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Optimal Fiscal Adjustment under Uncertainty

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Abstract

The paper offers a non-probabilistic framework for representation of uncertainty in the context of a simple linear-quadratic model of fiscal adjustment. Instead of treating model disturbances as random variables with known probability distributions, it is only assumed that they belong to some pre-specified compact set. Such an approach is appropriate when the decision maker does not have enough information to form probabilistic beliefs or when considerations for robustness are important. Solution of the model in the minimax sense when disturbance sets are ellipsoids is obtained and the application of the method is illustrated using the example of Portugal.

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1 Introduction

Uncertainty is inherent to economic decision-making; in many situations choices need to be made in the absence of complete information about their consequences. A classical example is the permanent income model where uncertainty about future incomes gives rise to precautionary savings.¹ The decision of how much to save and when does not pertain only to the individual or the household. Governments of resource-rich countries face a similar, if not a more challenging problem of managing resource wealth given highly volatile and unpredictable commodity prices (see IMF, 2015a). Uncertainty is a key factor in the conduct of monetary and fiscal policy and in climate change economics, among others.

This paper is concerned with the design of optimal policy in the presence of general uncertainty. The proposed analytical framework is applied to a small-scale model of fiscal adjustment but the method has considerable generality and can be used to address any decision problem of similar nature. The focus on adjustment is motivated by the prominent role that fiscal policy has played in the response to the financial crisis and the growing attention fiscal instruments have received in recent policy debates on stabilization and growth (see IMF, 2015b).

With monetary policy constrained by the zero lower bound of interest rates, many governments, especially in advanced countries, resorted to discretionary fiscal policy to provide short-term support to aggregate demand during the crisis. Stimulus packages included temporary tax cuts, increased spending on unemployment and social benefits and investment in infrastructure (IMF, 2013). In a number of cases, public interventions involved substantial support to the financial sector in the form of capital injections, asset purchases or extension of guarantees. Against the background of difficult economic conditions and declining revenue, fiscal balances deteriorated sharply and public debt soared. In advanced economies, the (simple) average debt-to-GDP ratio increased from 48 percent in 2007 to 75 percent in 2014.² Debt has been on a rising path in some emerging markets as well.

The expansion of fiscal deficits and rapid accumulation of government liabilities has raised concerns about the sustainability of public finances. Unwinding of the fiscal stimulus measures alone has not been sufficient to put debt on a declining path. Large-scale fiscal adjustment is still needed in many advanced

¹For instance, Mody et al. (2012), have attributed a significant part of the increase in savings rates in the aftermath of the financial crisis to the precautionary motive. Their result is explained with the heightened uncertainty about labor income and investment returns.

²October 2015 Fiscal Monitor data base.

economies to ensure solvency and minimize the drag of debt on growth.³

Designing a fiscal consolidation program that minimizes the negative impact on growth entails difficult choices. These include the targeted level of debt, the timing and size of the primary balance improvement, as well as the composition of adjustment (the mix of revenue and expenditure measures). Decision-making becomes all the more challenging when the effects of fiscal policy on output and debt are not known precisely due to parameters uncertainty, measurement errors and various shocks that may affect the outcome.

Blanchard and Leigh (2013) provide a brief non-technical summary of the arguments as to when it is better to adjust earlier rather than later and when backloading of adjustment could pay off. To a large extent the debate about the optimal pace of adjustment is organized around the size of fiscal multipliers – the change in GDP resulting from a unit discretionary change in taxes or government expenditure. Recent empirical evidence suggests that fiscal multipliers are not constant over the business cycle and they are typically bigger during downturns. Based on US data, Auerbach and Gorodnichenko (2012) find that spending multipliers vary from 0 - 0.5 in expansions to 1-1.5 in recessions. This implies that a reduction in government spending or an increase in taxes would have a large negative impact on output during a downturn and hence, postponing consolidation would be the more prudent approach. Additional considerations in favor of a smaller initial adjustment in bad times are related to possible non-linear and persistent effects on growth. For example, DeLong and Summers (2012) argue that in a depressed economy cyclical output shortfalls may affect future potential output. The presence of such hysteresis effects has important implications for the conduct of fiscal policy.

In some cases, however, a front-loaded consolidation may be warranted. As noted by Blanchard and Leigh (2013), if a country finds itself in a situation of debt overhang, it would likely face rising interest rates which could render its public debt unsustainable. Also, higher sovereign bond spreads could feed into the spreads of private borrowers and contribute to a further slowdown of growth. In such circumstances, a larger upfront adjustment that reduces the level of debt and restores confidence of investors could prove beneficial. Indeed, some studies suggest that fiscal adjustments can be associated with expansions in private demand. Using cross-country data for OECD countries, Giavazzi and Pagano (1995) find that changes in fiscal policy (both expansions and contractions) can have significant non-Keynesian effects if they are sufficiently

³Some authors find a negative impact of high public debt on growth (e.g., Kumar and Woo, 2010). The effect becomes pronounced only after a certain threshold of the debt-to-GDP ratio is reached. This threshold varies by country and depends on a number of factors, such as level of development and investors base.

large and persistent.

The foregoing discussion makes it clear that a "one-size-fits-all" approach would not be productive and the adjustment program in each case should reflect country-specific circumstances. Analysts and policy makers would generally be able to form views based on previous consolidation episodes and experience of other countries with similar characteristics. Nonetheless, it may be useful to complement the qualitative arguments with a quantitative measure of an optimal fiscal response. The potential benefit from a formalized approach is perhaps best illustrated by the example of a highly indebted economy facing low or negative growth. In such circumstances, the debt overhang story and the high multipliers story tend to work in opposite directions as regards the appropriate fiscal action. Should a larger adjustment be undertaken initially to reduce the fiscal deficit or should automatic stabilizers be allowed to operate? If consolidation is postponed, for how long and at what cost? While these questions do not admit simple answers, a model which incorporates the various trade-offs and accounts for some of the uncertainties, could help to guide the policy decisions.

Quantitative policy rules can be obtained as solutions to suitable optimization problems involving minimization of a loss function (or maximization of a utility function) subject to constraints. In the economic literature, loss functions are typically assumed to be quadratic and constraints are linear. The quadratic functional form is either specified ad hoc, based on considerations for stability, or it is derived as a second order Taylor approximation of some "true" objective function. Benigno and Woodford (2012) argue that linear-quadratic (LQ) problems can be employed as approximations to exact optimal policy problems in a broad range of cases. Furthermore, under certain conditions the linear decision rules that result from solving LQ models represent local linear approximations to the actual optimal policy rule.

In economic applications, the preferred framework for tackling optimal decision problems is stochastic LQ control; it has been extensively studied and its properties are well known. One feature that makes it particularly attractive is that when disturbances are normal, replacing the uncertain quantities with their mathematical expectations yields the same solution as the corresponding deterministic problem (the so called "certainty equivalence" principle).⁴

While certainty equivalence is a convenient analytical property, it is not

⁴The concept of certainty equivalence was introduced by Simon (1956) in the one-dimensional case and later generalized by Theil (1957). Most results related to certainty equivalence are obtained for LQ problems with additive Gaussian disturbances. Deviations from this framework may cause the principle to fail (see Chapter 10 in Chow(1986)).

necessarily a good representation of how decisions are made in practice. In many real world situations uncertainty is explicitly taken into account in decision-making, and actions are chosen such as to yield acceptable outcomes in a range of circumstances. Thus, when policy makers decide about specific measures, they would consider not just the most likely scenario, but also alternative, perhaps less likely, but still plausible scenarios. An approach entirely based on expected values would fail to capture this feature. Moreover, sometimes it is not practical, or even feasible, to assign probabilities to the various possible outcomes, which essentially precludes the use of stochastic methods. This point has been made on various occasions at meetings of the Federal Open Market Committee, as the following quotes attest.⁵

*A. Santomero: "At this point, it might be useful for us to recognize again the difference between risk and uncertainty. With risk, as we know, one can assign probabilities to the list of outcomes and act appropriately given the distribution. With uncertainty, it is difficult to assign probabilities to outcomes...Today we are operating in a world of increased uncertainty."*⁶

*D. Kohn: "I don't feel as though I know enough to say that the risks are balanced. I don't know. The range of outcomes is just too wide, and there's very little central tendency in it. So I'd be very uncomfortable with a statement saying that I kind of thought the risks were balanced. I am much more comfortable with a statement that says there is a lot of uncertainty out there and that's uncertainty around the economic outlook."*⁷

As noted earlier, the goal of this paper is to present a framework that allows for the derivation of optimal policy in the presence of general uncertainty. Unlike the stochastic set-up where model disturbances are random variables with given probability distributions, here they are only assumed to belong to some compact set. This set-membership approach to modeling uncertainty is relatively well established in the engineering literature and can be traced back to the works of Witsenhausen (1966), Bertsekas (1971) and Bertsekas and Rhodes (1973), among others.

In economics, similar ideas have been pioneered by Hansen and Sargent (2008) who take the view of robust control to address issues with model specification.

⁵For more examples, see Nelson and Katzenstein (2014).

⁶FOMC (2003), p.45

⁷FOMC (2007), p.110

In Hansen and Sargent (2008) and in other contributions, the decision maker expresses a preference for robustness through the inclusion of a penalty term in his loss function (the multiplier problem). An alternative formulation is to constrain the sum of the squared future disturbances (the constraint problem). Under certain conditions, the solution of the two problems coincide and in the LQ case optimal feedback rules can be derived explicitly.

The approach adopted in this paper is in the spirit of robust control in the sense that no specific structure is imposed on the model disturbances, except that they are bounded. The way the bounds are specified, however, is different from the standard robust control literature. Rather than postulating a single constraint on the squared sum of disturbances as in the constraint problem, we consider the case where the bounds for the uncertain quantities are provided separately in each period (instantaneous constraints). We take these bounds to be given by ellipsoids in R^n .

The instantaneous constraints formulation allows for a more flexible specification of uncertainties and is of greater practical relevance. However, the increased flexibility comes at the cost of higher technical complexity. Even with ellipsoidal constraints for disturbances, it does not seem feasible to obtain a closed form solution as it is the case with LQ robust control. The dynamic programming algorithm provided by Bertsekas (1971) is applicable but its implementation is cumbersome and the calculation of optimal controls is not straightforward. Instead, we obtain a solution using a minimax maximum principle. The resulting boundary value problem is solved numerically with a shooting method. The shooting method takes advantage of the finite horizon of the problem, a feature which, along with the possibility for time-varying coefficients, distinguishes our model from most of the literature. In the particular case of fiscal adjustment, the focus on finite horizons is natural; governments aim to bring debt to desirable levels within a given time frame. Often the credibility of the consolidation plan will depend on the clear specification of this time frame.

From the point of view of policy making, the applied aspects of the proposed approach are of interest. To give a flavor of how the model can be used to assess the pace of fiscal consolidation of a particular country, we take the example of Portugal in 2011. We perform several experiments assuming different fiscal multipliers and weights in the loss function and examine how the optimal response, measured by the change in the primary balance, reacts to the model assumptions. Overall, the results confirm the intuition that when multipliers are high and growth is low, the adjustment should be smaller. The value of the analysis is that it provides a numerical estimate of the optimal primary balance at each point in time. As a side tool, we show how external ellipsoidal

approximations of the reachable sets of a dynamical system can be used to infer where debt and the output gap may end up under the influence of disturbances. The reachable set technique can be considered as the analogue of fan charts in the case when no probabilistic assumptions are made about the shocks.

The paper is organized as follows. Section 2 provides a brief overview of the literature on decision-making under uncertainty; Section 3 develops the model and discusses its solution in the general case; Section 4 reports the results from the policy experiments for Portugal, and finally, Section 5 offers some conclusions and directions for further work.

2 Decision-making under uncertainty: literature review

The problem of selecting the best course of action in the face of uncertainty is a problem of decision theory. A typical decision problem comprises several elements: (i) an unknown quantity – usually interpreted as the state of nature; (ii) a set of possible actions; and (iii) a criterion (e.g. utility or loss function) to evaluate the outcome. When uncertainty can be plausibly described in probabilistic terms, the tools of statistical decision theory can be employed.

There are two main branches of statistical decision theory – Bayesian and frequentist, which mainly differ in how information is treated, in particular prior information. A Bayesian decision maker would minimize the expected loss, where the expectation is taken with respect to a prior probability distribution. A frequentist would minimize a risk function that is consistent with the worst possible outcome if a certain decision rule is applied.

