. With smaller changes, there is a better chance that the hypothetical value-added series will remain positive. This seems consistent with the advice of Burns (1930, p. 256) on this topic. Under certain circumstances, Bowley (1922, p. 256) raised the possibility of a negative nominal value added. Burns (1930, p. 257) noted that this anomaly will generally disappear with higher aggregations across establishments or industries.

[47]Burns (1930, pp. 244–47) noted that the Laspeyres, Paasche, and Fisher value-added deflators could be used to deflate nominal net output or value added into real measures. Burns (1930, p. 247) also noted that that a Fisher ideal production aggregate built up as the product of the Laspeyres and Paasche quantity indices (the "index" method) would give the same answer as deflating the nominal value-added ratio by the Fisher price index (the "deflating" method).

value-added deflators.[48] However, the details are omitted here because to derive these results, rather restrictive separability restrictions are required on the technology of the establishment.[49]

17.87 Attention now is turned to the problems involved in aggregating over establishments to form national output, intermediate input, and value-added deflators.

E. Approximations to Superlative Indices: Midyear Indices

17.88 A practical problem with superlative indices is that they always require current-period information on *quantities* as well as prices to be implemented. In the following section, a recent suggestion is looked at for approximating superlative indices when information on current-period quantities is not available.

17.89 Recall equations (15.18) and (15.19) in Section C.2 of Chapter 15, which defined the Walsh (1901, p. 398; 1921a, p. 97) and Marshall (1887) Edgeworth (1925) price index between periods 0 and 1, $P_W(p^0,p^1,q^0,q^1)$ and $P_{ME}(p^0,p^1,q^0,q^1)$, respectively. In Section C.4 it was indicated that the Walsh price index is a superlative index. On the other hand, although the Marshall-Edgeworth price index is not superlative, Diewert (1978, p. 897) showed that it will approximate any superlative index to the second order around a point where the base- and current-period price and quantity vectors are equal,[50] so that P_{ME} usually will approximate a superlative index fairly closely. In this section, some recent results credited to Schultz (1999) and Okamoto (2001) will be drawn on to show how various *midyear price indices* can approximate Walsh or Marshall-Edgeworth indices fairly closely under certain conditions. As shall be seen, midyear indices do not rely on quantity weights for the current and base periods; rather, they use quantity weights from years that lie between the base period and current period, and, hence, they can be produced on a timely basis. It is noted that the account is given in terms of using midperiod *quantity* weights, although equivalent indices could also be defined using midperiod *revenue shares* using appropriate definitions of indices in the terms given, for example, for Laspeyres and Paasche in equations (15.8) and (15.9), respectively.

17.90 Let t be an even positive integer. Then Schultz (1999) defined a *midyear price index*, which compares the price vector in period t, p^t, with the corresponding price vector in period 0, p^0, as follows:

$$(17.49)\ P_S(p^0,p^t,q^{t/2}) \equiv \sum_{n=1}^{N} p_n^t q_n^{t/2} \Big/ \sum_{n=1}^{N} p_n^0 q_n^{t/2} ,$$

where $q^{t/2}$ is the quantity vector that pertains to the intermediate period, $t/2$. The definition for a midyear price index when t is odd (and greater than 2) is a bit trickier. Okamoto (2001) defined *arithmetic-type* and *geometric-type midyear price indices* comparing prices in period 0 with period t, where

[48]The value-added aggregator function that corresponds to equation (17.55) is now $f^r(y,x)$. For this functional form, all quantities must be positive, and hence the prices of the outputs must be taken to be positive and the prices of intermediate inputs must be negative for the exactness result of equation (17.56) to hold. For the unit net revenue function that now corresponds to equation (17.27), all prices must be positive, output quantities positive, and intermediate input quantities negative for the exactness result (17.29) to hold.

