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Debt Maturity: Does It Matter for Fiscal Space?

by Jun Il Kim

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Debt Maturity: Does It Matter for Fiscal Space?

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Abstract

This paper examines how debt maturity affects the debt limit, defined as the maximum amount of debt a government can afford without defaulting. We develop a model where investors are risk neutral, the primary balance is stochastic but exogenous, and default occurs solely due to the government's inability to pay. We find that debt limit is higher for long-term debt. Underlying this finding is the intrinsic advantage of long-term debt to price in future upside potential in fiscal outcomes in its current price. Such advantage makes long-term debt effectively cheaper than short-term debt at the margin, and leads to a higher debt limit. Simulation results suggest that the effect of debt maturity on debt limit could be substantial—particularly, if fiscal outcomes are subject to large uncertainty.

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I. INTRODUCTION

In the aftermath of the global financial crisis, public debt sustainability has come to the forefront of policy discourse, as many advanced and emerging market economies experienced a sharp increase in public debt (as percent of GDP). Specifically, the (weighted) average ratio of public debt to GDP of advanced economies increased from 79 percent in 2008 to 107 percent by 2012 and has since remained little changed (IMF, 2015). As part of policy efforts to counter aggregate demand shortfall, a number of major countries implemented large fiscal stimulus in the early stage of the crisis. But such efforts were soon confronted with widespread concerns about debt sustainability amid tepid economic growth, and rising uncertainty about the economic outlook. More recently, public debt sustainability has received particular attention in bailout programs of several Eurozone countries where debt was already high in the run-up to the global financial crisis, and spiked sharply afterwards.

As such, recent debates on public debt sustainability have mostly been cast in terms of "fiscal space"—the degree to which countries have room for fiscal maneuver. While the concept has been used variously in academic literature, as well as in policy discussions, Ostry et al. (2010) and Ghosh et al. (2013) pin down a technical, but simple, definition—namely, as the difference between the current level of public debt and the debt *limit* implied by the country's historical record of fiscal adjustment. In other words, debt limit is defined as the maximum amount of debt that a country can afford without defaulting. Beyond this limit, debt dynamics become explosive and the government necessarily defaults. Assuming one-period debt and a stochastic fiscal reaction function, which explicitly incorporates the possibility of fiscal fatigue, they provide a range of estimates of fiscal space for advanced countries.

In this paper, we extend their framework to examine whether, and how, the debt maturity structure plays a role in determining the debt limit (in turn influencing the fiscal space, and public debt sustainability) of countries. The issue is of central importance as short-term debt is considered to be vulnerable to rollover risks, and governments (especially those of emerging market economies) are often advised to reduce the share of short-term debt in their debt portfolio. But standard debt sustainability analysis pays little attention to debt maturity, partly because of a lack of analytical foundation to link debt maturity to debt limit. Moreover, existing studies provide limited guidance as they mostly focus on the relationship between *actual* debt maturity and the *actual* level of debt (e.g., Chatterjee and Eyigungor, 2012, Greenwood et al., 2015). Our focus on debt limit is thus intended to contribute to debt sustainability analysis, and inform debt management policies.

The basic setup of our model closely follows that of Ghosh et al. (2013): investors are risk neutral, the primary balance is exogenously given but uncertain (subject to an *i.i.d*. shock), and default occurs solely due to insolvency (i.e., the government's inability to pay). For tractability, long-term debt is introduced in the convenient form of a long-duration bond (à la Hatchondo and Martinez, 2009), while short-term debt is represented by one-period

(discount) bond. Although maturity and duration differ conceptually, they are positively correlated. Since debt contract is featured as bond issuance in the model, a default event is isomorphic to an event of bond price falling to zero. Finally, we take debt maturity as given, and examine how it affects the debt limit.

Using this framework, we find that debt limit is higher for long-term debt. Simulation results suggest that the effect of debt maturity on debt limit could be substantial—particularly, if fiscal outcomes are subject to large uncertainty. Underlying the finding is the asymmetry between short- and long-term debts in terms of pricing in downside risks and upside potentials in future fiscal outcomes. Assuming that the recovery value upon default is pre-determined at the time of debt issuance, the downside risk (of unfavorable fiscal outcomes in the future) is reflected in the default risk premium for short- and long-term debts alike. But long-term debt has an intrinsic advantage in pricing in future upside potential in fiscal outcomes into the current price—which is absent in short-term debt.

The advantage of long-term debt arises from the feedback from future prices to the current price. The current price affects future prices as it determines the pace at which debt increases, thereby affecting the future default risks (given the exogenous primary balance). In turn, the price in the next period *conditional on no default* affects the pace of debt increases and default risks beyond the next period. Since the current price reflects default risks in all future periods until maturity, the expected price in the next period matters for the current price. For example, a higher expected price in the next period buoys the current price which, in turn, reduces the default risk in the future as it leads to a smaller increase in debt than otherwise. And the same story holds for all future prices. Given that it is the expected future price *conditional on no default* what matters for the current price, long-term debt is well positioned to bring forward future upside potential in fiscal outcomes into the current price.

Such feedback is absent in short-term debt simply because short-term investors, when they lend, do not care about what price will prevail in the next period if no default occurs (in which case they are fully repaid regardless of the price). In this respect, the price of short-term debt is only coarsely related to the underlying fundamentals, while long-term debt resembles equity claims at least on the upside. It is clear that the very reason why the feedback is at work for long-term debt but not for short-term debt is the long maturity itself, suggesting that the pricing advantage of long-term debt is intrinsic. It does not matter whether long-term debt involves coupon payment obligations or not. The same feedback would operate for pure discount bonds as long as the maturity is longer than one period, although the quantitative effects may differ depending on the specifics of the repayment schedule of long-term debt.

The feedback from future prices to the current price is what makes long-term debt effectively cheaper than short-term debt at the margin despite that both long- and short-term debts are actuarially fair under the assumed risk neutrality. As long as the default probability is less than unity (which must be the case at the debt limit), there is always room to price in future

upside potential in fiscal outcomes. It is obvious that such advantage has no relevance if there is no uncertainty, or if debt is sufficiently low that no default risk is present for all possible values of the primary balance. This explains why the *natural* debt limit—at or below which no default risk exists for all possible values of the primary balance—is identical for all maturities. We conjecture that these findings hold for a broad class of long-term debt because the basic intuition for the results is not specific to the assumed specification of long-duration bond.

The model is incomplete in that it addresses only solvency risk while ignoring other legs of default risk—liquidity risk in particular—and no welfare analysis is made available. By switching off other risks, however, it provides a clearer picture on how solvency risk and debt maturity interact. It also provides a good basis to think through how other risks matter for determining the debt maturity in the real world. It has been claimed that long-term debt is safer but more expensive than short-term debt if term premium is positive. Our finding suggests that long-term debt is not as costly as implied by positive term premium itself, and by implication the tradeoff between safety and cost of debt servicing is in fact more favorable to long-term debt than implied by the term premium alone.