While the Bayesian approach seems to have gained dominant position in modern decision theory, there are situations in which the (frequentist) minimax criterion would be more appropriate.⁸ One such situation is when the state of nature is determined by an intelligent opponent. In problems of this kind, the minimax approach is justified on the grounds of preference for conservative behavior. For example, in deciding whether to invest in a risky or in a safe bond, the safe bond could be chosen even though it has lower expected return (Berger, 2010). The minimax approach is perhaps most useful when no prior information is available. A closely related notion is that of Knightian uncertainty which refers to a form of uncertainty ”not susceptible to

⁸Influenced by game theory, the minimax criterion has been part of decision theory since its very early days (see Wald (1950), also Savage (1951) for an intuitive, non-technical explanation).

measurement and hence to elimination”,⁹ unlike ”measurable uncertainty” or ”risk” which can be represented by numerical probabilities.

Ellsberg (1961) in his influential paper highlights the distinction between the two types of uncertainty. He considered the willingness to take bets as revealing the degree of belief for events for which no statistical information is available and designed experiments showing that some decisions are incompatible with the axioms of rational behavior. This occurs in situations of information ambiguity. According to Ellsberg, minimax decision rules are consistent with complete lack of information about probabilities, whereas the Bayesian approach is undoubtedly preferable when one has a definite view on a particular distribution. The difficulty arises in the intermediate cases when uncertainty can neither be characterised as ignorance, nor as risk.

Arrow and Hurwicz (1972) proposed a modification to the minimax model to tackle problems where no *a priori* information is available.¹⁰ They showed that a rational criterion of choice under complete ignorance would take into account only the worst and the best outcome among all possible outcomes. A decision rule that formalizes this framework is a weighted average of the two, with the weight representing the degree of pessimism of the decision maker. Thus, under the Arrow-Hurwicz approach, a conservative decision maker would likely adopt a criterion closer to the minimax.

Recent developments in decision theory that offer solutions to the ambiguity problem include the Choquet expected utility (Schmeidler, 1989) and prospect theory (Kahneman and Tversky, 1979). A comprehensive survey of the advances in the field can be found in Etner et al. (2012). We shall only briefly refer to the work of Gilboa and Schmeidler (1989) which can be related to the application of robust control to economic problems (see below). Gilboa and Schmeidler (1989) recall Ellsberg’s experiment and confirm that no probability measure could support the empirically observed preferences through expected utility maximization. They provide an explanation to the paradox based on the observation that the decision maker has too little information to form a prior and therefore, he or she considers a set of priors. ”Being uncertainty averse (s)he takes into account the minimal expected utility (over all priors in the set) while evaluating a bet.”¹¹ Gilboa and Schmeidler establish an axiomatic foundation of the minimax expected utility theory which covers Wald’s minimax criterion. A further generalization is obtained by Maccheroni et al. (2006)

⁹Knight (1964), p.232.

¹⁰The approach of Arrow and Hurwicz (1972) is in spirit closer to the non-probabilistic framework proposed by Shackle than the decision theory based on subjective probability developed by Savage (see Zappia (2014) for an extensive account of Shackles work.)

¹¹Gilboa and Schmeidler (1989), p. 142.

through the introduction of variational preferences which include multiple prior and multiplier preferences as special cases.

While decision theory was initially developed in a static framework, its usefulness for dynamic problems was quickly appreciated. Witsenhausen (1966) is an example of systematic application of decision theory to a discrete time optimal control problem and is also a good source of references to earlier work.¹²

In a dynamic setup, the decision problem is often associated with the need to achieve robust performance of a perturbed control system. The concept of robust control originated in the engineering literature in recognition of the fact that classical control techniques did not deliver the desired outcomes (e.g. stability) under small perturbations of the system dynamics. Theory evolved over time and stochastic control methods that prevailed in the 1960s and 1970s gave way to alternative approaches. An example is the so-called H^∞ control which was initially developed in the frequency domain but as the link with dynamic games became known, efforts were directed to obtaining solutions in the time domain (see Basar and Bernhard (1995) for more details).

In economics, robust control methods have been actively promoted by Hansen and Sargent (2008). In their framework, the main justification of robustness comes from the need to address the issue of model misspecification. The essence of Hansen and Sargent's approach is that the decision maker has a model that is only an approximation of the true model that generates the data. This model is surrounded by a set of alternative models, such that their relative entropy is bounded by some number. Relative entropy is used as a measure of the distance between alternative probability distributions. Thus, the decision maker deals with misspecification by seeking a rule that will perform well across all models that satisfy the relative entropy constraint. Relating this idea to the previous discussion, the formulation of the decision problem in terms of multiple priors naturally leads to the Gilboa-Schmeidler paradigm and the minimax solution concept. Consequently, the problem can be formalized as a two-player dynamic game where a malevolent agent chooses a disturbance that minimizes the utility function which the decision maker attempts to maximize (see Hansen and Sargent, 2011).

The dynamic game approach to controlling uncertain systems has been applied to non-linear problems as well. Baras and Patel (1998), for example, obtained results for non-linear systems represented by difference inclusions, where the

¹²It is interesting to note that two decades before Gilboa and Schmeidler, Witsenhausen discussed the expected minimax utility criterion. He argued that the minimax decision rule, or "guaranteed performance evaluator" in his terminology, is a special case of an "expected guaranteed performance evaluator" which essentially combines expectations with the minimax.

decision maker is concerned with the rejection of bounded disturbances on a regulated output. The problem gives rise to a dynamic game where the controller plays against the set-valued system. The authors offer a specific example involving both parametric uncertainty and additive disturbances.

More recently, Moitie et al. (2002) considered a rather general discrete-time optimal control problem with uncertainty both in the dynamics and the measurement of the system's state, as well as imperfect measurement of the initial condition. They derived an optimal output-feedback solution in the minimax sense and applied it to a pursuit-evasion game with incomplete information of the current state of the evader.

Robust control methods have also been used in relation to econometrically estimated models. Onatski and Williams (2003), for instance, take a small estimated macroeconomic model and derive Taylor-type monetary policy rules that are robust to various forms of uncertainty. One important conclusion from their analysis is that policy rules designed to perform well under one type of uncertainty may fail and lead to instability if the uncertainty is of different nature. Onatski and Williams (2003) implement both parametric and non-parametric approaches to modeling model errors. The parametric specification results in a probabilistic description of uncertainty and use of the Bayesian decision criterion. The non-parametric specification imposes less structure (only empirically simulated bounds on the uncertainty) and the relevant criterion is minimax. They find that for many specifications the Bayesian and minimax criteria produce similar results.

As discussed above, the decision criterion used in this paper is minimax but the instantaneous constraints on the uncertain variables differentiate our model from the standard (constraint) robust control problem. Therefore, it is not obvious if an equivalent multiplier problem can be devised that would fit into the decision-theoretic framework of Maccheroni et al. (2006), for example. While establishing an axiomatic foundation for our model is of interest on its own, this task goes beyond the scope of the paper. Yet, we mention one possible interpretation which is due to Witsenhausen (1966). He considered a criterion of the form $\inf_{d \in D} \sup_{a \in A} E_{\mu_a(n)} L(d, n)$, where A is a set of probability measures on a common σ -algebra on the set of states of nature N ($n \in N$), $\mu_a(n)$ is the probability measure for $a \in A$, D is the action space or the set of possible decisions ($d \in D$) and $L(d, n)$ is a loss function. In this setting, expected performance corresponds to the case when A consists of a single element and guaranteed or minimax performance is associated with the case when the σ -algebra is the set of all subsets of N and $A = N$ with μ_n the atomic measure

at point n .¹³

3 A simple model of fiscal adjustment

We consider a model where the decision maker (government) is interested in the paths of output and public debt. With no discretionary fiscal policy and no uncertainty, output and debt would evolve according to the following equations:

$$\begin{aligned} Y_{t+1} &= (1 + g_{t+1})Y_t \\ D_{t+1} &= (1 + r_{t+1})D_t + P_{t+1}, \end{aligned}$$

where Y_t denotes nominal GDP at time t , g_{t+1} is the nominal GDP growth rate in $t + 1$, D_t stands for the nominal stock of debt, r_{t+1} is the nominal interest rate paid on debt and P_{t+1} is the primary deficit. A fiscal action, defined here as a change in the primary deficit $\Delta\tilde{P}_{t+1}$, influences the dynamics of both output and debt.¹⁴ The effect on output would depend on the size of the fiscal multiplier α_t , so GDP would change relative to the baseline by $\Delta Y_{t+1} = \alpha_t \Delta\tilde{P}_{t+1}$. Assuming that α_t is positive (Keynesian effect), an increase in the primary deficit would result in a higher GDP but would also increase government debt by the amount of the stimulus. Thus, with discretionary fiscal policy the above equations become:

$$Y_{t+1} = (1 + g_{t+1})Y_t + \alpha_t \Delta\tilde{P}_{t+1} \tag{1}$$

$$D_{t+1} = (1 + r_{t+1})D_t + P_{t+1} + \Delta\tilde{P}_{t+1}. \tag{2}$$

System (1)-(2) describes the dynamics of output and debt in nominal terms. It is often more convenient to use scaled variables, e.g. obtained by dividing both sides of the above equations by potential output in period $t + 1$, denoted by Y_{t+1}^p and assumed to grow at a rate g_{t+1}^p that is not affected by fiscal policy. The latter assumption is an important one as it rules out possible hysteresis effects as discussed in DeLong and Summers (2012).¹⁵ Since we are primarily concerned with medium-term dynamics, this assumption does not seem overly restrictive.

¹³See Witsenhausen (1966), p. 12.

¹⁴In practice, the primary balance may not be fully controllable given its endogeneity to growth and implementation lags. Since we are primarily concerned with annual observations, we can assume that most of the effects of fiscal measures will take place within the year.

¹⁵Admitting the possibility that fiscal policy can influence g_{t+1}^p will make the problem non-linear which precludes the use of the solution methods discussed below.

Setting $y_t := Y_t/Y_t^p$, $d_t := D_t/Y_t^p$, $p_t := P_{t+1}/Y_{t+1}^p$, $u_t := \Delta\tilde{P}_{t+1}/Y_{t+1}^p$, and using that $Y_{t+1}^p = Y_t^p(1 + g_{t+1}^p)$, we can rewrite the equations as:

$$y_{t+1} = \frac{(1 + g_{t+1})}{(1 + g_{t+1}^p)}y_t + \alpha_t u_t$$

$$d_{t+1} = \frac{(1 + r_{t+1})}{(1 + g_{t+1}^p)}d_t + p_t + u_t$$

In the absence of uncertainty, for given parameters and initial values of the output gap \bar{y}_0 and debt to potential GDP ratio \bar{d}_0 , one could calculate explicitly the state trajectories for any sequence of policy actions since the system is linear in both the state and control variables. The deterministic setting, however, is a simplifying assumption which will fail to hold in most practical applications. In this particular example, key parameters of the model are not known with certainty, namely the fiscal multiplier α_t , the interest and growth rates. Furthermore, there are various shocks that could affect the behavior of the system at any point in time. Finally, direct measurements of potential GDP, and hence the output gap, are not available and this variable can only be estimated with some error. Explicit modeling of each of these types of uncertainty is, of course, preferable but it entails significant technical complications. We leave this for future work. Here the focus is on the case when model uncertainty can be reasonably represented by additive disturbance terms to the system dynamics. This assumption gives rise to the following system of first-order difference equations:

$$y_{t+1} = \frac{(1 + g_{t+1})}{(1 + g_{t+1}^p)}y_t + \alpha_t u_t + w_{1,t} \quad (3)$$

$$d_{t+1} = \frac{(1 + r_{t+1})}{(1 + g_{t+1}^p)}d_t + p_t + u_t + w_{2,t} \quad (4)$$

$$y_0 = \bar{y}_0, \quad d_0 = \bar{d}_0.$$

As indicated earlier, of interest is the case when the unknown shocks $(w_{1,t}, w_{2,t})$ are only assumed to belong to some compact set, without imposing any structure on them. For concreteness, we take this set to be the ellipsoid

$$\mathcal{W}_t = \{w_t : (w_t - \bar{w}_t)'W_t^{-1}(w_t - \bar{w}_t) \leq 1\}, \quad (5)$$

where the matrices W_t^{-1} are positive-definite.