[49]The counterpart to the earlier separability assumption in equation (17.15) is now $z_1 = F^t(y,x,z_2,...,z_K) = G^t(f(y,x),z_2,...,z_K)$ for $t = 0,1$, where the output and intermediate input aggregator function f is linearly homogeneous and independent of t. This type of separability assumption was first made by Sims (1969). Under this separability assumption, the family of value-added deflators defined by equation (17.48) simplify to $r(p^1) / r(p^0)$, where the *unit net revenue function* is defined by $r(p) \equiv \max_q \left\{\sum_{n=1}^{N+M} p_n q_n : f(q_1,...,q_{N+M}) = 1\right\}$. Note that these deflators are independent of quantities. Under this separability assumption, the quantity index that corresponds to this real value-added price index is $f(y^1,x^1)/f(y^0,x^0)$, and thus this index depends *only* on quantities. Sims (1977, p. 129) emphasizes that if measures of real net output are to depend only on the quantity vectors of outputs produced and intermediate inputs used, then it will be necessary to make a separability assumption. Since these separability assumptions are very restrictive from an empirical point of view, the economic approaches to the PPI have been developed so they do not rely on separability assumptions.

[50]As usual, this result can be generalized to points of approximation where $p^1 = \alpha p^0$ and $q^1 = \beta q^0$; that is, points where the period 1 price vector is proportional to the period 0 price vector and where the period 1 quantity vector is proportional to the period 0 quantity vector.

t is odd by equations (17.50) and (17.51), respectively:

$$(17.50)\ P_{OA}(p^0,p^t,q^{(t-1)/2},q^{(t+1)/2}) \equiv \frac{\sum_{n=1}^{N} p_n^t \left(\tfrac{1}{2}\right)\left(q_n^{(t-1)/2}+q_n^{(t+1)/2}\right)}{\sum_{n=1}^{N} p_n^0 \left(\tfrac{1}{2}\right)\left(q_n^{(t-1)/2}+q_n^{(t+1)/2}\right)}$$

$$(17.51)\ P_{OG}(p^0,p^t,q^{(t-1)/2},q^{(t+1)/2}) \equiv \frac{\sum_{n=1}^{N} p_n^t \left(q_n^{(t-1)/2}+q_n^{(t+1)/2}\right)^{1/2}}{\sum_{n=1}^{N} p_n^0 \left(q_n^{(t-1)/2}+q_n^{(t+1)/2}\right)^{1/2}}.$$

Each of the price indices defined by equation (17.50) and equation (17.51) is of the fixed-basket type. In the arithmetic-type index defined by (17.50), the fixed-basket quantity vector is the simple arithmetic average of the two quantity vectors that pertain to the intermediate periods, $(t-1)/2$ and $(t+1)/2$, whereas in the geometric-type index defined by equation (17.51), the reference quantity vector is the geometric average of these two intermediate period quantity vectors.

17.91 Okamoto (2001) used the above definitions to define the following sequence of *fixed-base (arithmetic-type) midyear price indices*:

$$(17.52)\ 1, P_{ME}(p^0,p^1,q^0,q^1), P_S(p^0,p^2,q^1), P_{OA}(p^0,p^3,q^1,q^2), P_S(p^0,p^4,q^2), P_{OA}(p^0,p^5,q^2,q^3), \ldots .$$

Thus, in period 0, the index is set equal to 1. In period 1, the index is set equal to the Marshall-Edgeworth price index between periods 0 and 1, $P_{ME}(p^0,p^1,q^0,q^1)$ (which is the only index number in the above sequence that requires information on current-period quantities). In period 2, the index is set equal to the Schultz midyear index, $P_S(p^0,p^2,q^1)$, defined by equation (17.49), which uses the quantity weights of the prior period 1, q^1. In period 3, the index is set equal to the arithmetic Okamoto midyear index, $P_{OA}(p^0,p^3,q^1,q^2)$, defined by equation (17.50), which uses the quantity weights of the two prior periods, q^1 and q^2, and so on.

17.92 Okamoto (2001) also used the above definitions to define the following sequence of *fixed-base (geometric-type) midyear price indices*:

$$(17.53)\ 1, P_W(p^0,p^1,q^0,q^1), P_S(p^0,p^2,q^1), P_{OG}(p^0,p^3,q^1,q^2), P_S(p^0,p^4,q^2), P_{OG}(p^0,p^5,q^2,q^3), \ldots .$$

Thus, in period 0, the index is set equal to 1. In period 1, the index is set equal to the Walsh price index between periods 0 and 1, $P_W(p^0,p^1,q^0,q^1)$ (which is the only index number in the sequence that requires information on current period quantities). In period 2, the index is set equal to the Schultz midyear index, $P_S(p^0,p^2,q^1)$. In period 3, the index is set equal to the (geometric-type) Okamoto midyear index, $P_{OG}(p^0,p^3,q^1,q^2)$, defined by equation (17.51), which uses the quantity weights of the two prior periods, q^1 and q^2 , and so on.