Our paper is related to but distinct in focus from the existing literature on sovereign default where debt maturity or the decision to default is modeled as an optimal choice of the government. The early literature on debt maturity highlights debt or moral hazard in the determination of debt maturity. For example, Missale and Blanchard (1994) develop a reputational equilibrium model of the maximum debt maturity in which the borrowing government would want to shorten debt maturity to signal its commitment not to inflate debt away. Jeanne (2009) presents a model of the maturity of international sovereign debt in which the need to roll over external debt disciplines the policies of debtor countries but makes them vulnerable to unwarranted debt crises due to bad shocks.

More recent studies on sovereign debt maturity focus on the tradeoff between the rollover risk and the borrowing cost or service yields that public debt offers. Broner et al. (2013) highlight the trade-off between safer longer-term debt and cheaper short-term debt and find empirical evidence that EMEs pay a positive term premium and debt issuance shifts towards shorter maturities during crises. Greenwood et al. (2015) abstract from the default risk (assuming debt sustainability is always ensured) and focus on the comparative advantage of debts of different maturities—i.e., lower rollover risk and correspondingly smaller welfare loss of long-term debt and larger monetary service yields of short-term debt. Their model implies a positive correlation between debt maturity and debt level as observed in the US historical data.

Although different in the underlying motivation for default, our paper is closely related to several studies pioneered by Aguiar and Gopinath (2006) that incorporate sovereign debt in a quantitative model of interest rate spreads, business cycles, and endogenous default. Alfaro and Kanczuk (2007) compare different rationales for or against short-term debt including

maturity premium, sustainability, and service smoothing and take their model to the Brazil data to find that short-term debt offers higher welfare. Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) find that incorporating long-maturity or long-duration bonds enables their model to better replicate business cycle properties when calibrated for Argentine data. Their results also suggest that EM governments would have an incentive to issue short-term debt because the latter decreases the default frequency (and associated output cost) by weakening debt dilution incentives.

While these two studies take debt maturity as given, Arellano and Ramanarayanan (2012) develop a dynamic model that can account for the observed patterns of the term structure of spreads and debt issuances by emerging markets within an optimizing framework. At the heart of their model lies the tradeoff between benefits of long- and short-term debts—i.e., long-term debt provides a hedge against volatility in spreads whereas short-term debt is less vulnerable to the borrower's disincentives to repay.

The rest of the paper is structured as follows. Section II sets up the model by introducing long-duration bond into the model developed by Ghosh et al (2013). Section III sketches the determination of the debt limit in a non-stochastic setting—which offers useful insight as to the equilibrium solutions of the model under uncertainty. Section IV solves the model in a stochastic setting to establish equilibrium conditions that can be used to pin down the debt limit for long- and short-term debt, and discusses policy implications of the model. Section V presents numerical solutions of the model. Section VI concludes.

II. THE MODEL

Consider a long-duration bond that promises to pay an infinite stream of coupons which decay geometrically at a constant rate $\delta \in [0, 1]$. Specifically, a bond issued in period t promises to pay one dollar in period t+1 and $(1-\delta)^{k-1}$ dollars in period t+*k* for $k \geq 2$. The Macaulay duration of such a bond is given by $(1+r)/(\delta+r)$ where *r* is the one-period risk-free interest rate. Thus, the duration is inversely related to δ and one-period (discount) bond corresponds to $\delta = 1$. With this structure, the law of motion for coupon payment obligations, denoted by *d*, is characterized by

$$d_{t+1} = (1-\delta)d_t + n_t \tag{1}$$

where n_t is the number of bonds newly issued (or purchased) in period t.

The government issues bond in each period to meet the gross financing need (GFN), *x*, which is comprised of coupon payment obligations (henceforth, outstanding debt) and primary deficit. Then,

$$n_t = q_t^{-1}x_t = q_t^{-1}(d_t - s_t) \tag{2}$$

where q_t and s_t are the bond price and the primary balance, respectively. The primary balance (or the fiscal reaction function) is stochastic and characterized by

$$s_t = s^* + e_t, \tag{3}$$

where $s^* > 0$ and e_t is an *i.i.d.* shock distributed according to $G(e)$ over a finite support $[-\overline{e},\ \overline{e}\ ]$.[1]

Investors are risk neutral and, for simplicity, assumed to recover nothing upon default. The bond price satisfies the zero-profit condition, which is given by

$$q_t = [(1 - p_{t+1})/(1 + r)][1 + (1 - \delta)q^e_{t+1}] \tag{4}$$

where p_{t+1} is the one-period-ahead default probability perceived in period t and $q^e_{t+1} = E_t[q_{t+1}|\ q_{t+1} > 0]$ is the expected value of the bond price that would prevail in the next period *conditional on* no default. Note that $q_t \leq q^f = 1/(\delta + r)$ where q^f is the risk-free bond price. In case where there are multiple bond prices that satisfy (4), we assume the highest such price is the equilibrium solution.

The model is closed by setting the default rule. We define default as an event in which the bond price falls to zero and hence the government cannot meet the GFN. We also assume a cross-default rule in which a failure to meet the GFN in any given period triggers default on entire debt obligations. The default rule states that the government defaults (and the bond price falls to zero) if and only if the GFN exceeds the *unadjusted* debt limit (see below for further discussion). Denoting by $\overline{x}_t$ the possibly time-varying unadjusted debt limit, the default rule formally takes the form:

$$D_t = \begin{cases} 1 & \text{iff } x_t > \overline{x}_t \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where *D* is an indicator function for the default event.

Given this setup, the model is stationary in the sense that each period is fully characterized by two state variables, $\Omega_t = \{d_t, s_t\}$. Consequently, period *t* and period *t+j* are identical in all respects if $\Omega_t = \Omega_{t+j}$. Moreover, the primary balance is exogenous and subject to an *i.i.d.* shock which is the only source of uncertainty in the model. This structure of the model suggests that in equilibrium the bond price should be a time-invariant function of the state variables. Specifically,

$$q_t = q(d_t, s_t), \quad q_1 \leq 0 \ \text{ and } \ q_2 \geq 0 \tag{6}$$

[1] The autonomous component of the primary balance (s^*) is assumed to be constant only for simplicity. This specification of the primary balance satisfies the technical restrictions for fiscal fatigue in Ghosh et al. (2013).

The signs of q_1 and q_2 are implied by the fact that in the model default occurs solely due to the government's inability to pay. For given primary balance, a larger coupon payment obligation should be associated with a higher default probability and hence a lower bond price. A better outturn of the primary balance improves the government's debt servicing capacity and thus should support a higher bond price in equilibrium.