The choice of ellipsoids as a class of sets where model disturbances lie is partly driven by technical convenience since an ellipsoid is fully specified by only two

parameters – its center and shape matrix. This allows for analytical derivations of first-order optimality conditions with respect to the disturbance vector. An additional advantage of the ellipsoidal sets is their potential link to statistical inference.¹⁶

To simplify notation it is convenient to set $x_t = (y_t, d_t)'$, $e_t = (0, p_t)'$. Thus, the output and debt dynamics can be written in a compact form as

$$x_{t+1} = A_t x_t + B_t u_t + e_t + G_t w_t \quad (6)$$

$$x_0 = \bar{x}_0 \text{ (given)}. \quad (7)$$

where the time-varying matrices A_t and B_t can be inferred from equations (3)-(4). Note that the formulation in (6) is slightly more general than in (3)-(4) as it allows both components of the disturbance vector to influence both variables if the matrix G_t is not diagonal (e.g. a shock on debt also affects growth). Further, since often in practice factors other than the primary deficit influence the debt dynamics, such stock-flow adjustments, when known, should be reflected in the free term e_t . Finally, we highlight that obtaining a solution based on the general system (6)-(7) enhances the applicability of the method; it can be applied to an arbitrary problem characterized with linear dynamics, including to estimated models, for example.¹⁷

It is worth noting that while in equation (6) the additive nature of disturbances w_t is specified ad hoc, an equivalent representation can be obtained by linearizing a non-linear model. Consider, for instance, the following general model:

$$x_{t+1} = f_t(x_t, u_t),$$

where $x_t \in R^n$, $u_t \in R^m$ and the vector-valued function $f(\cdot, \cdot)$ is smooth, with $f(0, 0) = 0$. The above non-linear system can be expressed as a sum of a linear and a non-linear term around the origin:

$$x_{t+1} = A_t x_t + B_t u_t + R_t(x_t, u_t),$$

¹⁶Suppose we are given a model with Gaussian errors where the variables of interest have been estimated statistically and are known to be distributed normally with mean μ and covariance matrix Σ . The level sets of the normal distribution, or the surfaces of constant probability density, are the sets of points x such that $(x - \mu)' \Sigma^{-1} (x - \mu) = c^2$. If $c^2 = \chi_p^2(a)$, the probability that the random vector is inside the the ellipsoid $(x - \mu)' \Sigma^{-1} (x - \mu) \leq \chi_p^2(a)$ is $1 - a$. This is the $(1 - a)$ confidence region for x – the multivariate analogue of the confidence interval.

¹⁷One possible extension of the fiscal consolidation model is to consider two control variables – taxes and expenditure– to calculate both the optimal pace and composition of adjustment. However, if the revenue and spending multipliers differ significantly, it may be necessary to modify the solution so as to constrain the controls. Otherwise, the model might suggest unrealistically large swings in the tax and expenditure ratios.

where A_t and B_t are the derivatives of f with respect to the first and the second argument, respectively, and the vector-valued function

$R_t = (R_{1,t}(x_t, u_t), R_{2,t}(x_t, u_t))$ summarizes all non-linear terms. Setting $w_t = (w_{1,t}, w_{2,t}) = (R_{1,t}(x_t, u_t), R_{2,t}(x_t, u_t))$ leads to a linear system as in (6).

We next turn to the objective of the decision maker. The decision maker chooses a sequence of controls u_t (fiscal actions) that minimize a loss function under the assumption of maximum disturbances (within the given bounds), i.e. he solves

$$\inf_{u_t} \sup_{w_t \in \mathcal{W}_t} J(u, w) \quad (8)$$

subject to constraints (5) -(7), where

$$J(u, w) = \sum_{t=0}^{T-1} \frac{\beta^t}{2} [(x_t - \bar{x}_t)' Q_t (x_t - \bar{x}_t) + (u_t - \bar{u}_t)' R_t (u_t - \bar{u}_t)] \\ + \frac{\beta^T}{2} (x_T - \bar{x}_T)' Q_T (x_T - \bar{x}_T)$$

is a quadratic loss function which depends on the state x and the control u , and involves some reference values for these variables \bar{x} and \bar{u} ; $\beta < 1$ is a discount factor, and Q_t and R_t are positive definite weight matrices. The reference values for the state variable x are exogenous to the model. While it would be natural to select the value of 1 as a benchmark for the ratio of actual to potential output (zero output gap), the choice of target debt ratio is less obvious. For some countries the decision may be guided by the existence of formal fiscal rules, such as the Maastricht debt criterion for EU member states, for example. However, strict adherence to such rules may not always be a prudent choice, especially if the initial debt is far off the reference value and the adjustment period is relatively short.

More generally, setting too ambitious goals could result in unrealistic adjustment paths. The presence of the quadratic term $(u_t - \bar{u}_t)^2$ in the objective function would attenuate this effect to some extent since it punishes large deviations from the baseline. However, if the unconstrained solution still points to significant changes in the primary balance, a formal constraint could be imposed on the control variable, i.e. $u_t \in \mathcal{U}_t = [\underline{u}_t, \bar{u}_t]$. The rationale is that too large an adjustment may not be politically feasible (or too much stimulus could trigger debt sustainability concerns).

In control theory problem (5)-(8) is known as the "linear-quadratic tracking problem" and it has been widely used in engineering applications. One

difference from the engineering literature is the presence of the discount factor β in the loss function (8). Adding a discount factor does not lead to any significant complication in terms of solving the model but is important from an economic point of view; for longer planning horizons when adjustment needs are high, it would counteract very large upfront debt reductions that lead to excessive output gaps in the initial periods.

The formulation of the tracking problem in this paper is fairly general and allows for the consideration of various alternatives. For instance, the goal of the policy maker could be to steer the system as close as possible to a desired end state \bar{x}_T with no reference to intermediate states. This is a special case of the above model where the running cost function is set equal to zero and only the final term remains. Also, the time-varying coefficients provide significant flexibility in terms of assumptions that can be incorporated in the model, e.g. the user can specify different fiscal multipliers in different periods of time.

It is customary to solve problems like (5)-(8) by applying the dynamic programming method (DP). For a model with linear dynamics and more general loss function and disturbance sets, Bertsekas (1971) provides a solution method by dynamic programming. The implementation of the DP algorithm for this particular problem, however, leads to technical difficulties related to the need to solve a complex equation for the Lagrange multiplier associated with the ellipsoidal constraint (see Appendix A.2). An alternative is to employ some form of a minimax principle along the lines of the maximum principle for discrete-time problems. Such minimax principle has been established by Bertsekas (1971) but its applicability is relatively limited since it rests on rather strong assumptions. It turns out, however, that these assumptions are satisfied for our model and we can use the result to calculate the optimal controls and trajectories. The following set of necessary conditions obtains:

$$\begin{aligned}
 w_t^* &= \bar{w}_t + \frac{1}{2\lambda_t} W_t G_t' p_{t+1} \\
 u_t^* &= \bar{u}_t - \beta R_t^{-1} B_t' p_{t+1} \\
 p_t &= A_t' p_{t+1} + \beta^t Q_t (x_t^* - \bar{x}_t) \\
 p_T &= \beta^T Q_T (x_T^* - \bar{x}_T) \\
 x_{t+1}^* &= A_t x_t^* + B_t u_t^* + e_t + G_t w_t^* \\
 x_0 &= \bar{x}_0.
 \end{aligned}$$

where p_t is the adjoint variable and λ_t is the Lagrange multiplier associated with the ellipsoidal constraint.

Thus, application of the minimax principle leads to a boundary value problem with an initial value for the state x and a transversality condition at the right end for p . Since the model is in finite time, we can use the so-called shooting method to find the solution. The method comprises guessing the initial value of the adjoint vector, solving the system with this initial value and after comparing the end value with the one obtained from the transversality condition, updating the initial guess. Appendix A.3 gives the necessary details of the procedure and explains how a multivariate secant method can be used to numerically solve the problem.

4 Application: the case of Portugal

As an illustration of how the model developed in the previous section can be applied to assess a specific fiscal adjustment program, we consider the case of Portugal at the time of the country's request for IMF assistance. The purpose of the exercise is to compare the paths of the primary balance, output gap and debt under the program with those implied by the model and examine how the key variables respond to changes in assumptions. Although the illustrative simulations refer to an advanced economy, the data requirements of the method are relatively modest and it can be used for emerging and low-income countries as well.¹⁸

After becoming part of the Euro area, Portugal enjoyed a period of low interest rates which contributed to a significant fiscal expansion prior to the global financial crisis. The increase in current expenditure (mostly social benefits and health care) outpaced the decline in the interest bill and public debt rose from 48 percent in 2000 to 93 percent in 2010 (IMF, 2011). At the same time, real exchange rate appreciation and unaddressed long-standing structural problems lead to loss of competitiveness and dimmed the country's growth prospects. Facing sharply worsening financing conditions with sovereign spreads at record high levels, in May 2011, the Portuguese authorities requested a three-year arrangement under the IMF's Extended Fund Facility (EFF).

One of the objectives of the program was restoring market confidence through bold fiscal reforms, while mitigating the impact of consolidation on economic activity. The authorities were committed to reaching a deficit of 3 percent of GDP and stabilizing debt by 2013. To ensure credibility of the consolidation plan, the adjustment was front-loaded with discretionary measures in 2011

¹⁸In fact, one could argue that the proposed approach is more appealing for countries with poor data availability or a track record of erratic fiscal behavior, given that in such cases the probabilistic framework would be difficult to apply.

amounting to 5.7 percent of GDP and another 5 percent in 2012-13 (IMF, 2011). Both revenue-enhancing measures and expenditure cuts were envisaged with a view to minimizing the drag on growth and protect vulnerable groups.

Table 1 presents a set of indicators pertaining to output and fiscal sector developments in the base year (2010) and IMF staff's projections under the baseline scenario.¹⁹

Table 1: Portugal: Selected indicators

	2010	2011	2012	2013	2014	2015	2016
<i>in billions of euro</i>							
Nominal GDP	172.5	170.6	169.8	174.0	180.7	187.2	193.5
Potential nominal GDP	173.6	175.5	178.7	183.0	188.3	193.8	199.5
Primary deficit	10.6	2.9	-0.5	-3.6	-5.1	-5.9	-6.4
Public debt	160.5	181.5	190.5	200.7	207.8	211.4	214.8
Stock-flow adjustments	4.8	11.0	1.4	5.0	2.8	0.0	0.0
<i>in percent</i>							
Nominal GDP growth	2.3	-1.1	-0.5	2.5	3.8	3.6	3.4
Potential GDP growth	0.4	1.1	1.8	2.4	2.9	2.9	3.0
Effective interest rate	3.7	4.4	4.5	4.6	4.6	4.6	4.7

Source: IMF (2011)

The information in the table is used to construct the matrices A_t and the initial values for the output gap and debt to potential GDP ratio. In the model, the baseline primary deficit is captured in the term e_t and there we also include the stock-flow adjustments that are implicit in IMF staff's projections. These liabilities, mainly incurred in relation to provision of government support to the financial sector, are quite significant in the initial years and constitute an important driver of public debt. To pin down the rest of the model parameters, the following assumptions are made:

(i) The fiscal multiplier α is set equal to 0.5, a value that is consistent with the findings for advanced economies in Mineshima et al. (2014).²⁰ For simplicity, it is kept constant throughout the projection period. (In one of the simulations a

¹⁹Potential nominal GDP is calculated based on the output gap shown in Table 1 of IMF (2011). Stock-flow adjustments are obtained as the difference between actual debt in the current period and the sum of the debt in the previous period, interest payments and the primary deficit. It is important to note that the actual fiscal adjustment path (as well as growth and debt) were different from the ones envisaged in 2011.

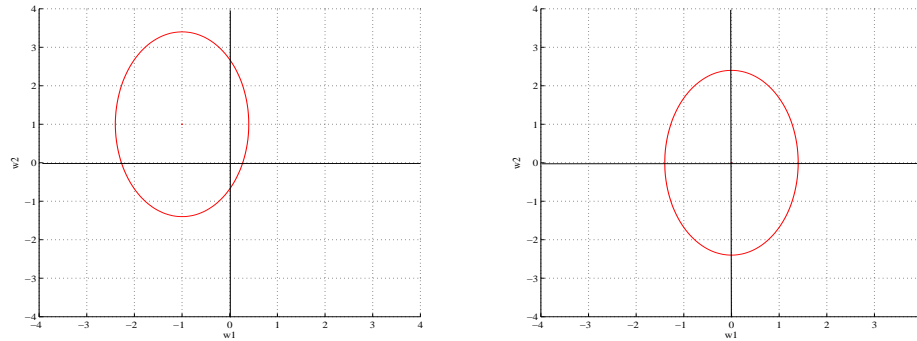
²⁰The authors find that the average first-year spending multiplier is about 0.75 and the revenue multiplier is about 0.25.

fiscal multiplier of 1.5 is assumed to explore the sensitivity of the optimal deficit path to the multiplier value.)

(ii) The weighting matrices Q_t and R_t are assumed constant and equal to the identity matrix; for Q_t this implies that equal weights are assigned to debt reduction and closing of the output gap. This is equivalent to minimizing the Euclidean distance between the actual state vector and the target vector. Other choices can be accommodated easily and the results would generally be sensitive to the weights, as demonstrated below.

(iii) The matrix G_t is also set equal to the identity matrix, implying that the first equation is affected only by the first disturbance w_1 and the second equation is perturbed only by w_2 .