17.93 It is also possible to define *chained* sequences[51] of midyear indices that are counterparts to the fixed-base sequences defined by equations (17.52) and (17.53). Thus a chained counterpart to equation (17.52) can be defined as follows:

$$(17.54)\ 1, P_{ME}(p^0,p^1,q^0,q^1), P_S(p^0,p^2,q^1), P_{ME}(p^0,p^1,q^0,q^1)P_S(p^1,p^3,q^2), P_S(p^0,p^2,q^1)P_S(p^2,p^4,q^3), P_{ME}(p^0,p^1,q^0,q^1)P_S(p^1,p^3,q^2)P_S(p^3,p^5,q^4), P_S(p^0,p^2,q^1)P_S(p^2,p^4,q^3)P_S(p^4,p^6,q^5), \ldots .$$

A chained counterpart to equation (17.53) can be defined as follows:

$$(17.55)\ 1, P_W(p^0,p^1,q^0,q^1), P_S(p^0,p^2,q^1), P_W(p^0,p^1,q^0,q^1)P_S(p^1,p^3,q^2), P_S(p^0,p^2,q^1)P_S(p^2,p^4,q^3), P_W(p^0,p^1,q^0,q^1)P_S(p^1,p^3,q^2)P_S(p^3,p^5,q^4), P_S(p^0,p^2,q^1)P_S(p^2,p^4,q^3)P_S(p^4,p^6,q^5), \ldots .$$

Note that equations (17.54) and (17.55) differ only in the use of the Marshall-Edgeworth index, $P_{ME}(p^0,p^1,q^0,q^1)$, to compare prices in period 1 with period 0, versus the Walsh index, $P_W(p^0,p^1,q^0,q^1)$, which is also used to compare prices for the same two periods. Otherwise, only the basic Schultz midyear formula, $P_S(p^t,p^{t+2},q^{t+1})$, is used in both equations (17.54) and (17.55).

[51]See Section F in Chapter 15 for a review of chained indices.

17.94 Using Canadian and Japanese data, Schultz (1999) and Okamoto (2001) showed that midyear index number sequences like those defined by equations (17.54) and (17.55) are reasonably close to their superlative Fisher ideal counterparts.

17.95 In addition to the above empirical results, some theoretical results can be generated that support the use of midyear indices as approximations to superlative indices.[52] The theoretical results presented rely on specific assumptions about how the quantity vectors q^t change over time. Two such specific assumptions will be made.

17.96 It now is assumed that there are *linear trends in quantities* over the sample period; that is, it is assumed that

$$(17.56)\ q^t = q^0 + t\alpha\ ;\ t = 1,...,T,$$

where $\alpha \equiv [\alpha_1,..., \alpha_N]$ is a vector of constants. Hence, for t even, using equation (17.56), it follows that

$$(17.57)\ (½)q^0 + (½)q^t = (½)q^0 + (½)\left[q^0 + t\alpha\right] = q^0 + \left(t/2\right)\alpha = q^{t/2}.$$

Similarly, for t odd (and greater than 2), it follows that

$$(17.58)\ (½)q^0 + (½)q^t = (½)q^0 + (½)\left[q^0 + t\alpha\right] = (½)q^0 + (½)\left[q^0 + \left\{(½)(t-1) + (½)(t+1)\right\}\alpha\right] = (½)\left[q^0 + ½(t-1)\alpha\right] + (½)\left[q^0 + ½(t+1)\alpha\right] = (½)q^{(t-1)/2} + (½)q^{(t+1)/2}.$$

Thus, under the linear time trends in quantities equation (17.56), it can be shown, using equations (17.57) and (17.58), that the Schultz midyear and the Okamoto arithmetic-type midyear indices all equal their Marshall-Edgeworth counterparts; that is,

$$(17.59)\ P_S(p^0,p^t,q^{t/2}) = P_{ME}(p^0,p^t,q^0,q^t) \text{ for } t \text{ even};$$

$$(17.60)\ P_{OA}(p^0,p^t,q^{(q-1)/2},q^{(q+1)/2}) = P_{ME}(p^0,p^t,q^0,q^t) \text{ for } t \text{ odd}.$$