Before solving the model, we need to make sure that the comparison of debt limit between long- and short-term bonds is made on the same conceptual basis. To this end, it is important to recognize that outstanding debt (d) refers to a *flow* of coupon payment obligations for long-duration bond while it is the nominal value of the *stock* of outstanding debt (before the primary balance) for one-period bond. Likewise, the gross financing need (x) as defined in the model is a flow variable for long-duration bond while it is essentially a stock variable for one-period bond. It would thus be wrong if one compares these two variables between long- and short-term debts as they stand in the model.

For this reason, we solve the model as it is but make adjustments on the variables of our interest to make them comparable between long-duration and one-period bonds. Specifically, we define *adjusted* outstanding debt and gross financing need for long-duration bond (dropping time subscript) as follows:

$$d_A = [1 + (1 - \delta)q]d \quad \text{and} \quad x_A = d_A - s \tag{7}$$

Conceptually, d_A is the present value or buy-back value of the infinite stream of coupon payment obligations and hence corresponds to a stock variable comparable to d of one-period bond.[2] Similarly, x_A is comparable to x of one-period bond.

In what follows, short-term debt and long-term debt refer to one-period bond ($\delta = 1$) and long-duration bond ($\delta < 1$), respectively. For short-term debt, debt limit refers to $\overline{x}$ as directly obtained from the model. For long-term debt, debt limit refers to $\overline{x}_A$ while $\overline{x}$ is labeled as *unadjusted* debt limit. Finally, we drop time subscripts while denoting a variable in the next period by using a prime given the stationary structure of the model and for notational convenience.

III. NON-STOCHASTIC CASE

We first solve the model assuming no uncertainty. This exercise offers useful insight as to the equilibrium solutions of the model under uncertainty. To this end, we assume that $\overline{e} = 0$, so that $s = s^*$.

[2] The amount of outstanding debt in period t can be thought of d_t units of long-duration bond that promises to pay $(1 - \delta)^{s-t}$ dollars for all $s \geq t$ (i.e., starting from period t). The market value of such a bond is equal to $1 + (1 - \delta)q_t$.

As there is no uncertainty, *x* and d move one for one and differ by a constant. Thus, debt limit defined for *x* is intrinsically related to whether d is on a sustainable (i.e., non-explosive) path. Specifically, default occurs (and the bond price falls to zero) as soon as d is expected to grow without bound in *all* circumstances. The law of motion for d as shown in (1) and (2) implies

$$d' \leq d \quad \Leftrightarrow \quad (d - s^*)/q \leq \delta d \tag{8}$$

Solving the latter inequality for d assuming $q > 0$ yields,

$$d \ \leq \ s^*/(1 - \delta q)$$

This inequality means that *d* is non-increasing whenever it remains at or below $s^*/(1 - \delta q)$ but otherwise grows without bound with certainty. Therefore, there is no default risk whenever the above inequality holds and therefore the corresponding bond price should be equal to the risk-free price. Otherwise, the bond price falls to zero.

Substituting $q = q^f$ into the above inequality yields,

$$d \leq \overline{d} = [(\delta + r)/r]s^* \quad \text{or, equivalently,} \quad x \leq \overline{x} = (\delta/r)s^* \tag{9}$$

These are familiar results for one-period bond ($\delta = 1$). To be specific, $\overline{x} = s^*/r$ is the *natural* debt limit of one-period debt at or below which debt is non-increasing and the government never defaults. The same intuition holds for long-duration bond. It is straightforward to show

$$\overline{d}_A = [(1 + r)/r]s^* \quad \text{and} \quad \overline{x}_A = s^*/r \tag{10}$$

Both $\overline{d}_A$ and $\overline{x}_A$ are identical to their respective one-period bond counterparts that can be obtained by substituting $\delta = 1$ into the expressions for $\overline{d}$ and $\overline{x}$ in (9). The results in (9) and (10) establish that debt limit, if adjusted appropriately, is identical for long- and short-term debts in the absence of uncertainty. This result implies that both long- and short-term debts are equally expensive from the perspective of the government when there is no uncertainty in fiscal outcomes.

IV. STOCHASTIC CASE

We now assume that the primary balance is stochastic. We first solve the model for short-term debt. The analytical results provide a useful basis to find equilibrium solutions for long-term debt.

A. Debt Limit of Short-term Debt (δ = 1)

For short-term debt, the government's budget constraint is characterized by

$$x' = x/q - s', \qquad q = (1 - p')/(1 + r) \tag{11}$$

This is the classic case studied by Ghosh et al. (2013) where the debt limit is constant. The constancy of the debt limit is implied by the fact that the evolution of the GFN is a self-generating process. We solve the model assuming a constant debt limit denoted by $\overline{x}$.

According to the default rule, the one-period-ahead default probability is defined as $p' = \Pr[x' > \overline{x}]$, which can be rewritten into

$$p' = G(H) \tag{12}$$

where $H = x/q - s^* - \overline{x}$ and $q = (1 - p')/(1 + r)$. Given the dependence of H on p' via q, the equality in (12) constitutes a fixed-point problem for p'. Note that $p' = 1$ is always a (corner) solution to the fixed-point problem in which case the government defaults with unit probability. In equilibrium, the lowest solution to the fixed-point problem for given debt limit yields the one-period-ahead default probability and the equilibrium price of bond at each level of the GFN.

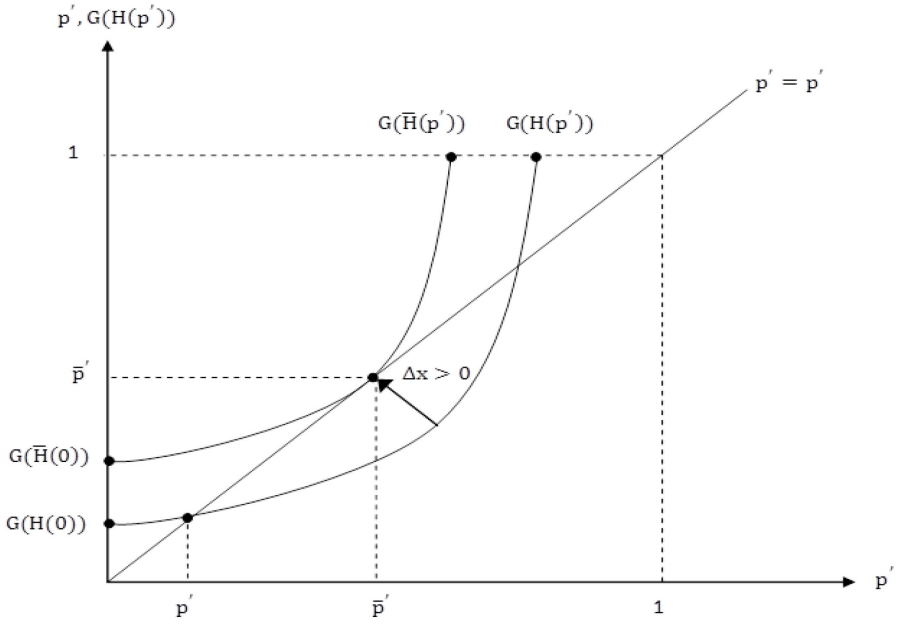
The debt limit is determined as the maximum GFN beyond which no interior solution exists to the fixed-point problem in (12) so that $p' = 1$ is the only (corner) solution. Denoting by $\overline{p}'$ the one-period-ahead default probability *at the debt limit*, the pair $\{\overline{p}', \overline{x}\}$ is essentially pinned down by the two equilibrium conditions given by

$$\begin{aligned} &\text{(i)} \quad \overline{p}' = G(\overline{H}) \\ &\text{(ii)} \quad \partial \mathrm{G}(\overline{H})/\,\partial \overline{p}' = 1 \end{aligned} \tag{13}$$

where $\overline{H} = \overline{x}/\overline{q} - s^* - \overline{x}$ and $\overline{q} = (1 - \overline{p}')/(1 + r)$. The second condition is the slope condition that ensures that $\overline{p}'$ is the *maximum* interior solution to the fixed point problem.