Figure 1: Disturbance ellipsoids (left panel for $t=0,1,2$, right panel for $t=3,4,5$)



(iv) For illustrative purposes the disturbance ellipsoid is centered at the point $(-1, 1)$ in the first three years of the projection horizon to reflect concerns for possible negative shocks to the output gap and positive (debt-increasing) shocks to debt. For the remainder of the period, the anticipated shocks are neutral, so the disturbance ellipsoids are centered at the origin. (Note that this does not eliminate the uncertainty.) The shape matrix W_t is a constant diagonal matrix for all $t = 0, \dots, T - 1$ with entries of 1.96 and 5.76 in the main diagonal (a diagonal shape matrix relates to the case of uncorrelated shocks in a stochastic setting). This approximately corresponds to an uncertainty region between -2.4 and 0.4 for the output gap shock and between -1.4 and 3.4 for the debt shock in each period for $t = 0, 1, 2$.²¹ The choice of parameters for the shape matrix is

²¹As discussed in Appendix A.4, an ellipsoid in R^2 with a diagonal shape matrix is contained in a rectangle with sides equal to two times the square root of the main diagonal entries. In this case, this is a rectangle centered at $(-1, 1)$ with sides $2\sqrt{1.96}$ and $2\sqrt{5.76}$.

largely driven by past data. Using different vintages of the World Economic Outlook (WEO) database, we compared the one-year ahead forecasts for the output gap and debt ratio in the period 2001-2010 (information available at the time of the EFF request). The median absolute forecast errors are estimated at 1.4 percentage points for the output gap and 2.4 percentage points for the debt to potential GDP ratio, respectively.²² The diagonal elements of the shape matrix are the squares of these two numbers. Figure 1 plots the respective disturbance sets for $t = 0, 1, 2$ (left panel) and $t = 3, 4, 5$ (right panel). Uncertainty is higher for the debt ratio which explains the shape of the ellipsoids. It is important to emphasize that potential GDP used to calculate the output gap in simulations is based on Table 1 and reflects available information as of the projections date. Output gaps are notoriously difficult to estimate, especially in financial booms and busts, and potential outputs for many countries affected by the financial crisis, including Portugal, have been revised downwards subsequently. In principle, uncertainties regarding the future path of potential GDP should be taken into account when determining the uncertainty bounds.

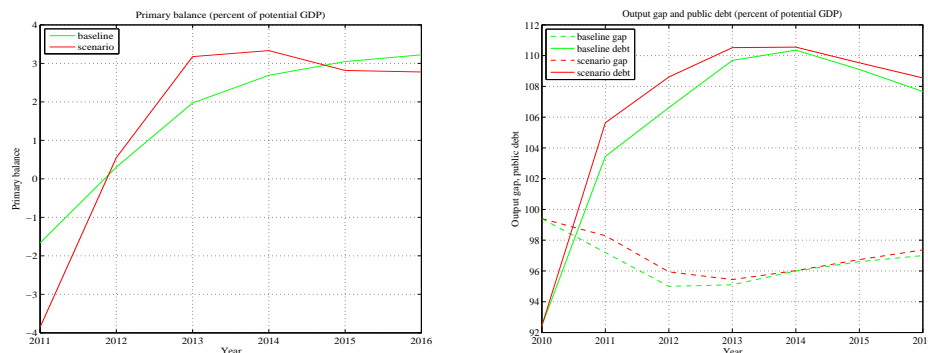
(v) Finally, it is assumed that the government targets a debt to potential GDP ratio of 107.7 percent (the same as in IMF (2011)) and zero output gap at the end of the planning period.

As an initial step of the scenario analysis we calculate the optimal fiscal path without uncertainty. It would be useful to compare the IMF baseline, which presumably is the most likely scenario, with the solution of a deterministic version of our model where all disturbances are set equal to zero. As shown in Appendix A.1, the deterministic model admits an explicit solution – a type of fiscal Taylor rule where the primary balance responds to the output gap and the debt level.

Figure 2 shows the optimal fiscal path implied by the deterministic model, along with the baseline IMF staff projections. The figure suggests that while the overall size of the required adjustment for the period is similar, the time profile of the optimal primary balance is different. In particular, the model implies a smaller adjustment in the initial period by about 2 percent of potential GDP and a larger adjustment in the third and fourth periods. Looking at the output gap and debt to potential GDP trajectories, they diverge slightly after the first period but come close together toward the end of the projection horizon. It should be stressed that these results are conditional on the model

²²The disturbance ellipsoids were calibrated based on past data just for illustrative purposes and the median absolute deviation was chosen as a robust measure of variability. Of course, if more detailed and forward-looking information is available to the decision maker, it would be natural to incorporate it and possibly specify different ellipsoidal bounds for each period.

Figure 2: Primary balance, output gap and public debt in the deterministic case (fiscal multiplier of 0.5; equal weights)

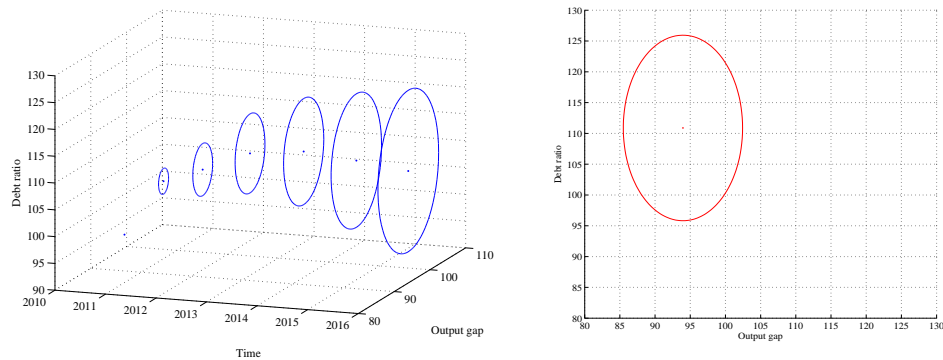


assumptions. If, for instance, a different target for the debt ratio is set, the two paths will likely end up being farther apart.

Adding uncertainty to the model changes the analysis in important ways; the trajectory of the state vector is no longer a sequence of points in R^2 as in the deterministic case. Instead, the evolution of the output gap and the debt ratio is represented by a sequence of sets. Each of these sets comprises all possible states that can be reached at a particular point in time starting from the initial condition and taking into account all possible realizations of the disturbance vector. The exact calculation of the reachable sets is a demanding task even for simple problems (if at all feasible), so it is a common practice to use approximations. Although not central to the analysis, obtaining reachable sets approximations and their visualization can provide useful information about where the system may end up if no discretionary action is taken. In a way, this technique can be thought of as the set-membership analogue of the fan chart in the stochastic setup, except that no probabilities are assigned to the different states.

In most applications, first a specific class of sets is chosen and the best approximations (external and internal) of the reachable sets are sought in the respective class. Here we use external ellipsoidal approximations which guarantees that the actual state lies within the approximating ellipsoid (see Appendix A.4 for technical details). Figure 3 shows the evolution of the reachable set approximations over the planning horizon (left panel), as well as approximation of the reachable set in the last period (right panel).²³

²³The plots are generated using the Matlab Ellipsoidal Toolbox by Alex Kurzanskiy.

Figure 3: Reach tube (left panel) and reachable set in final period

Based on the reachable set analysis, if no action is taken, the debt ratio could reach as high as 125 percent of potential GDP at the end of the planning horizon. By increasing or reducing the baseline primary balance, the policy maker can avoid such an undesirable outcome. Decisions about the amount and timing of the primary balance change can be guided by the model solution when discretionary policy is pursued.

The primary balance and state trajectories in an active policy scenario are shown in Figure 4. Public debt is higher and the output gap is larger in comparison with both the deterministic case and the IMF baseline, but this is to be expected given the pessimistic assumptions about disturbances and the relatively wide range of uncertainty. However, relative to the possible realizations under a "do nothing" strategy, the active policy scenario can bring some improvement. The optimal reaction of the primary balance according to the model, is to adjust slightly less in the first period, compared to the baseline, but much more aggressively thereafter. The model-based cumulative adjustment, as measured by the sum of primary balances throughout the projection period, is close to 15 percent of potential GDP, compared to about 10 percent under the IMF baseline and the deterministic scenarios. This is not surprising in view of the bounds for the debt shock; recall that under the worst case scenario shocks can add up to 3.4 percentage points to the debt ratio in each period during the first three years and up to 2.4 percentage points per year in the last three years.

Various experiments can be run within this framework. For example, one can examine how the optimal primary balance and the associated debt and output gap would change if the fiscal multiplier is larger, say 1.5 instead of 0.5. The dynamics of the variables in Figure 5 confirm the intuition that less adjustment

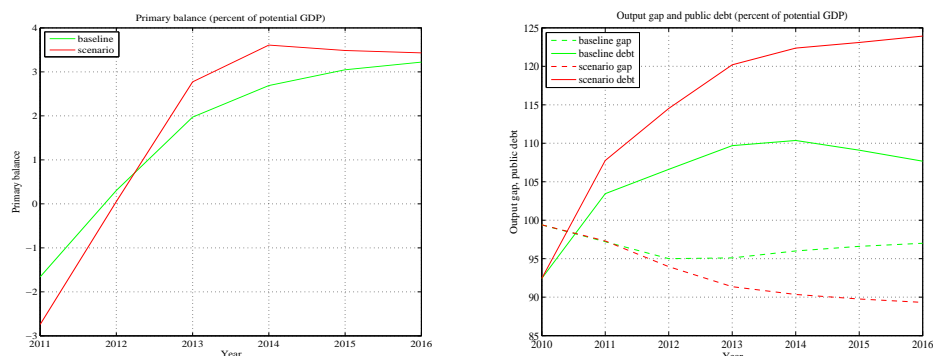
Figure 4: Primary balance, output gap and public debt under uncertainty (fiscal multiplier of 0.5; equal weights)



should be done when the fiscal multiplier is high, especially in periods when growth is low. Thus, the model suggests relaxation the primary balance in the first period by about 1 percentage point of potential GDP relative to the baseline and tightening after the third period when growth picks up, with a cumulative adjustment of about 10.5 percentage points over the six-year horizon. Under this scenario, however, debt remains on a rising path and approaches 125 percent of potential GDP in the final period. Such debt dynamics may not be acceptable, including due to concerns about financing. Therefore, the policy maker may attach a high value to quickly reducing debt and regaining markets' confidence. In the model, this preference for faster debt reduction can be accommodated by increasing the relative weight of debt in the objective function. An experiment where the weighting matrix Q has entries $1/3$ and $2/3$ in the main diagonal, yields the outcome shown in Figure 6. Not surprisingly, the recommended adjustment is larger and more front-loaded but this comes at the cost of widening the output gap relative to the case of equal weights.

To sum up, the simulation results presented above are in agreement with the conventional wisdom laid out in the introduction: when multipliers are high, a smaller and less front-loaded adjustment is warranted; however, if financing conditions are tight, more needs to be done in terms of reducing the deficit initially. What makes the formalized approach valuable in our view is its capability to produce a quantitative response which is optimal with respect to the given criterion. In the deterministic case, this is a relatively simple feedback rule and when uncertainty is involved the optimal reaction function can be computed numerically. Finally, we note that the above framework has the flexibility of accommodating a wide range of assumptions, including multipliers

Figure 5: Primary balance, output gap and public debt under uncertainty (fiscal multiplier of 1.5; equal weights)



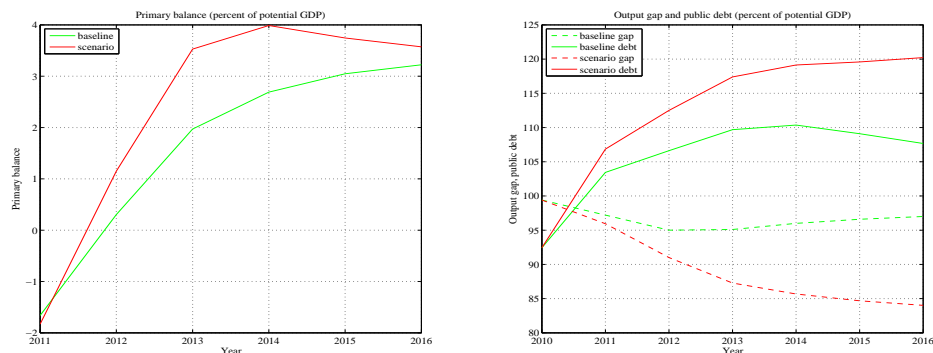
that vary with the business cycle, time-varying and correlated shocks, time-varying targets and weights, etc. The scenarios discussed above are purely illustrative and the user can experiment with different specifications that reflect more accurately the available information and his or her beliefs.

5 Conclusion and possible extensions

The paper provides an implementable solution to a linear-quadratic decision problem under general description of uncertainty when system disturbances are contained in ellipsoids. It applies the analytical framework to assess the fiscal consolidation program of Portugal for the period 2011-2016 based on uncertainty bounds derived from past data. Results suggest that if reducing debt and closing the output gap are given equal weights in the government's loss function, the optimal time profile of the primary balance implies a smaller adjustment in the initial year, when growth was negative, and larger adjustments thereafter. This is particularly the case when fiscal multipliers are big. A primary balance in the first period that is consistent with the baseline program would be optimal if, for example, the government attaches twice as much weight to reducing debt than to keeping actual GDP close to potential.