Thus, under the linear trends equation (17.56), the fixed-base and chained arithmetic-type sequences of midyear indices, equations (17.52) and (17.54), respectively, become the following sequences of Marshall-Edgeworth indices:[53]

$$(17.60)\ 1, P_{ME}(p^0,p^1,q^0,q^1), P_{ME}(p^0,p^2,q^0,q^2), P_{ME}(p^0,p^3,q^0,q^3), P_{ME}(p^0,p^4,q^0,q^4), ...;$$

$$(17.61)\ 1, P_{ME}(p^0,p^1,q^0,q^1), P_{ME}(p^0,p^2,q^0,q^2), P_{ME}(p^0,p^1,q^0,q^1)P_{ME}(p^1,p^3,q^1,q^3), P_{ME}(p^0,p^2,q^0,q^2)P_{ME}(p^2,p^4,q^2,q^4), P_{ME}(p^0,p^1,q^0,q^1)P_{ME}(p^1,p^3,q^1,q^3)P_{ME}(p^3,p^5,q^3,q^5), ...$$

17.97 The second specific assumption about the behavior of quantities over time is that quantities change at *geometric rates* over the sample period; that is, it is assumed that

$$(17.62)\ q_n^t = (1+g_n)^t q_n^0\ n = 1,...,N\ ;\ t = 1,...,T,$$

where g_n is the geometric growth rate for quantity n. Hence, for t even, using equation (17.62),

$$(17.63)\left[q_n^0 q_n^t\right]^{1/2} = (1+g_n)^{t/2} q_n^0 = q_n^{t/2}.$$

For t odd (and greater than 2), again using (17.62),

$$(17.64)\ \left[q_n^0 q_n^t\right]^{1/2} = (1+g_n)^{t/2} q_n^0 = (1+g_n)^{(1/4)\left[(t-1)(t+1)\right]} q_n^0 = \left[q_n^{(t-1)/2} q_n^{(t+1)/2}\right]^{1/2}.$$

Using equations (17.63) and (17.64), it can be shown that, if quantities grow geometrically, then the Schultz midyear and the Okamoto geometric-type midyear indices all equal their Walsh counterparts; that is,

[52]Okamoto (2001) also makes some theoretical arguments relying on the theory of Divisia indices to show why midyear indices might approximate superlative indices.

[53]Recall that Marshall-Edgeworth indices are not actually superlative, but they will usually approximate their superlative Fisher counterparts fairly closely using "normal" time-series data.

(17.65) $P_S(p^0,p^t,q^{t/2}) = P_W(p^0,p^t,q^0,q^t)$ for t even;

(17.66) $P_{OG}(p^0,p^t,q^{(q-1)/2}, q^{(q+1)/2}) = P_W(p^0,p^t,q^0,q^t)$ for t odd.

Thus, under the geometric growth rates equation (17.62), the fixed-base and chained geometric-type sequences of midyear indices, equations (17.53) and (17.55), respectively, become the following sequences of Walsh price indices:

(17.67) $1, P_W(p^0,p^1,q^0,q^1), P_W(p^0,p^2,q^0,q^2), P_W(p^0,p^3,q^0,q^3), P_W(p^0,p^4,q^0,q^4), \ldots$;

(17.68) $1, P_W(p^0,p^1,q^0,q^1), P_W(p^0,p^2,q^0,q^2),$
$P_W(p^0,p^1,q^0,q^1)P_W(p^1,p^3,q^1,q^3),$
$P_W(p^0,p^2,q^0,q^2)P_W(p^2,p^4,q^2,q^4),$
$P_W(p^0,p^1,q^0,q^1)P_W(p^1,p^3,q^1,q^3)P_W(p^3,p^5,q^3,q^5), \ldots$.

17.98 Since the Walsh price indices are superlative, the results in this section show that if quantities are trending in a very smooth manner, then it is likely that superlative indices can be approximated fairly closely without having a knowledge of current-period quantities (but provided that lagged quantity vectors can be estimated on a continuous basis).

17.99 It seems very likely that the midyear indices will approximate superlative indices to a much higher degree of approximation than chained or fixed-base Laspeyres indices.[54] However, the real choice may not be between computing Laspeyres indices versus midyear indices but in producing midyear indices, in a timely manner versus waiting a year or two to produce actual superlative indices. However, there is always the danger that when price or quantity trends suddenly change, the midyear indices considered could give rather misleading advanced estimates of a superlative index. However, if this limitation of midyear indices is kept in mind, it seems that it would generally be useful for statistical agencies to compute midyear indices on an experimental basis.[55]

[54]It is clear that the midyear index methodologies could be regarded as very simple forecasting schemes to *estimate* the current period quantity vector based on past time series of quantity vectors. Viewed in this way, these midyear methods could be greatly generalized using time-series forecasting methods.