Figure 1 illustrates the determination of the one-period-ahead default probability for short-term debt as an interior solution to the fixed point problem in (12). In the Figure, $G(H(p'))$ is monotonically increasing in p' before it becomes flat at unity, and shifts upward with an increase in the GFN ($\Delta x > 0$). For a given debt limit ($\overline{x}$), there would in general be two or more interior solutions (obtained at the intersection of $G(H(p'))$ and the 45-degree line denoted by $p' = p'$), the lowest of which is by assumption the equilibrium solution for p' (which yields the highest bond price). As x increases from some initial level below $\overline{x}$, the lowest interior solution increases (while the upper interior solution decreases). It should be noted that the fixed-point problem has always a corner solution which is given by $p' = 1$.

Figure 1. Determination of Default Probability: Short-term Debt



The maximum interior solution for $\overline{p}'$ obtains at the tangency point between $G(H(p'))$ and the 45-degree line, and the corresponding level of x is the largest GFN that the given debt limit can sustain. Any further increase in x will lead to the corner solution $p' = 1$ in which case the bond price collapses to zero and the government necessarily defaults. But this is the very definition of debt limit according to the default rule. At the tangency point, therefore, actual GFN must coincide with the given debt limit (i.e., $x = \overline{x}$) and the corresponding default probability must be the default probability at the debt limit (i.e., $p' = \overline{p}'$). The tangency point is completely pinned down by the equilibrium conditions in (13).

Several equilibrium properties of the debt limit and the price of short-term debt are worth summarizing for future references. First, the bond price is uniquely determined by x given the constant debt limit. Formally,

(14) $$q = q(x) > 0 \quad \text{and} \quad q(x_1) \geq q(x_2) \ \text{ if } x_1 \leq x_2$$

Second, the bond price falls below the risk-free price over a short range of the GFN less than the support of the primary balance shock. More specifically, there exists $x_0 < \overline{x}$ such that $p' = 0$ if $x \leq x_0$ and $p' > 0$ otherwise. It can be easily shown that $x_0 \geq \overline{x} - 2\overline{e}$.[3] Third, the maximum outstanding debt consistent with the debt limit is given by $\overline{d} = \overline{x}/\overline{q}$ (see the Appendix for the proof).

[3] See the Appendix in Ghosh et al. (2013) for the proof.

B. Debt Limit for Long-term Debt ($\delta < 1$)

Identifying debt limit for long-term debt is complicated by the interdependence of bond prices across time periods. The current price affects future price via its effect on the pace at which debt increases which in turn affects the default risk in the future. Conversely, future prices matter for the current price as the latter reflects default risk in all future periods until maturity. Such interdependence introduces enormous complications into the model making it virtually intractable. Given the complexity of the model, it is beyond the scope of this paper to present a full-fledged solution for the debt limit of long-term debt. As shown later, we focus instead on a specific debt limit of long-term debt which is most interesting and relevant for fiscal space and debt sustainability assessment.

If the government issues long-term debt ($\delta < 1$), its budget constraint is given by

$$x' = q^{-1}x + (1-\delta)d - s', \quad q = [(1-p')/(1+r)][1+(1-\delta)q^e] \tag{15}$$

Unlike in case of short-term debt, the evolution of the GFN is governed not only by itself but also outstanding debt (d). As such, debt limit of long-term debt is not constant but varies depending on d. This also implies that the bond price would no longer be uniquely determined by x alone. For this reason, we conjecture that (unadjusted) debt limit is characterized by

$$\overline{x} = \overline{x}(d) \quad \text{and} \quad \partial\overline{x}(d)/\partial d \leq 0$$

Given this conjecture, the one-period-ahead default probability is then defined as

$$p' = \Pr[x' > \overline{x}(d')]$$

Let us first look into the evolution of d and its implication for q. As discussed earlier, the law of motion for d implies,

$$d' \leq d \quad \Leftrightarrow \quad q^{-1}(d-s) \leq \delta d \tag{16}$$

Let us define $\pi(d) = \Pr[d' > d]$ which refers to the probability that d increases between two adjacent periods. It is then straightforward to find $d^* > 0$ such that $\pi(d) = 0$ if $d \leq d^*$ and $\pi(d) > 0$ otherwise. If d is never increasing, there would be no default risk and hence the bond price should equal the risk-free price. Substituting $q = q^f$ and $s = s^* - \overline{e}$ (i.e., the worst possible primary balance) into the latter inequality in (16) and rearranging terms yield,

$$d \leq d^* = [(\delta + r)/r](s^* - \overline{e}) \text{ or, equivalently, } x \leq x^* = (\delta/r)(s^* - \overline{e}) \tag{17}$$

As discussed in Section III, these results suggest that the natural debt limit is identical for both long- and short-term debts and is given by $x^*(1) = x_A^*(\delta) = (s^* - \overline{e})/r$.

If $d > d^*$, default risk rises above zero because $\pi(d) > 0$ and the bond price correspondingly falls below the risk-free price. Formally,

$$\pi(d) > 0 \quad \text{and} \quad q < q^f \quad \text{iff} \quad d > d^* \tag{18}$$

Note that $\pi(d) > 0$ implies that the probability that d grows without bound is positive. This in turn implies that default risk begins to rise above zero as soon as the natural debt limit is passed and, therefore, $q^e < q^f$ must also hold if $d > d^*$.