One natural extension of the analysis in this paper would be to incorporate non-linear dynamics in the model. For instance, the interest rate which the government pays on its new debt is not independent of the level of indebtedness. Indeed, the higher the stock of public debt, the larger the

Figure 6: Primary balance, output gap and public debt under uncertainty (fiscal multiplier of 1.5; weight on debt=2/3, weight on output gap=1/3)



premium that markets will require to continue to finance the borrower.²⁴ In the deterministic case, the solution of the corresponding non-linear model can be obtained using dynamic programming or a maximum principle for discrete time problems. Extending the non-linear model to incorporate uncertainty, however, poses considerable difficulties. One possible approach is to seek a more general form of the minimax principle which can be applied to a broader class of functions.²⁵ Alternatively, one can treat the control problem under uncertainty as a dynamic game, where the decision maker plays against an adversary (Nature) who tries to maximize the loss function. In order to be able to pursue this approach, an appropriate equilibrium concept needs to be specified. It would appear that a Stackelberg game, where Nature is the leader and the decision maker is the follower, would be the proper setup; however, obtaining feedback Stackelberg solutions is generally difficult.

Another direction for future work is to deal specifically with parametric uncertainty and uncertainty about the initial condition. In the model considered above, uncertainty is represented by additive terms which can be interpreted as shocks to the system state. However, from a practical point of view it is preferable to be able to model explicitly uncertainty in the coefficients— in this case bounds for the fiscal multipliers, interest rates and growth rates of actual and potential GDP. A related issue of interest pertains to imperfect

²⁴Empirical evidence, based on a model where the feedback of debt introduces non-linearity, can be found in Cherif and Hasanov (2012).

²⁵Boltyanskii and Cebotaru (1974) have proved such a maximum principle for both discrete and continuous time problems but in their set-up the uncertainty features in the criterion, not in the dynamics.

measurement. Potential GDP is not observable and can only be estimated with an error. Ideally, this feature should be incorporated in the analysis as well but again, it will make the problem technically much more challenging.

Finally, the setup in the paper is in finite time. This seems appropriate in view of the specific goal of bringing the debt ratio down over the medium term to some level deemed prudent. Assuming that the medium-term target is achieved, the question remains open as to what policies should be pursued in the long run to ensure that debt remains sustainable under small perturbations, i.e. the analysis needs to be extended to include stability considerations (infinite-time behavior).

Addressing these questions will make the model more realistic and will provide decision makers with a richer tool for policy evaluation and design.

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Appendix A

A.1 Solution of the deterministic linear-quadratic tracking problem

In the absence of uncertainty, the problem of minimizing the discounted deviations of the state and control vectors from given target levels under linear dynamic constraints is stated as follows:

$$\sum_{t=0}^{T-1} \frac{\beta^t}{2} [(x_t - \bar{x}_t)' Q_t (x_t - \bar{x}_t) + (u_t - \bar{u}_t)' R_t (u_t - \bar{u}_t)] + \frac{\beta^T}{2} (x_T - \bar{x}_T)' Q_T (x_T - \bar{x}_T) \rightarrow \min \quad (9)$$

s.t.

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t u_t + e_t. \\ x_0 &= \bar{x}_0, \text{ given.} \end{aligned} \quad (10)$$

Versions of this problem (known as the linear-quadratic tracking problem), without the discount factor, have been extensively used in the engineering literature. For this specific formulation, however, which involves discounting, time-varying coefficients, a free term in the dynamics equation and non-zero reference levels, the solution is not easy to find in the literature. For a variant of the model with constant matrices a solution is provided in Engwerda (1990). Chow (1986) considers a similar problem but without explicit controls. For convenience, the solution of (9)-(10) is derived below.

Proposition 1 *Let $x_t \in R^n$, $u_t \in R^m$ and let the matrices $Q_t \in R^{n \times n}$ and $R_t \in R^{m \times m}$ be symmetric positive definite. Given sequences of vectors $\{\bar{x}_t\}_{t=1}^T$, $\{\bar{u}_t\}_{t=0}^{T-1}$, $\{e_t\}_{t=0}^{T-1}$ and real matrices $\{A_t\}_{t=0}^{T-1}$, and $\{B_t\}_{t=0}^{T-1}$ of appropriate dimensions, the solution to problem (9)-(10) is given by the pair of sequences $(\{u_t^*\}, \{x_t^*\})$ with*

$$u_t^* = K_t (\beta^t R_t \bar{u}_t - B_t' (P_{t+1} A_t x_t^* + P_{t+1} e_t + h_{t+1})),$$

$$x_{t+1}^* = A_t x_t^* + B_t u_t^* + e_t, \quad x_0 = \bar{x}_0$$

where

$$K_t = (\beta^t R_t + B_t' P_{t+1} B_t)^{-1}$$

and the matrix P_{t+1} and the vector h_{t+1} satisfy the following equations:

$$P_t = \beta^t Q_t + A_t' P_{t+1} A_t - A_t' P_{t+1} B_t K_t B_t' P_{t+1} A_t$$

$$h_t = A_t' P_{t+1} B_t K_t (\beta^t R_t \bar{u}_t - B_t' (P_{t+1} e_t + h_{t+1})) + A_t' (P_{t+1} e_t + h_{t+1}) - \beta^t Q_t \bar{x}_t$$

for $t = 0, 1, \dots, T-1$ and

$$P_T = \beta^T Q_T$$

$$h_T = -\beta^T Q_T \bar{x}_T.$$

Proof. There are different ways to prove the above proposition. This is a convex finite-dimensional problem, so we can use the method of Lagrange. We write the Lagrangian function as:

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{T-1} \mu_0 \frac{\beta^t}{2} [(x_t - \bar{x}_t)' Q_t (x_t - \bar{x}_t) + (u_t - \bar{u}_t)' R_t (u_t - \bar{u}_t)] \\ &+ \mu_0 \frac{\beta^T}{2} (x_T - \bar{x}_T)' Q_T (x_T - \bar{x}_T) + \sum_{t=0}^{T-1} \lambda_{t+1}' (A_t x_t + B_t u_t + e_t - x_{t+1}). \end{aligned}$$

Let $(\{u_t^*\}, \{x_t^*\})$ be the point that delivers the minimum of (9) subject to (10). Then there exist numbers $\mu_0, \lambda_1, \dots, \lambda_T$, not all equal to zero, such that $\mu_0 \geq 0$ and

$$\mu_0 \beta^t R_t (u_t^* - \bar{u}_t) + B_t' \lambda_{t+1} = 0, \quad t = 0, \dots, T-1 \quad (11)$$

$$\mu_0 \beta^t Q_t (x_t^* - \bar{x}_t) + A_t' \lambda_{t+1} - \lambda_t = 0, \quad t = 0, \dots, T-1 \quad (12)$$

$$\mu_0 \beta^T Q_T (x_T^* - \bar{x}_T) - \lambda_T = 0. \quad (13)$$

(The above equations follow from setting the partial derivatives of the Lagrangian equal to zero.) Clearly, $\mu_0 \neq 0$ (otherwise it follows that $\lambda_t = 0$ for all t), so we can set $\mu_0 = 1$. Suppose that λ_t is of the form

$$\lambda_t = P_t x_t^* + h_t$$

for some matrix $P_t \in R^{n \times n}$ and vector $h_t \in R^n$. Thus, from equation (11) we obtain:

$$\begin{aligned} \beta^t R_t (u_t^* - \bar{u}_t) + B_t' \lambda_{t+1} &= \beta^t R_t u_t^* - \beta^t R_t \bar{u}_t + B_t' (P_{t+1} x_{t+1}^* + h_{t+1}) = \\ &= \beta^t R_t u_t^* - \beta^t R_t \bar{u}_t + B_t' (P_{t+1} (A_t x_t^* + B_t u_t^* + e_t) + h_{t+1}) \end{aligned}$$

or, after some algebra:

$$u_t^* = (\beta^t R_t + B_t' P_{t+1} B_t)^{-1} (\beta^t R_t \bar{u}_t - B_t' (P_{t+1} A_t x_t^* + P_{t+1} e_t + h_{t+1})).$$

We set $K_t = (\beta^t R_t + B_t' P_{t+1} B_t)^{-1}$, so

$$u_t^* = K_t(\beta^t R_t \bar{u}_t - B_t'(P_{t+1} A_t x_t^* + P_{t+1} f_t + h_{t+1})).^{26}$$

From (12) we have

$$\begin{aligned} \lambda_t &= \beta^t Q_t(x_t^* - \bar{x}_t) + A_t' \lambda_{t+1} \\ P_t x_t^* + h_t &= \beta^t Q_t(x_t^* - \bar{x}_t) + A_t'(P_{t+1} x_{t+1}^* + h_{t+1}) \\ P_t x_t^* + h_t &= \beta^t Q_t(x_t^* - \bar{x}_t) + A_t'(P_{t+1}(A_t x_t^* + B_t u_t^* + e_t) + h_{t+1}) \\ P_t x_t^* + h_t &= \beta^t Q_t(x_t^* - \bar{x}_t) + A_t' P_{t+1} A_t x_t^* + A_t' P_{t+1} B_t K_t (\beta^t R_t \bar{u}_t - \\ &\quad - B_t'(P_{t+1} A_t x_t^* + P_{t+1} e_t + h_{t+1})) \\ &\quad + A_t' P_{t+1} e_t + A_t' h_{t+1}. \end{aligned}$$

By grouping the terms before x_t^* we obtain the discrete-time Riccati equation for this problem:

$$P_t = \beta^t Q_t + A_t' P_{t+1} A_t - A_t' P_{t+1} B_t K_t B_t' P_{t+1} A_t.$$

Similarly, by grouping the free terms we get:

$$h_t = A_t' P_{t+1} B_t K_t (\beta^t R_t \bar{u}_t - B_t'(P_{t+1} e_t + h_{t+1})) + A_t' P_{t+1} e_t + A_t' h_{t+1} - \beta^t Q_t \bar{x}_t.$$

The above two equations hold for all $t = 0, \dots, T - 1$ and allow us to recursively compute the matrix P_t and the vector h_t provided that we know their values at the last point T . The right end conditions are obtained from (13):

$$\begin{aligned} P_T &= \beta^T Q_T \\ h_T &= -\beta^T Q_T \bar{x}_T. \end{aligned}$$

This completes the proof. ■

An example of a Matlab program that calculates the optimal controls and trajectories for this problem is presented in Appendix B.1.1.

²⁶The matrix $\beta^t R_t + B_t' P_{t+1} B_t$ is invertible since R_t is positive definite and P_{t+1} is also positive definite as can be seen from the recursive relationship below.

A.2 Dynamic programming algorithm and a minimax principle for uncertain systems

Considers the following linear discrete system:

$$x_{t+1} = A_t x_t + B_t u_t + G_t w_t \quad (14)$$

where $x_t \in R^n$, $u_t \in R^m$, $w_t \in \mathcal{W}_t \subset R^q$, and A_t , B_t and G_t are real matrices of appropriate dimensions. The goal is to find controls u_t^* , such that the functional

$$J(u_0, u_1, \dots, u_{T-1}) = \sup_{w_t \in \mathcal{W}_t} \sum_{t=1}^T [f_t(x_t) + g_{t-1}(u_{t-1})] \quad (15)$$

is minimized subject to the constraint given by (14), where the functions f_t, g_{t-1} are proper convex functions. Bertsekas (1971) proposes the following dynamic programming (DP) algorithm:

$$\begin{aligned} J_T(x_T) &= f_T(x_T) \\ E_{t+1}(x) &= \sup_{w_t \in \mathcal{W}_t} J_{t+1}(x + G_t w_t), \quad t = 0, 1, \dots, T-1 \\ H_t(x_t) &= \inf_{u_t} [E_{t+1}(A_t x_t + B_t u_t) + g_t(u_t)], \quad t = 0, 1, \dots, T-1 \\ J_t(x_t) &= H_t(x_t) + f_t(x_t), \quad t = 1, 2, \dots, T-1 \\ J_0(x_0) &= H_0(x_0). \end{aligned}$$