Appendix 17.1: Relationship Between Divisia and Economic Approaches

17.100 Divisia's approach to index number theory relied on the theory of differentiation. Thus, it does not appear to have any connection with economic theory. However, starting with Ville (1946), a number of economists[56] have established that the Divisia price and quantity indices *do* have a connection with the economic approach to index number theory. This connection is outlined in the context of output price indices.

17.101 The approach taken to the output price index is similar to that taken in Section C.1. Thus, it is assumed that there is a linearly homogeneous *output aggregator function*, $f(q) = f(q_1,\ldots,q_N)$, that aggregates the N individual outputs that the establishment produces into an aggregate output, $q = f(q)$.[57] It is assumed further that in period t, the producer maximizes the revenue that it can achieve, given that it faces the period t aggregator constraint, $f(q) = f(q^t)$, where q^t is the observed period t output vector produced by the establishment. Thus, the observed period t production vector q^t is assumed to solve the following period t revenue maximization problem:

$$\text{(A17.1)}\ R(Q^t,p^t) \equiv \max_q \left\{ \sum_{i=1}^{N} p_i^t q_i \ : f(q_1,\ldots,q_N) = Q^t \right\} = \sum_{i=1}^{N} p_i^t q_i^t \ ;\ t = 0,1,\ldots,T,$$

where the period t output aggregate Q^t is defined as $Q^t \equiv f(q^t)$, and $q^t \equiv [q_1{}^t,\ldots,q_N{}^t]$ is the establishment's period t observed output vector. The period

[55]Okamoto (2001) notes that in the 2000 Japanese CPI revision, midyear indices and chained Laspeyres indices will be added as a set of supplementary indices to the usual fixed-base Laspeyres price index.

[56]See, for example, Malmquist (1953, p. 227), Wold (1953, pp. 134–47), Solow (1957), Jorgenson and Griliches (1967), and Hulten (1973). See Balk (2000) for a comprehensive survey of work on Divisia price and quantity indices.

[57]Recall the separability assumptions (17.15).

t price vector for the N outputs that the establishment produces is $p^t \equiv [p_1^t,...,p_N^t]$. Note that the solution to the period t revenue maximization problem defines the *producer's revenue function*, $R(Q^t,p^t)$.

17.102 As in Section B.4, it is assumed that f is (positively) linearly homogeneous for strictly positive quantity vectors. Under this assumption, the producer's revenue function, $R(Q,p)$, decomposes into $Qr(p)$, where $r(p)$ is the *producer's unit revenue function*; see equation (17.16) in Section B.4. Using this assumption, it is found that the observed period t revenue, $\sum_{i=1}^{N} p_i^t q_i^t$, has the following decomposition:

(A17.2) $$\sum_{i=1}^{N} p_i^t q_i^t = r\left(p^t\right) f\left(q^t\right) \text{ for } t = 0,1,\ldots,T.$$

Thus, the period t total revenue for the N commodities in the aggregate, $\sum_{i=1}^{N} p_i^t q_i^t$, decomposes into the product of two terms, $r(p^t)f(q^t)$. The period t unit revenue, $r(p^t)$, can be identified as *the period t price level* P^t, and the period t output aggregate, $f(q^t)$, as *the period* t *quantity level* Q^t.

17.103 The economic price level for period t, $P^t \equiv c(p^t)$, defined in the previous paragraph now is related to the Divisia price level for time t, $P(t)$, that was defined in Chapter 15 by the differential equation (15.29). As in Section D.1 of Chapter 15, now think of the prices as being continuous, differentiable functions of time, $p_i(t)$ say, for $i = 1,\ldots,N$. Thus, the unit revenue function can be regarded as a function of time t as well; that is, define the unit revenue function as a function of t as

(A17.3) $r^*(t) \equiv r[p_1(t),p_2(t),\ldots,p_N(t)]$.