As d grows further, the bond price declines toward zero since $q_1 \leq 0$ while $\pi(d)$ increases toward unity. A natural extension of this argument is that there must exist $\overline{d} > d^*$ such that

$$q = \begin{cases} > 0 & \text{iff } d \leq \overline{d} \\ = 0 & \text{otherwise} \end{cases} \tag{19}$$

Conceptually, $\overline{d}$ is the upper bound of d above which $\pi(d) = 1$ for *all* possible values of x so that d grows without bound with certainty and, as a result, the bond price necessarily falls to zero.[4]

The bond price as characterized in (19) implies that if $d < \overline{d}$, there is always room for d to increase up to $\overline{d}$ without triggering default. This intuition can be used to characterize the debt limit schedule, $\overline{x}(d)$. Specifically, substituting $d' = \overline{d}$ and $x = \overline{x}(d)$ into (1) and (2) and rearranging terms yields,

$$\overline{x}(d) = \overline{q}(d)[\overline{d} - (1-\delta)d] \quad \text{if } d \leq \overline{d} \tag{20}$$

where $\overline{q}(d) = q(\overline{x}(d))$ is the bond price evaluated at the debt limit. It can be easily shown that

$$\partial\overline{x}(d)/\partial d = -(1-\delta)\overline{q}(d)/[1 - (\partial\overline{q}/\partial\overline{x})\{\overline{d} - (1-\delta)d\}] \leq 0 \tag{21}$$

where the inequality follows from $\partial\overline{q}/\partial\overline{x} < 0$. This result has intuitive appeal. As d approaches to $\overline{d}$ from below, the bond price falls while fiscal space measured by $\overline{d} - d$ shrinks. Declining bond price means that a larger volume of debt issuance is required to meet a given amount of GFN while shrinking fiscal space means that the affordable volume of debt issuance is decreasing. Thus, $\overline{x}(d)$ must decline as d increases.

[4] It is easy to show that $\overline{d} \leq \hat{d} = [(\delta + r)/r](s^* + \overline{e})$ where $\hat{d}$ is the largest d that can be sustained if the primary balance remains at the best possible value and the bond price equals the risk-free price (despite positive default risk) at *all* times.

Figure 2. Determination of Debt Limit for Long-term Debt

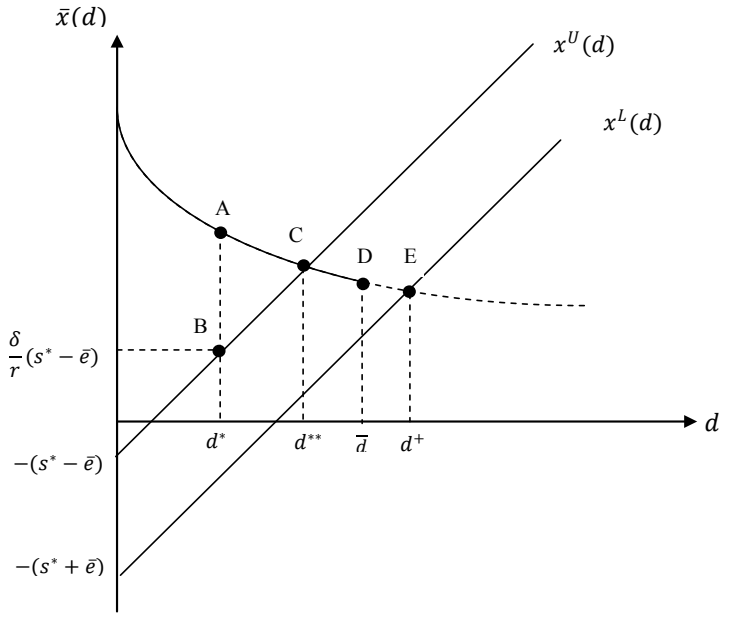


Figure 3. Bond Price and Default Probability by Region of Debt

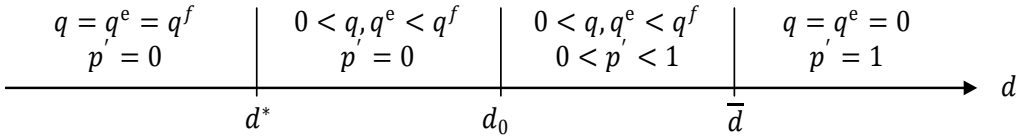


Figure 2 provides a graphic illustration of the schedule $\overline{x}(d)$ and the feasible set of the gross financing need. In the Figure it is assumed without loss of generality that the worst possible primary balance is positive. The area bounded by $x^U(d)$ and $x^L(d)$ represents the feasible set of x for each level of d where $x^U(d)$ and $x^L(d)$ are defined as follows:

$$x^U(d) = d - (s^* - \overline{e}) \quad \text{and} \quad x^L(d) = d - (s^* + \overline{e})$$

In the Figure, point A is on the debt limit schedule and lies above point B which corresponds to the (unadjusted) natural debt limit shown in (17). It is important to recall that d grows with positive probability ($\pi(d) > 0$) and default risk rises above zero as soon as d exceeds d^*.

Consider next point C which is the intersection of $\overline{x}(d)$ and $x^U(d)$. At point C, the one-period-ahead default probability p' is positive (see the Appendix).

Although not shown in Figure 2, there exists $d_0 \in (d^*, d^{**})$ such that $p' = 0$ if $d \leq d_0$ and $p' > 0$ otherwise.[5] For $d > d^{**}$, debt limit begins to bind with positive probability because it lies within the feasible set of x. Point E is the intersection of $\overline{x}(d)$ and $x^L(d)$ at which default occurs with certainty because $\overline{x}(d) \leq x$. As such, point E cannot be a legitimate debt limit.[6] This requires that $\overline{d}$ must lie between d^{**} and d^+. Point D is such a point which corresponds to the unadjusted debt limit associated with $\overline{d}$.

Figure 3 summarizes how the bond price and the one-period-ahead default probability evolve across regions of d demarcated by several thresholds discussed above. Both q and q^e are equal to q^f until d^* and then fall below q^f but remain positive between d^* and $\overline{d}$, before falling discretely to zero if $d > \overline{d}$. In contrast, p' remains at zero until d_0 after which it becomes positive and then reaches to unity if $d > \overline{d}$. Since the adjusted natural debt limit of long-term debt is identical to that of short-term debt, default risk begins to rise above zero earlier for long-term debt.

Let us now return to the model. As noted earlier, we are unable to identify the entire (unadjusted) debt limit schedule $\overline{x}(d)$ because the model is not fully tractable. However, we are able to find equilibrium conditions that can be used to determine $\overline{x}(\overline{d})$. Our focus on $\overline{x}(\overline{d})$ is well justified because it is associated with the maximum amount of outstanding debt (d) a country can afford without defaulting and therefore most relevant for fiscal space and debt sustainability analysis.

At $d = \overline{d}$, the fixed-point problem for p' is characterized by

$$p' = G(Z) \tag{22}$$

where $Z = q^{-1}x + (1-\delta)\overline{d} - s^* - \overline{x}(d')$. There are four unknowns in this fixed point problem which are given by $\{\overline{p}', \overline{d}, \overline{q}^e, d'\}$ where $\overline{q}^e$ is the expected bond price conditional on no default in the next period which evaluated at $x = \overline{x}(\overline{d})$. Once these unknowns are determined, $\overline{x}(\overline{d})$ can be uncovered by using (20). Thus, we need four equilibrium conditions to pin down $\overline{x}(\overline{d})$.