Below it is shown how this DP algorithm can be implemented to solve the optimal fiscal adjustment problem considered in this note. Although in our case the dynamical system is slightly different due to the free vector e_t , the algorithm still works. The problem we need to solve is:

$$J = \min_{u_t} \max_{w_t \in \mathcal{W}_t} \left\{ \sum_{t=0}^{T-1} \frac{\beta^t}{2} [(x_t - \bar{x}_t)' Q_t (x_t - \bar{x}_t) + (u_t - \bar{u}_t)' R_t (u_t - \bar{u}_t)] + \frac{\beta^T}{2} (x_T - \bar{x}_T)' Q_T (x_T - \bar{x}_T) \right\} \quad (16)$$

subject to

$$x_{t+1} = A_t x_t + B_t u_t + G_t w_t + e_t \quad (17)$$

$$x_0 = \bar{x}_0, \quad (18)$$

where the set \mathcal{W}_t is the ellipsoid $\mathcal{W}_t = \{w_t : (w_t - \bar{w}_t)'W_t^{-1}(w_t - \bar{w}_t) \leq 1\}$. The DP algorithm can be implemented as follows:

$$\begin{aligned}
J_T(x_T) &= \frac{\beta^T}{2}(x_T - \bar{x}_T)'Q_T(x_T - \bar{x}_T) \\
E_{t+1}(x) &= \sup_{w_t \in \mathcal{W}_t} J_{t+1}(x + G_t w_t), \quad t = 0, 1, \dots, T-1 \\
H_t(x_t) &= \inf_{u_t} [E_{t+1}(A_t x_t + B_t u_t + e_t) + \frac{\beta^t}{2}(u_t - \bar{u}_t)'R_t(u_t - \bar{u}_t)], \quad t = 0, 1, \dots, T-1 \\
J_t(x_t) &= H_t(x_t) + \frac{\beta^t}{2}(x_t - \bar{x}_t)'Q_t(x_t - \bar{x}_t), \quad t = 1, 2, \dots, T-1 \\
J_0(x_0) &= H_0(x_0).
\end{aligned}$$

It will be instructive to illustrate how the above DP algorithm works for a simpler problem. For the purpose, suppose that $\bar{x}_t = 0, \bar{w}_t = 0, \bar{u}_t = 0, \bar{e}_t = 0$. Then, the first step of the algorithm is:

$$\begin{aligned}
E_T(x) &= \max_{w_{T-1} \in \mathcal{W}_{T-1}} J_T(x + G_{T-1} w_{T-1}) = \\
&= \max_{w_{T-1} \in \mathcal{W}_{T-1}} \frac{\beta^T}{2}(x + G_{T-1} w_{T-1})'Q_T(x + G_{T-1} w_{T-1})
\end{aligned}$$

which is equivalent to the following problem:

$$\max_{w_{T-1}} \frac{\beta^T}{2} [(x + G_{T-1} w_{T-1})'Q_T(x + G_{T-1} w_{T-1})]$$

s.t.

$$w_{T-1}' W_{T-1}^{-1} w_{T-1} \leq 1.$$

We set a Lagrangian

$$\mathcal{L} = \frac{\beta^T}{2} [(x + G_{T-1} w_{T-1})'Q_T(x + G_{T-1} w_{T-1})] + \lambda_{T-1} (1 - w_{T-1}' W_{T-1}^{-1} w_{T-1})$$

and find that the necessary conditions for this problem are:

$$\begin{aligned}
\beta^T G_{T-1}' Q_T (G_{T-1} w_{T-1}^* + x) &= 2\lambda_{T-1} W_{T-1}^{-1} w_{T-1}^*, \\
\lambda_{T-1} \geq 0, \quad \lambda_{T-1} (1 - w_{T-1}^{*'} W_{T-1}^{-1} w_{T-1}^*) &= 0.
\end{aligned}$$

and therefore

$$w_{T-1}^* = \left(\frac{2\lambda_{T-1}}{\beta^T} (G'_{T-1} Q_T)^{-1} W_{T-1}^{-1} - G_{T-1} \right)^{-1} x.$$

We can find λ_{T-1} from the last necessary condition:

$$\begin{aligned} x' \left(\left(\frac{2\lambda_{T-1}}{\beta^T} (G'_{T-1} Q_T)^{-1} W_{T-1}^{-1} - G_{T-1} \right)^{-1} \right)' W_{T-1}^{-1} \\ \left(\frac{2\lambda_{T-1}}{\beta^T} (G'_{T-1} Q_T)^{-1} W_{T-1}^{-1} - G_{T-1} \right)^{-1} x = 1 \end{aligned} \quad (19)$$

This is for any admissible x . Note, that the above equation may have more than one non-negative solution. We now introduce the auxiliary matrices S_T and M_{T-1} and set $S_T := Q_T$ and

$$M_{T-1} := \left(\frac{2\lambda_{T-1}}{\beta^T} (G'_{T-1} S_T)^{-1} W_{T-1}^{-1} - G_{T-1} \right)^{-1}.$$

Then,

$$w_{T-1}^* = M_{T-1} x$$

and

$$E_T(x) = \frac{\beta^T}{2} [(I + G_{T-1} M_{T-1}) x]' Q_T (I + G_{T-1} M_{T-1}) x].$$

We set $N_{T-1} = (I + G_{T-1} M_{T-1})' Q_T (I + G_{T-1} M_{T-1})$, so

$$E_T(x) = \frac{\beta^T}{2} x' N_{T-1} x.$$

Note that N_{T-1} is symmetric since Q_T is symmetric. For the next step, the DP algorithm requires that $x = A_{T-1} x_{T-1} + B_{T-1} u_{T-1}$. So, substituting in the above we obtain

$$E_T(A_{T-1} x_{T-1} + B_{T-1} u_{T-1}) = \frac{\beta^T}{2} (A_{T-1} x_{T-1} + B_{T-1} u_{T-1})' N_{T-1} (A_{T-1} x_{T-1} + B_{T-1} u_{T-1}).$$

To find u_{T-1}^* we need to minimize:

$$E_T(A_{T-1} x_{T-1} + B_{T-1} u_{T-1}) + \frac{\beta}{2} u_{T-1}' R_{T-1} u_{T-1} =$$

$$\frac{\beta^T}{2}(A_{T-1}x_{T-1} + B_{T-1}u_{T-1})'N_{T-1}(A_{T-1}x_{T-1} + B_{T-1}u_{T-1}) + \frac{\beta^{T-1}}{2}u_{T-1}'R_{T-1}u_{T-1}$$

The necessary (and sufficient) condition for a minimum is

$$\beta B_{T-1}'N_{T-1}(A_{T-1}x_{T-1} + B_{T-1}u_{T-1}^*) + R_{T-1}u_{T-1}^* = 0.$$

$$u_{T-1}^* = -(\beta B_{T-1}'N_{T-1}B_{T-1} + R_{T-1})^{-1}\beta B_{T-1}'N_{T-1}A_{T-1}x_{T-1}$$

or

$$u_{T-1}^* = -K_{T-1}x_{T-1}$$

where we have set

$$K_{T-1} := (\beta B_{T-1}'N_{T-1}B_{T-1} + R_{T-1})^{-1}\beta B_{T-1}'N_{T-1}A_{T-1}.$$

It will be convenient to introduce another auxiliary matrix:

$$P_{T-1} = (A_{T-1} - B_{T-1}K_{T-1})'N_{T-1}(A_{T-1} - B_{T-1}K_{T-1}).$$

Then,

$$H_{T-1}(x_{T-1}) = \frac{\beta^T}{2}x_{T-1}'P_{T-1}x_{T-1} + \frac{\beta^{T-1}}{2}x_{T-1}'K_{T-1}'R_{T-1}K_{T-1}x_{T-1}.$$

We now move to the next time period. We have

$$\begin{aligned} J_{T-1}(x_{T-1}) &= H_{T-1}(x_{T-1}) + \frac{\beta^{T-1}}{2}x_{T-1}'Q_{T-1}x_{T-1} = \\ &= \frac{\beta^{T-1}}{2}x_{T-1}'[\beta P_{T-1} + K_{T-1}'R_{T-1}K_{T-1} + Q_{T-1}]x_{T-1}. \end{aligned}$$

Let

$$S_{T-1} = \beta P_{T-1} + K_{T-1}'R_{T-1}K_{T-1} + Q_{T-1}.$$

Then,

$$J_{T-1}(x_{T-1}) = \frac{\beta^{T-1}}{2}x_{T-1}'S_{T-1}x_{T-1}.$$

and

$$E_{T-1}(x) = \max_{w_{T-1} \in \mathcal{W}_{T-2}} J_{T-1}(x + G_{T-2}w_{T-2}) =$$

$$= \max_{w_{T-2} \in \mathcal{W}_{T-2}} \frac{\beta^{T-1}}{2} (x + G_{T-2}w_{T-2})' S_{T-1} (x + G_{T-2}w_{T-2}).$$

The necessary conditions for this problem are:

$$\begin{aligned} \beta^{T-1} G'_{T-2} S_{T-1} (x + G_{T-2}w_{T-2}) &= 2\lambda_{T-2} W_{T-2}^{-1} w_{T-2}^*, \\ \lambda_{T-2} &\geq 0, \quad \lambda_{T-2} (1 - w_{T-2}^{*'} W_{T-2}^{-1} w_{T-2}^*) = 0. \end{aligned}$$

The first condition can be rewritten as

$$\begin{aligned} w_{T-2}^* &= \left(\frac{2\lambda_{T-2}}{\beta^{T-1}} (G'_{T-2} S_{T-1})^{-1} W_{T-2}^{-1} - G_{T-2} \right)^{-1} x \\ &= M_{T-2} x, \end{aligned}$$

where as before, we set

$$M_{T-2} = \left(\frac{2\lambda_{T-2}}{\beta^{T-1}} (G'_{T-2} S_{T-1})^{-1} W_{T-2}^{-1} - G_{T-2} \right)^{-1}.$$

With that,

$$\begin{aligned} E_{T-1}(x) &= \frac{\beta^{T-1}}{2} x' (I + G_{T-2} M_{T-2})' S_{T-1} (I + G_{T-2} M_{T-2}) x = \\ &= \frac{\beta^{T-1}}{2} x' N_{T-2} x \end{aligned}$$

where we have set

$$N_{T-2} := (I + G_{T-2} M_{T-2})' S_{T-1} (I + G_{T-2} M_{T-2}).$$

We need to minimize with respect to u_{T-2} :

$$\begin{aligned} E_{T-1}(A_{T-2}x_{T-2} + B_{T-2}u_{T-2}) &+ \frac{\beta^{T-2}}{2} u'_{T-2} R_{T-2} u_{T-2} = \\ \frac{\beta^{T-1}}{2} (A_{T-2}x_{T-2} + B_{T-2}u_{T-2})' &N_{T-2} (A_{T-2}x_{T-2} + B_{T-2}u_{T-2}) + \\ \frac{\beta^{T-2}}{2} u'_{T-2} R_{T-2} u_{T-2}. & \end{aligned}$$

The above is minimized when u_{T-2}^* satisfies:

$$\beta B'_{T-2} N_{T-2} (A_{T-2}x_{T-2} + B_{T-2}u_{T-2}^*) + R_{T-2} u_{T-2}^* = 0.$$

Hence

$$\begin{aligned} u_{T-2}^* &= -[\beta B'_{T-2} N_{T-2} B_{T-2} + R_{T-2}]^{-1} \beta B'_{T-2} N_{T-2} A_{T-2} x_{T-2} = \\ &= -K_{T-2} x_{T-2}, \end{aligned}$$

where

$$K_{T-2} = [\beta B'_{T-2} N_{T-2} B_{T-2} + R_{T-2}]^{-1} \beta B'_{T-2} N_{T-2} A_{T-2},$$

and so on.

To summarize, the implementation of the DP algorithm leads to a sequence of controls and disturbances pairs (u_t^*, w_t^*) computed as

$$\begin{aligned} u_t^* &= -K_t x_t \\ w_t^* &= M_t (A_t x_t + B_t u_t^*), \end{aligned}$$

where the matrices M_t and K_t and the auxiliary matrices N_t , P_t and S_t are obtained as follows:

$$S_T = Q_T$$

and for $t = T - 1, T - 2, \dots, 0$

$$\begin{aligned} M_t &= \left(\frac{2\lambda_t}{\beta^{t+1}} (G'_t S_{t+1})^{-1} W_t^{-1} - G_t \right)^{-1} \\ N_t &= (I + G_t M_t)' S_{t+1} (I + G_t M_t) \\ K_t &= (\beta B'_t N_t B_t + R_t)^{-1} \beta B'_t N_t A_t \\ P_t &= (A_t - B_t K_t)' N_t (A_t - B_t K_t) \\ S_t &= \beta P_t + K'_t R_t K_t + Q_t. \end{aligned}$$

For a given λ_t , all the above matrices can be precomputed and this would allow us to calculate u_t^* and w_t^* starting from x_0 . However, the Lagrange multipliers λ_t , associated with the ellipsoidal constraints, are not known in advance and furthermore, they depend on x through equation (19). This makes it very difficult in practice, if at all possible, to solve the problem based on the above scheme. The DP algorithm, however, is still useful as it provides an explicit characterization of the value function $J_t(x_t)$ —information that will be used later.