Assuming that the first-order partial derivatives of the unit revenue function r exist, the logarithmic derivative of $r^*(t)$ can be calculated as follows:

(A17.4) $$d \ln r^*(t)/dt \equiv \left[1/r^*(t)\right] dr^*(t) dt$$

$$= \left[1/r^*(t)\right] \sum_{n=1}^{N} r_i\left[p_1(t), p_2(t),..., p_N(t)\right]$$

using equation (A17.3),

where

$$r_i[p_1(t),p_2(t),\ldots,p_N(t)] \equiv \partial r[p_1(t),p_2(t),\ldots,p_N(t)]/\partial p_i$$

is the partial derivative of the unit revenue function with respect to the ith price, p_i, and $p_i'(t) \equiv dp_i(t)/dt$ is the time derivative of the ith price function, $p_i(t)$. Using Hotelling's (1932, p. 594) lemma, the producer's revenue-maximizing supply for commodity i at time t is

(A17.5) $q_i(t) = Q(t)\, r_i[p_1(t),p_2(t),\ldots,p_N(t)]$
for $i = 1,\ldots,N$,

where the aggregate output level at time t is $Q(t) = f[q_1(t),q_2(t),\ldots,q_N(t)]$. The continuous-time counterpart to equation (A17.2) is that total revenue at time t is equal to the output aggregate, $Q(t)$, times the period t unit revenue, $r^*(t)$; that is,

(A17.6) $$\sum_{i=1}^{N} p_i(t) q_i(t) = Q(t) r^*(t)$$

$$= Q(t) r\left[p_1(t), p_2(t),..., p_N(t)\right].$$

Now the logarithmic derivative of the Divisia price level $P(t)$ can be written as (recall equation 15.29 in Chapter 15)

(A17.7) $$P'(t) / P(t) = \frac{\sum_{i=1}^{N} p_i'(t) q_i(t)}{\sum_{i=1}^{N} p_i(t) q_i(t)} \sum_{i=1}^{N} p,$$

$$= \frac{\sum_{i=1}^{N} p_i'(t) q_i(t)}{Q(t) r^*(t)}, \text{ using (A17.6)}$$

$$= \frac{\sum_{i=1}^{N} p_i(t)\left\{Q(t) r_i\left[p_1(t), p_2(t),..., p_N(t)\right]\right\}}{Q(t) r^*(t)},$$

using equation (A17.5)

$$= \sum_{i=1}^{N} r_i\left[p_1(t), p_2(t),..., p_N(t)\right] p_i' \Big/ r^*(t)$$

$$= \left[1/r^*(t)\right] dr^*(t)/dt,$$

using equation (A17.4)

$$\equiv r^{*\prime}(t)/r^*(t).$$

Thus, under the above continuous-time revenue-maximizing assumptions, the Divisia price level, $P(t)$, is essentially equal to the unit revenue function evaluated at the time t prices, that is,

$r^*(t) \equiv r[p_1(t),p_2(t),\ldots,p_N(t)]$.

17.104 If the Divisia price level $P(t)$ is set equal to the unit revenue function $r^*(t) \equiv r[p_1(t),p_2(t),\ldots,p_N(t)]$, then from equation (A17.2) it follows that the Divisia quantity level $Q(t)$ defined in Chapter 15 by equation (15.30) will equal the producer's output aggregator function regarded as a function of time, $f^*(t) \equiv f[q_1(t),\ldots,q_N(t)]$. Thus, under the assumption that the producer is continuously maximizing the revenue that can be achieved given an aggregate output target where the output aggregator function is linearly homogeneous, it has been shown that the Divisia price and quantity levels $P(t)$ and $Q(t)$, defined implicitly by the differential equations (15.29) and (15.30) in Chapter 15, are essentially equal to the producer's unit revenue function $r^*(t)$ and output aggregator function $f^*(t)$, respectively.[58] These are rather remarkable equalities since, in principle, given the functions of time, $p_i(t)$ and $q_i(t)$, the differential equations can be solved numerically,[59] and hence $P(t)$ and $Q(t)$ are in principle observable (up to some normalizing constants).

17.105 For more on the Divisia approach to index number theory, see Vogt (1977; 1978) and Balk (2000).

[58]The scale of the output aggregator and unit revenue functions are not uniquely determined by the differential equations (15.29) and (15.30); that is, given $f(q)$ and $r(p)$, one can replace these functions by $\alpha f(q)$ and $(1/\alpha)r(p)$, respectively, and still satisfy equations (15.29) and (15.30) in Chapter 15.

[59]See Vartia (1983).