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[5] Conceptually, point C corresponds to x_0 of short-term debt as discussed in Section IV.A.

[6] $\overline{x}(d)$ is drawn using a dotted line beyond point D in order to highlight that it is not well defined for $d > \hat{d}$.

The first equilibrium condition is characterized by

$$\overline{x}(\overline{d})/\overline{q} = \delta\overline{d} \tag{23}$$

This condition is equivalent to requiring that $d' = \overline{d}$ when $x = \overline{x}(\overline{d})$ (see (1) and (2)). To understand this requirement, it is important to note that all variables that appear in (23) are known at the time of debt issuance and hence no uncertainty is involved. With this understanding, suppose that $d' > \overline{d}$. According to (19), $q' = 0$ with certainty. This in turn implies that $q = 0$ because no investor would lend to the government knowing that it will certainly default in the next period. This result contradicts the definition of debt limit at which the bond price must be positive. Suppose alternatively that $d' < \overline{d}$. Then there always exists $\varepsilon > 0$ such that $d' = \overline{d}$ and $q > 0$ at $x = \overline{x}(\overline{d}) + \varepsilon$. This means that no default occurs in the current period even if $x > \overline{x}(\overline{d})$. Again, this result contradicts the definition of debt limit and violates the default rule. Thus, in equilibrium, the condition in (23) must hold. Note that the same condition implies that $\overline{x}(d') = \overline{x}(\overline{d})$.

The second and third equilibrium conditions are akin to those in (13) for short-term debt. Evaluating Z in (22) at $x = \overline{x}(\overline{d})$ while imposing (23) yields,

$$\overline{Z} = \overline{x}(\overline{d})/\delta\overline{q} - s^* - \overline{x}(\overline{d}) \tag{24}$$

It should be highlighted that $\overline{Z}$ does not involve $\overline{d}$ as a separate conditioning variable. This special feature enables us to apply the same solution technique as used for short-term debt. Specifically, the second and third equilibrium conditions are characterized by

$$\begin{aligned} &\text{(i)} \ \ \overline{p}' = G(\overline{Z}) \\ &\text{(ii)} \ \ \partial \mathrm{G}(\overline{Z})/\,\partial\overline{p}' = 1 \end{aligned} \tag{25}$$

These conditions determine uniquely the pair $\{\overline{p}', \overline{x}(\overline{d})\}$ for given $\overline{q}^e$.

The last equilibrium condition obtains by requiring that $\overline{q}^e$ be a rational expectations equilibrium solution. Specifically, it is characterized by

$$\overline{q}^e = (1 - \overline{p}')^{-1} \int_{\overline{Z}}^{\overline{e}} q(\overline{d}, s)\, g(e) de \ > \ \overline{q} \tag{26}$$

where $g(e)$ is the densitiy function of the primary balance shock. Since $d' = \overline{d}$ in equilibrium, it follows that $q'(d', s') = q(\overline{d}, s)$ if $s' = s$. Moreover, $\overline{q}^e$ must be greater than $\overline{q}$ in equilibrium because $\overline{q}^e$ is the expected value of q' *conditional on no default*. Consequently, the lower limit of the integral is equal to $\overline{Z}$.

These four equilibrium conditions characterized by (23), (25) and (26) can jointly pin down the equilibrium values of $\{\overline{p}', \overline{x}(\overline{d}), \overline{q}^e, \overline{d}\}$. Although no closed form solution is available, these conditions can be used to characterize the adjusted debt limit of long-term debt. For clarity of comparison and notational convenience, we use the notation $\overline{x}(\delta)$ for $\overline{x}(\overline{d})$ of long-term debt and denote the debt limit of short-term debt discussed earlier by $\overline{x}(1)$.

By using (7) and (23), it straightforward to show that the *adjusted* debt limit of long-term debt can be expressed as

$$\overline{x}_A = \overline{d}_A - \overline{s} = \overline{x}(\delta)/\delta \tag{27}$$

Utilizing this result, we can show that (adjusted) debt limit is higher for long-term debt. Formally,

$$\overline{x}_A > \overline{x}(1) \tag{28}$$

The Appendix provides formal proof of this result.

What drives this result is an intrinsic advantage of long-term debt over short-term debt in pricing in future upside potential in fiscal outcomes (i.e., possibility of favorable shocks to the primary balance in the future) into the current price. Indeed, central to the proof of (28) is the strict inequality given by $\overline{q}^e > \overline{q}$. This inequality implies that there always exists future upside potential in fiscal outcomes to be priced in as long as the default probability is less than unity. For example, all else equal, a higher expected price in the next period buoys the current price which, in turn, reduces the default risk in the future as it leads to a smaller increase in debt than otherwise. Given that it is the expected future price *conditional on no default* that matters for the current price, long-term debt is well positioned to bring forward future upside potential into the current price.

Such feedback from future prices to the current price is, however, absent in short-term debt. Short-term investors do not care about what price will prevail in the next period if no default occurs, simply because it does not matter for the return on their investment.[7] In this respect, the price of short-term debt is only coarsely related to the underlying fundamentals while long-term debt resembles equity claims at least on the upside. This implies that the very reason why the feedback is at work for long-term debt but not for short-term debt is the long maturity itself. It does not matter whether long-term debt involves coupon payment

[7] Technically, this is why the price of short-term debt ($\delta = 1$) does not involve q^e as can be seen from (4). While the downside risk (of unfavorable shocks to the primary balance in the future) is reflected in the default probability for short- and long-term debts alike, the future upside potential in fiscal outcomes is priced in only for long-term debt.

obligations or not. The same feedback operates for a pure discount bond as long as the maturity is longer than one period.

The pricing advantage of long-term debt is intrinsic because it arises from the long maturity itself. And it is what makes long-term debt effectively cheaper than short-term debt at the margin despite that both debts are actuarially fair under the assumed risk neutrality. It is obvious that such advantage has no relevance if there is no uncertainty or if debt is low enough so that no default risk is present for all possible values of the primary balance. This explains why the *natural* debt limit—at or below which no default risk is present for all possible realizations of the primary balance shock—is identical for all maturities. We conjecture that these findings hold for a broad class of long-term debt because the basic intuition for the results is not specific to the assumed specification of long-duration bond.

The model has several policy implications that can shed light on the existing empirical evidence as to the maturity structure of sovereign debt and debt sustainability analysis.

First, higher debt limit for long-term debt under risk neutrality does not necessarily imply that long-term debt should dominate short-term debt in real data. Rather it suggests that other legs of default risk than solvency risk should matter for the determination of actual debt maturity. In fact, the average public debt maturity varies significantly among advanced economies and tends to be shorter for emerging market economies. If term premium is negligible, our finding suggests that long-term debt would likely dominate short-term debt as the former is more resilient to solvency risk. This may explain why the average debt maturity is longer for advanced economies than for emerging market economies. If term premium is positive and large, however, long-term debt may become more expensive than short-term debt despite its resilience to solvency risk and, as a result, the average debt maturity would likely be shorter than otherwise (Broner et al., 2013).