In view of the technical difficulties in implementing the DP algorithm, an alternative approach is sought to solve the LQ minimax problem. In fact, for the more general problem (14)-(15), Bertsekas (1971) provides a minimax

maximum principle which holds under certain assumptions. Before stating the main result, it would be useful to introduce some terminology.

A sequence of controls and disturbances $\{u_0^*, w_0^*, \dots, u_{T-1}^*, w_{T-1}^*\}$ associated with problem (14)-(15) will be called a *minimax sequence* and the associated trajectory $\{\bar{x}_0, x_1^*, \dots, x_T^*\}$ will be called a *minimax trajectory* if:

$$H_t(x_t^*) = E_{t+1}(A_t x_t^* + B_t u_t^*) + g_t(u_t^*) = \inf_{u_t} [E_{t+1}(A_t x_t^* + B_t u_t) + g_t(u_t)]$$

$$E_{t+1}(A_t x_t^* + B_t u_t^*) + g_t(u_t^*) = J_{t+1}(x_{t+1}^*) = \sup_{w_t \in \mathcal{W}_t} J_{t+1}(A_t x_t^* + B_t u_t^* + G_t w_t).$$

For a fixed t , $t = 1, \dots, T$, a point x is called *non-singular* if for every vector w_t^* for which the supremum of (15) is attained, we have

$\partial E_{t+1}(x) = \partial J_{t+1}(x + G_t w_t^*)$, where the symbol $\partial f(x)$ denotes the subdifferential (in the sense of convex analysis) of the function $f(x)$. Recall that a vector $\psi \in R^n$ is called a subgradient of the convex function $f(x)$ at the point $x \in R^n$ if the inequality $f(y) - f(x) \geq \langle \psi, y - x \rangle$ holds for all $y \in R^n$. The set of all subgradients is called the subdifferential of f at x and it is a closed convex set. If the function is differentiable, its subdifferential consists of a single element – the gradient. For the minimax problem, however, we cannot hope in general to be able to work with differentiable functions due to the maximization condition in (15) which explains the use of $\partial f(x)$. Finally, the initial point x_0 is called *regular* if $\partial J_0(x_0)$ is non-empty.

Proposition 2 (*Bertsekas (1971), p. 41*) *Let $\{u_0^*, w_0^*, \dots, u_{T-1}^*, w_{T-1}^*\}$ be a minimax sequence and let $\{\bar{x}_0, x_1^*, \dots, x_T^*\}$ be the corresponding minimax trajectory. Assume that the initial state \bar{x}_0 is regular and that the points $(A_t x_t^* + B_t u_t^*)$, $t = 0, \dots, T - 1$ are non-singular. Then, there exist vectors p_1, p_2, \dots, p_T and q_1, q_2, \dots, q_{T-1} satisfying the adjoint equation*

$$p_t = A_t' p_{t+1} + q_t$$

with

$$\begin{aligned} p_T &\in \partial f_T(x_T) \\ q_t &\in \partial f_t(x_t), \quad t = 1, \dots, T - 1 \end{aligned}$$

and such that

$$\langle p_{t+1}, B_t u_t^* + G_t w_t^* \rangle + g_t(u_t^*) = \min_{u_t} \max_{w_t \in \mathcal{W}_t} [\langle p_{t+1}, B_t u_t + G_t w_t \rangle + g_t(u_t)] \quad (20)$$

for $t = 0, 1, \dots, T - 1$.

The proof of the above proposition relies heavily on notions and results from convex analysis and will not be presented here. It would be useful though, to highlight the main ideas behind it as this would help clarify the conditions under which the minimax principle is valid.

If w_t^* maximizes $J_{t+1}(x + G_t w_t)$, then for any vector $p'_{t+1} \in \partial J_{t+1}$, we have that $\langle p'_{t+1}, G_t w_t^* \rangle = \max_w \langle p'_{t+1}, G_t w_t \rangle$. Similarly, it can be shown that if u_t^* delivers the minimum of $[E_{t+1} + g_t(u_t)]$, then for $p''_{t+1} \in \partial H_t$, we have that $\langle p''_{t+1}, B_t u_t^* \rangle + g_t(u_t^*) = \min_u [\langle p''_{t+1}, B_t u_t \rangle + g_t(u_t)]$. Thus, if there exists a vector $p_{t+1} \in \partial H_t(A_t x_t^*) \cap \partial J_{t+1}(A_t x_t^* + B_t u_t^* + G_t w_t^*)$, then the minimax condition in equation (20) will hold. In other words, for the validity of Proposition 2 it is necessary that the intersection of the two subdifferentials is non-empty. Further, the following inclusion and equality can be established:
 $\partial H_t(A_t x_t^*) \subset \partial E_{t+1}(A_t x_t^* + B_t u_t^*)$ and
 $\partial E_{t+1}(A_t x_t^* + B_t u_t^*) = \partial J_{t+1}(A_t x_t^* + B_t u_t^* + G_t w_t^*)$. It follows that for every point at which the function E_{t+1} is differentiable, the intersection condition will be satisfied. Since we obtained explicit expressions for the functions E_{t+1} and H_t as part of the DP algorithm and these functions are differentiable, we can apply Proposition 2. This implies the existence of vectors p_t and q_t as above.

A.3 The shooting method

Application of the minimax principle to the optimal fiscal adjustment problem leads to the following system of equations:

$$\begin{aligned} w_t^* &= \bar{w}_t + \frac{1}{2\lambda} W_t G_t' p_{t+1} \\ u_t^* &= \bar{u}_t - \beta R_t^{-1} B_t' p_{t+1} \\ p_t &= A_t' p_{t+1} + \beta^t Q_t (x_t^* - \bar{x}_t) \\ p_T &= \beta^T Q_T (x_T^* - \bar{x}_T) \\ x_{t+1}^* &= A_t x_t^* + B_t u_t^* + e_t + G_t w_t^* \\ x_0 &= \bar{x}_0. \end{aligned}$$

Substituting u_t^* and w_t^* into the equation for x_{t+1}^* and then using the equation for p results in the following system:

$$\begin{aligned} x_{t+1}^* &= A_t x_t^* + B_t (\bar{u}_t - \beta R_t^{-1} B_t' p_{t+1}) + e_t + G_t (\bar{w}_t + \frac{1}{2\lambda} W_t G_t' p_{t+1}) \\ p_t &= A_t' p_{t+1} + \beta^t Q_t (x_t^* - \bar{x}_t) \end{aligned}$$

$$p_T = \beta^T Q_T(x_T^* - \bar{x}_T)$$

$$x_0 = \bar{x}_0.$$

where $\lambda = 0.5((W_t G_t' p_{t+1})' G_t' p_{t+1})^{1/2}$. This is a boundary value problem with an initial condition for the state x and a transversality condition at the right end for the adjoint variable p .²⁷ To obtain a solution to the boundary value problem we can use the so-called "shooting method".

The essence of the shooting method is to guess an initial value for p and then solve the system for $t = 1, \dots, T$. Once x_T is found, the transversality condition is applied to obtain p_T . This value is compared to the value of p_T obtained by solving the adjoint equation with the guess for p_0 . If the two are different, the guess for p_0 is updated. In our implementation of the shooting method, the update is based on Broyden's secant formula.

Broyden's method is a quasi-Newton method which uses an approximation of the Jacobian $F'(x^*)$ of the system of equations $F(x) = 0$ at the solution x^* . The presentation below follows Kelly (1995).

If x_c and B_c are the current approximations of the solution and the Jacobian, respectively, then the next step of the algorithm is given by:

$$x_+ = x_c - B_c^{-1} F(x_c)$$

After x_+ is computed, the matrix B_c is updated as follows:

$$B_+ = B_c = \frac{(y - B_c s) s'}{s' s}$$

where $y = F(x_+) - F(x_c)$ and $s = x_+ - x_c$. The matrix B can be initialized with the identity matrix.

An example of a Matlab code that implements this procedure is given in Appendix B.1.2.

A.4 Reachable sets and their ellipsoidal approximations

Below we show how to calculate ellipsoidal approximations of reachable sets of discrete time dynamical systems. The presentation is based on Chernousko (1988).

²⁷The system of difference equation as stated above is implicit. In principle, it is possible to solve the second equation for p_{t+1} to derive an explicit system.

A.4.1 Ellipsoids

Ellipsoids are analogues of the ellipse in higher dimensions. An ellipsoid in R^n is defined as

$$E(a, Q) = \{x : (x - a)'Q^{-1}(x - a) \leq 1\},$$

where the n -dimensional vector a is the center of the ellipsoid and the positive definite matrix Q is its shape matrix. By rotating the coordinate system we can make its axes coincide with the principal axes of the ellipsoid in which case the matrix Q is diagonal. Then the defining inequality becomes:

$$\sum_{i=1}^n Q_{ii}^{-1}(x_i - a_i)^2 = \sum_{i=1}^n \frac{(x_i - a_i)^2}{c_i^2} \leq 1.$$

In some cases ellipsoids can be degenerate, e.g. if in the diagonal form one of the elements $c_i = 0$ (e.g. a disk in R^3).

In R^2 , the canonical equation of the ellipse centered at zero can be written as

$$\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_2^2} \leq 1$$

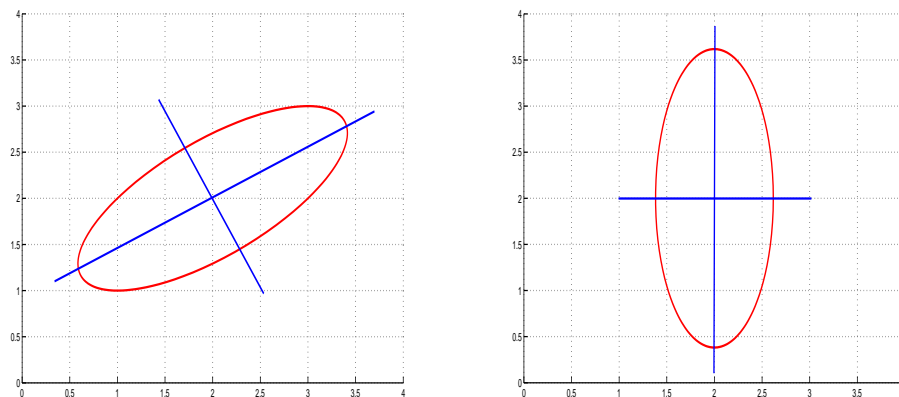
The coordinates (x_1, x_2) of any point of the ellipse satisfy: $|x_1| \leq c_1$ and $|x_2| \leq c_2$. These inequalities determine a rectangle with sides $2c_1$ and $2c_2$ and the ellipse is fully contained in this rectangle. This is a useful property which carries over to higher dimensions with the rectangle being replaced with a box.

Ellipsoids are invariant under affine transformations. Consider the mapping $y = Ax + b$, where $x \in E(a, Q)$. Then, $y \in E(Aa + b, AQA')$, i.e. an ellipsoid with center $Aa + b$ and shape matrix AQA' . The transformation can be chosen such that the ellipsoid is aligned with the coordinate axes, which, as noted above, implies that the matrix AQA' is diagonal (Figure 7). In particular, the ellipsoid $E(a, Q)$ can be transformed into the unit ball centered at the origin by selecting the matrix A and the vector b such that $Aa + b = 0$ and $AQA' = I$. These conditions will be satisfied if $A = Q^{1/2}$ and $b = -Q^{1/2}a$.

The volume of the unit ball in R^n is calculated according to the formula:

$$V_B = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}$$

where $\Gamma(\cdot)$ is the gamma function. Since we can transform an arbitrary ellipsoid into the unit ball, the volume of the original ellipsoid $E(a, Q)$ can be found by

Figure 7: Ellipsoids in R^2 

multiplying the volume of the unit ball by the determinant of the transformation matrix, i.e.

$$V_E = \frac{\pi^{n/2}(\det Q)^{1/2}}{\Gamma(n/2 + 1)}.$$

In general, any convex set D in R^n can be approximated internally and externally by an ellipsoid, i.e. there are ellipsoids $E(a, Q)$ and $E(a, n^2Q)$, such that $E(a, Q) \subset D \subset E(a, n^2Q)$.²⁸ For every bounded set D in R^n there exists a unique ellipsoid E^+ of minimum volume that contains D . Also, for every compact convex set D in R^n there exists a unique ellipsoid E^- of maximum volume that is contained in D .