Second, a country would be able to reap the benefit of fiscal reforms earlier than later if debt maturity is longer. In the context of our model, fiscal reforms to strengthen debt sustainability can be mapped into an expected permanent increase in the autonomous part of the primary balance ($\Delta s^* > 0$). A permanently higher primary balance in the future will affect the current price of long-term debt given the feedback discussed above. Assuming that fiscal reforms are credible, for example, long-term bond yields would fall immediately at an announcement of fiscal reforms (that will take time to implement) while short-term yields are less likely to respond to such news. By the same token, the price of long-term debt would also respond by more than that of short-term debt to a bad news about future fiscal performance (e.g., ($\Delta s^* < 0$). For example, long-term bond yields may rise by more than short-term bond yields in response to a downward revision in the medium-term projections of the fiscal balance.

Third, debt stress warnings may arrive earlier for long-term debt. The model suggests that for long-term debt, default risk begins to rise above zero and the bond price correspondingly

falls below the risk-free price as soon as the natural debt limit is passed. For short-term debt, however, the bond price remains at the risk-free price even after debt passes the same natural debt limit. Thus, an earlier detection of debt stress may be possible for long-term debt. Last but not least, standard debt sustainability analysis undertaken by assuming one-period debt would likely be too conservative if the average debt maturity is significantly longer than one period.

V. SIMULATION RESULTS

In this section, we simulate the model to get a feel for the effects of debt maturity on debt limit. The model can be solved numerically to find the debt limit of one-period bond but not for long-duration bond because major technical challenges arise with regard to the equilibrium condition in (26). Simply speaking, the functional form of $q(d,s)$ is unknown. As a result, we are unable to obtain exact numerical solutions for long-duration bond.

For this reason, we take $\overline{q}^e$ as given when simulating the model for long-duration bond. To be specific, simulation is undertaken in two steps. In the first, the model is numerically solved to find the value of $\overline{q}_L^e$ such that $\overline{q}_L^e = \overline{q}(\overline{q}_L^e)$. In the second, the model is simulated to generate a sample of 200 simulated debt limits by varying $\overline{q}^e$ incrementally over the interval $[\overline{q}_L^e,\ q^f]$ in which the equilibrium value of $\overline{q}^e$ must lie. Since in equilibrium $\overline{q} < \overline{q}^e < q^f$, the considered interval of $\overline{q}^e$ includes non-permissible values. Therefore, the simulated range of debt limits is correspondingly wider than should be.

In the simulation, a positive recovery value is introduced. This does not affect the qualitative results but produces more discernible quantitative variations in debt limits. To this end, the zero-profit condition in (4) is modified as follow:

$$q_t = [(1-p_{t+1})/(1+r)][1+(1-\delta)q_{t+1}^e] + p_{t+1}\theta q_t, \quad 0 < \theta < 1/(1+r)$$

which states that investors recover upon default a fraction $(1+r)\theta$ of the market value of their investment. For the baseline simulation, we set the values of key parameters as follows:

$$r = 0.02, \quad s^* = 4.0, \quad \overline{e} = 2.0, \quad \theta = 0.9$$

For sensitivity check with regard to the risk-free interest rate and the underlying uncertainty as to the primary balance, we also experiment with $r = 0.03$ and $\overline{e} = 3.0$ both of which are expected to lower the debt limit. Finally, we assume the triangular density function for the primary balance shock.

Table 1. Simulated Debt Limits

r	$\overline{e}$	δ	q^f	$\overline{q}$	$\overline{x}_A$	$\overline{x}_A/\overline{x}(1)$	Range
0.02	2.0	1.0	0.98	0.98	108.9	1.00	...
		0.5	1.92	1.91	112.2	1.03	[108.9, 115.8]
		0.2	4.55	4.41	118.3	1.09	[108.9, 129.4]
		0.1	8.33	7.73	123.2	1.13	[108.9, 141.0]
	3.0	1.0	0.98	0.97	76.9	1.00	...
		0.5	1.92	1.88	83.3	1.08	[76.9, 91.1]
		0.2	4.55	4.21	91.7	1.19	[76.9, 113.0]
		0.1	8.33	7.11	98.3	1.28	[76.9, 128.4]
0.03	2.0	1.0	0.97	0.97	75.0	1.00	...
		0.5	1.89	1.86	77.8	1.04	[75.0, 80.9]
		0.2	4.35	4.14	82.3	1.10	[75.0, 91.0]
		0.1	7.69	6.92	85.5	1.14	[75.0, 98.3]
	3.0	1.0	0.97	0.96	56.6	1.00	...
		0.5	1.89	1.83	61.4	1.09	[56.6, 67.2]
		0.2	4.35	3.92	67.2	1.19	[56.6, 82.0]
		0.1	7.69	6.10	69.2	1.22	[56.6, 87.6]

It should be reminded that any inferences from the simulation results are suggestive rather than conclusive because the reported debt limits for long-term debt are not the exact numerical solutions. With this understanding, Table 1 presents the simulation results undertaken for four different values of δ ranging from 0.1 to 1.0. These values of δ are chosen as their implied duration is broadly matched with popular maturities observed in real data for advanced economies. For instance, $\delta = 0.1$ and $\delta = 0.2$ correspond approximately to maturities of 10 and 5 years respectively if time period is measured in annual frequency. For long-term debt ($\delta < 1$), the reported debt limit ($\overline{x}_A$) refers to the adjusted debt limit and corresponds to the median of 200 simulated debt limits whose range is shown in the last column.

Simulation results are suggestive of potentially large gain in debt limit when debt maturity is lengthened. Several regularities emerge from the Table. First, the debt limit ratio $\overline{x}_A/\overline{x}(1)$ is greater than unity for all $\delta < 1$. Second, all else equal, the same ratio is inversely related to δ

suggesting that debt limit would likely be higher the longer is debt maturity. The reported ratios are ranged between 1.03 and 1.28 depending on debt duration. To put these results in perspective, one can imagine that a country can afford about 30 percent larger debt if it issues 10-year treasury than in case where all debt is issued in 1-year treasury.

Third, all else equal, the effect of debt maturity on debt limit is larger the larger is the uncertainty as to fiscal outcomes (i.e., the larger is the support of the primary balance shock) suggesting that government facing larger uncertainty in their fiscal performance would likely benefit more by issuing long-term debt. Given the nature of debt contract in which the downside risk matters more for the price of debt than the upside potential, an increase in uncertainty in fiscal outcomes—represented as a mean-preserving spread of the primary balance shock—would likely lead to a decline in debt limit for both short- and long-term debts. But the decline should be expected to be less in relative terms for long-term debt whose price is buoyed in part by grater upside potential in fiscal outcomes. The simulated debt limits are suggestive of this conjecture. For instance, in the upper panel of Table 1, simulated debt limit falls by 30 percent for short-term debt when $\overline{e}$ is increased from 2 to 3 while it falls by 20 percent for long-term debt with $\delta = 0.1$.