Finally, we note that the sum of two ellipsoids $S = E_1(a_1, Q_1) + E_2(a_2, Q_2)$ is defined as the set of all vectors $x = x_1 + x_2$, where $x_1 \in E_1$ and $x_2 \in E_2$. The set S is a closed convex set but generally it is not an ellipsoid. However, it can be approximated by an ellipsoid. For the external approximation $E^+(a^+, Q^+)$, the following result is valid :

Theorem 1 (Chernousko, 1988) *The parameters of the ellipsoid $E^+(a^+, Q^+)$ of minimal volume that contains the sum S of the ellipsoids E_1 and E_2 , one of which may be degenerate (the matrix Q_1 is positive semi-definite and Q_2 is*

²⁸For convex sets, possessing the central symmetry property, the approximation can be improved substantially: $E(0, Q) \subset D \subset E(0, nQ)$.

positive definite) are given by

$$a^+ = a_1 + a_2$$

$$Q^+ = (p^{-1} + 1)Q_1 + (p + 1)Q_2,$$

where $p > 0$ is the only positive root of the algebraic equation

$$\sum_{j=1}^n \frac{1}{p + \lambda_j} = \frac{n}{p(p + 1)}$$

and the numbers $\lambda_j \geq 0$, $j = 1, \dots, n$ are the roots of the characteristic equation $\det(Q_1 - \lambda Q_2) = 0$.

A.4.2 Reachable sets and their approximations

Consider the discrete dynamical system

$$x(t_{i+1}) = F(x(t_i), w(t_i), t_i)$$

$$t_0 < t_1, \dots, i = 0, 1, \dots$$

$$w(t_i) \in \mathcal{W}(x(t_i), t_i)$$

$$x(t_0) \in M.$$

The reachable set $D(t_k, t_0, M)$ of the above system for all $k \geq 0$ is the set of points $x(t_k)$ that represent ends of all state trajectories $x(\cdot)$ of this system satisfying the initial condition. Reachable sets possess the semi-group property

$$D(t_k, t_0, M) = D(t_k, t_j, D(t_j, t_0, M)), \quad 0 \leq j \leq k.$$

This means that the reachable set at moment t_k can be obtained by extending the ends of all trajectories taken at any moment t_j , where $0 \leq j \leq k$. In the process of constructing approximations of reachable sets we will need the notions of sub- and super-reachability. We shall call the family of sets $D^-(t_k)$, subreachable, if for all k the following inclusion is satisfied:

$D^-(t_{k+1}) \subset D(t_{k+1}, t_k, D^-(t_k))$ and $D^-(t_0) \subset M$. Similarly, the family of sets $D^+(t_k)$, will be called superreachable if for all k : $D^+(t_{t+1}) \supset D(t_{k+1}, t_k, D^+(t_k))$ and $D^+(s) \supset M$. These two types of sets give a two-sided approximation of the reachable set:

$$D^-(t_k) \subset D(t_k, t_0, M) \subset D^+(t_k).$$

Obtaining two-sided ellipsoidal approximations of the reachable set entails finding the parameters of the ellipsoids:

$$D^-(t_k) = E(a^-(t_k), Q^-(t_k))$$

$$D^+(t_k) = E(a^+(t_k), Q^+(t_k)).$$

In this paper we are primarily concerned with external approximations, i.e. in finding the ellipsoid D^+ which contains the set of all states of the system that can be reached at a given moment in time. Specifically, the following linear system is of interest:

$$\begin{aligned} x(t_{k+1}) &= A_k x(t_k) + G_k w(t_k) + e_k \\ w(t_k) &\in E(0, W_k), \quad k = 0, 1, \dots \end{aligned}$$

Here A_k are non-degenerate matrices of dimension $n \times n$, G_k are matrices with dimension $m \times n$, e_k are n -dimensional vectors and W_k are symmetric positive semi-definite matrices of dimension $m \times m$. All of these are assumed given. The fact that the center of the control ellipsoid is at the origin is not restrictive since we can always change it by changing the vector e_k . Let the initial set be given by

$$x(t_0) \in M = E(a_0, Q_0).$$

For the linear system, the relation describing the evolution of the reachable set has the following form:

$$D(t_{k+1}, t_0, M) = A_k D(t_k, t_0, M) + G_k E(0, W_k) + e_k,$$

which is a sum of two ellipsoids. From the definition of superreachable sets:

$$E(a^+(t_{k+1}), Q^+(t_{k+1})) \supset A_k E(a^+(t_k), Q^+(t_k)) + G_k E(0, W_k) + e_k.$$

The center and shape matrix of the approximating ellipsoid are found as follows (see pp. 115-116 in Chernousko, 1988)

$$a^+(t_{k+1}) = A_k a^+(t_k) + e_k$$

$$Q^+(t_{k+1}) = (p^{-1} + 1)Q_1 + (p + 1)Q_2^+$$

where

$$Q_1 = G_k W_k G_k'$$

$$Q_2^+ = A_k Q^+(t_k) A_k'$$

$$a(t_0) = a_0.$$

The ellipsoidal approximations of reachable sets above are discussed in the context of a disturbed system to match the presentation in the main text; the same arguments apply for a controlled system.

Appendix B

B.1 Examples of Matlab programs

B.1.1 Deterministic LQ problem

```

function [ x u ] = lq( n, m, T, A, B, e, Q, R, x0, ubar, xbar, bet )
%.....
%This program solves the linear-quadratic optimal tracking problem:
% min_{u_t} \sum_{t=0}^{T-1} 0.5\beta^t[(x_t - \bar{x}_t)'Q_t(x_t - \bar{x}_t)+
%+(u_t - \bar{u}_t)'R_t(u_t - \bar{u}_t)]+0.5\beta^T(x_T - \bar{x}_T)'Q_T
%(x_T - \bar{x}_T), subject to x_{t+1}=A_tx_t+B_tu_t+e_t.
%Here A_t, B_t are arrays of matrices of size (n, n, T) and
%(n, m, T), respectively. Q_t and R_t are arrays of weight
%matrices of size (n, n, T) and (n, m, T), respectively; \bar{u}_t and
%\bar{x}_t are target values for the state and control, respectively.
%n is the size of the state vector, m is the size of the control vector
%and T is the number of periods. x_0 is the initial state (given).
%.....

P=zeros(n,n,T);
P(:,:,T)=bet^(T)*Q(:,:,T);
h=zeros(n,1,T);
h(:,:,T)=-bet^(T)*Q(:,:,T)*xbar(:,:,T);
K=zeros(m,m,T-1);

%Solves the Ricatti equation and calculates various matrices and vectors
t=T-1;
while t>0
    %Auxiliary matrix
    K(:,:,t)=(bet^(t-1)*R(:,:,t)+B(:,:,t)'*P(:,:,t+1)*B(:,:,t))^(-1);
    %Ricatti equation
    P(:,:,t)=bet^(t-1)*Q(:,:,t)+A(:,:,t)'*P(:,:,t+1)*A(:,:,t)-A(:,:,t)'*...
    P(:,:,t+1)*B(:,:,t)*K(:,:,t)*B(:,:,t)'*P(:,:,t+1)*A(:,:,t);
    %Free term
    h(:,:,t)=A(:,:,t)'*P(:,:,t+1)*B(:,:,t)*K(:,:,t)*(bet^(t-1)*R(:,:,t)*...
    ubar(:,:,t) - B(:,:,t)'*(P(:,:,t+1)*e(:,:,t)+h(:,:,t+1))+A(:,:,t)'*...
    (P(:,:,t+1)*e(:,:,t)+h(:,:,t+1))-bet^(t-1)*Q(:,:,t)*xbar(:,:,t);
    t=t-1;
end

%Calculate the optimal control and trajectory
u=zeros(1,T-1);
x=zeros(n,1,T);
x(:,:,1)=x0;

for j=1:T-1;
    u(j)=-K(:,:,j)*B(:,:,j)'*P(:,:,j+1)*A(:,:,j)*x(:,j)+...
    K(:,:,j)*(bet^(j-1)*R(:,:,j)*ubar(:,:,j)-B(:,:,j)'*...
    (P(:,:,j+1)*e(:,:,j)+h(:,:,j+1)));

    x(:,:,j+1)=A(:,:,j)*x(:,j)+B(:,:,j)*u(j)+e(:,:,j);
    j=j+1;
end
end

```

B.1.2 Minimax LQ problem

```

function [ x w u it crt] = berts41( n,m, T, XT, x0, A0, W, A,B, G, ...
e, Q, R, xbar, wbar, ubar, bet, tol, stop )

%.....
% This function computes the solution of the linear-quadratic minimax
% problem:
% \min_{u} \max_{w} \sum_{t=0}^{T-1} 0.5 \beta^t [(x_t - \bar{x}_t)'
% Q_t (x_t - \bar{x}_t) + (u_t - \bar{u}_t)' R_t (u_t - \bar{u}_t)]
% + (x_T - \bar{x}_T)' Q_T (x_T - \bar{x}_T), s.t.
% x_{t+1} = A_t x_t + B_t u_t + G_t w_t + e_t, where the disturbance w_t is contained
% in the ellipsoid \{ \mathcal{W} \}_t = \{ w_t : (w_t - \bar{w}_t)' W_t^{-1} (w_t - \bar{w}_t) \le 1 \}.
% Here A_t, B_t are arrays of matrices of size (n, n, T) and (n, m, T),
% respectively. Q_t and R_t are arrays of weight matrices of size
% (n, n, T) and (n, m, T), respectively; \bar{u}_t and
% \bar{x}_t are target values for the state and control, respectively.
% n is the size of the state vector, m is the size of the control vector
% and T is the number of periods. x_0 is the initial state (given).
%.....

% Initialize matrices
Xk=zeros(n,1,T);
pk=zeros(n,1,T);
Xk1=zeros(n,1,T);
pk1=zeros(n,1,T);
uk=zeros(m,1,T-1);
wk=zeros(n,1,T-1);
uk1=zeros(m,1,T-1);
wk1=zeros(n,1,T-1);

it=0; %counter -- set to zero

Ak=A0;

Xk(:,:,T)=XT; %End value for the state
pk(:,:,T)=(bet^T)*Q(:,:,T)*(Xk(:,:,T)-xbar(:,:,T)); %Transversality cond.

%The block below implements the shooting method based on Broyden's secant
%algorithm

while stop>tol;

it=it+1;%Counts the number of iterations

%Calculate (x,p)
for t=T-1:-1:1;
lam=(1/2)*((W(:,:,t)*G(:,:,t)'*pk(:,:,t+1))*G(:,:,t)'*pk(:,:,t+1))^(1/2);
%Lagrange multiplier for the ellipsoidal constraint
wk(:,:,t)=wbar(:,:,t)+1/(2*lam)*W(:,:,t)*G(:,:,t)'*pk(:,:,t+1);
uk(:,:,t)=ubar(:,:,t)-(bet^t*R(:,:,t))^(1/2)*B(:,:,t)'*pk(:,:,t+1);
Xk(:,:,t)=A(:,:,t)^(-1)*(Xk(:,:,t+1)-B(:,:,t)*uk(:,:,t)-G(:,:,t)*...
wk(:,:,t)-e(:,:,t));
pk(:,:,t)=A(:,:,t)'*pk(:,:,t+1)+bet^t*Q(:,:,t)*(Xk(:,:,t)-xbar(:,:,t));
t=t-1;
end

%Initial point error estimate
Fk=Xk(:,:,1)-x0;

```

```

%Update end values for x and p
Sk=-Ak^(-1)*Fk;
XT=Xk(:, :, T)+Sk;
Xk1(:, :, T)=XT;
pk1(:, :, T)=(bet^T)*Q(:, :, T)*(Xk1(:, :, T)-xbar(:, :, T));

%Calculate (x,p) using the updated end conditions
for t=T-1:-1:1;
    lam1=(1/2)*((W(:, :, t)*G(:, :, t)'*pk1(:, :, t+1))'*G(:, :, t)'*...
    pk1(:, :, t+1))^(1/2);
    wk1(:, :, t)=wbar(:, t)+1/(2*lam1)*W(:, :, t)*G(:, :, t)'*pk1(:, :, t+1);
    uk1(:, :, t)=ubar(:, :, t)-(bet^t*R(:, :, t))^(-1)*B(:, :, t)'*pk1(:, :, t+1);
    Xk1(:, :, t)=A(:, :, t)^(-1)*(Xk1(:, :, t+1)-B(:, :, t)*uk1(:, :, t)-G(:, :, t)*...
    wk1(:, :, t)-e(:, :, t));
    pk1(:, :, t)=A(:, :, t)'*pk1(:, :, t+1)+bet^t*Q(:, :, t)*(Xk1(:, :, t)-xbar(:, :, t));
    t=t-1;
end
%Check initial condition again
Fk1=Xk1(:, :, 1)-x0;

%Update the Jacobian estimate
YXk=Fk1-Fk;
Ak1=Ak+((YXk-Ak*Sk)*Sk')/(Sk'*Sk);

%Use the new state-costate and new Jacobian as new end points
Xk(:, :, T)=Xk1(:, :, T);
pk(:, :, T)=(bet^T)*Q(:, :, T)*(Xk(:, :, T)-xbar(:, :, T));
Ak=Ak1;

stop=sum(abs(Fk1));%Calculate sum of absolute errors

end

%Assign values
x=Xk1;
w=wk1;
u=uk1;

crt=stop;
end

```