VI. CONCLUSION

We develop a model of sovereign debt default and show that debt limit is higher for long-term debt, and that the effect of debt maturity on debt limit could be substantial if the underlying uncertainty in fiscal outcomes is large. Key to these findings is the pricing advantage of long-term debt over short-term debt in terms of bringing forward upside potential in future fiscal outcomes into the current price. Such advantage makes long-term debt effectively cheaper than short-term debt at the margin. This result is interesting as it obtains under risk neutrality, and is in stark contrast to the findings of many existing quantitative models of endogenous sovereign default (which also assume risk neutrality) that short-term debt yields higher welfare unless some additional mechanism is introduced to make it particularly vulnerable to rollover risk (e.g., Chatterjee and Eyigungor, 2012).

The fact that long-term debt is cheaper than short-term debt under risk neutrality neither suggests that long-term debt should dominate short-term debt in actual debt maturity composition, nor that the optimal debt maturity should never be short. Rather, it underscores the importance of other legs of default risk than solvency risk in the determination of actual debt maturity. It also suggests that long-term debt may not be as expensive as implied by positive term premium itself. Another implication of our finding is that fiscal space estimates obtained by assuming one-period debt may be too conservative if the average maturity of public debt is significantly longer than one period.

The model can be further extended in several ways. First, the specification of the primary balance can be enriched by allowing fiscal fatigue more explicitly as studied by Ghosh et al (2013). Such an extension would not complicate the analysis by much as long as the primary

balance is modeled as a (nonlinear) function of the gross financing need. Second, the primary balance shock could be allowed to be serially correlated although this would complicate the analysis significantly. Finally, the possibility of multiple equilibria can be studied explicitly. In our model, the possibility of multiple equilibria arises from two sources. First, the fixed-point problem for the one-period-ahead default probability has in general two or more interior solutions. We simply rule out multiple equilibria by assuming that the best bond price would prevail if there are multiple prices that solve the fixed-point problem. Second, multiple equilibria may emerge in debt dynamics in case of long-term debt given the feedback from future prices into the current price (Lorenzoni and Werning, 2014). This possibility could be incorporated and explored.

APPENDIX

A. Proof of $\overline{d} = \overline{x}/\overline{q}$ for Short-term Debt

By definition, the maximum outstanding debt (before the primary balance), $\overline{d}$, consistent with the debt limit is characterized by

$$\overline{d} = \overline{x} - \overline{s} \tag{A.1}$$

The first condition in (13) implies that $\overline{s}$ is given by

$$\overline{s} = \overline{H} + s^* = (1 - \overline{q})\overline{x}/\overline{q} \tag{A.2}$$

Substituting (A.2) into (A.1) yields $\overline{d} = \overline{x}/\overline{q}$. This completes the proof.

B. Proof of $p' > 0$ at Point C in Figure 1

First note that the debt limit that matters for p' at point C in Figure 1 is $\overline{x}(d')$ and not $\overline{x}(d^{**})$. Therefore, it suffices for the proof to show that $d' > d^{**}$ because $\overline{x}(d') < x^U(d')$ if $d' > d^{**}$ as can be seen from Figure 1. The relations in (16) indicate that

$$d' > d \quad \Leftrightarrow \quad q^{-1}x > \delta d \tag{A.3}$$

At point C, $d = d^{**}$ and $x = \overline{x}(d^{**}) = q[\overline{d} - (1 - \delta)d^{**}]$ where the last expression follows from (20). Substituting these values into (A.3) yields,

$$q^{-1}\overline{x}(d^{**}) = (\overline{d} - d^{**}) + \delta d^{**} > \delta d^{**}$$

This completes the proof.

C. Proof of $\overline{x}_A(\delta) > \overline{x}(1)$

We use for clarity the notation $z(\delta)$ and $z(1)$ to denote variables associated with long-duration and one-period bonds, respectively. The result in (27) indicates that it suffices for the proof to show that $\overline{x}(\delta)/\delta > \overline{x}(1)$. To this end, let us compare $\overline{Z}$ with $\overline{H}$:

$$\begin{aligned} \overline{H} &= [(r + \overline{p}'(1))/(1 - \overline{p}'(1))]\overline{x}(1) - s^* \\ \overline{Z} &= [(1 - \delta\overline{q}(\delta))/\overline{q}(\delta)](\overline{x}(\delta)/\delta) - s^* \end{aligned} \tag{A.4}$$

where $\overline{q}(\delta) = [(1 - \overline{p}'(\delta))/(1 + r)][1 + (1 - \delta)\overline{q}^e(\delta)]$. It is straightforward to show that

$$\partial\overline{Z}/\partial\overline{x}(\delta) > 0 \tag{A.5}$$

Since $\overline{q}^{e}(\delta) > \overline{q}(\delta)$ must hold in equilibrium as discussed in Section IV, we posit that $\overline{q}^{e}(\delta) = (1+\alpha)\overline{q}(\delta),\ \alpha > 0$ with the understanding that α depends on $\overline{q}(\delta)$ and $\overline{x}(\delta)$ in equilibrium. Then,

(A.6) $$\overline{q}(\delta) = (1-\overline{p}'(\delta))/[(1+r)-\lambda(1-\overline{p}'(\delta))],\quad \lambda = (1-\delta)(1+\alpha)$$

Substituting (A.6) in $\overline{Z}$ in (A.4) yields,

(A.7) $$\overline{Z} = [(r+\overline{p}'(\delta))/(1-\overline{p}'(\delta))](\overline{x}(\delta)/\delta) - s^{*} - \alpha(1-\delta)(\overline{x}(\delta)/\delta)$$

Comparing (A.7) with $\overline{H}$ in (A.4) immediately suggests that if $\alpha = 0$, the maximum interior solution to the fixed-point problem $\overline{p}' = G(\overline{Z})$ is characterized by:

(A.8) $$\overline{x}(\delta)/\delta = \overline{x}(1) \quad \text{and} \quad \overline{p}'(\delta) = \overline{p}'(1)$$

Since $\partial\overline{Z}/\partial\alpha < 0$, it readily follows that if $\alpha > 0$,

(A.9) $$\overline{p}'(\delta) > G(\overline{Z}) \ \text{ at } \{\overline{x}(\delta)/\delta,\ \overline{p}'(\delta)\} = \{\overline{x}(1),\ \overline{p}'(1)\}$$

This result, together with (A.5), implies that $\overline{x}(\delta)/\delta > \overline{x}(1)$ must hold in order for $\overline{x}(\delta)$ to be the maximum interior solution to the fixed-point problem, $\overline{p}'(\delta) = G(\overline{Z})$. This completes the proof.

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