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Monetary Transaction Costs and the Term Premium

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Abstract

We show that, in a monetary equilibrium, trade and asset prices depend on both the supply of the liquidity by the Central Bank and the liquidity of assets and commodities. As a result, monetary aggregates are informative for the conduct of monetary policy. We also show asset prices are higher in liquidity-constrained states of nature. This generates a term premium even in absence of aggregate uncertainty. These results hold in any monetary economy with heterogeneous agents and short-term liquidity effects, where monetary costs act as transaction costs and the quantity theory of money is verified.

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I. INTRODUCTION

In the worst months of the 2008-2009 financial crisis, as market participants worried about asset valuation and counterparty risk, interbank markets froze, many assets became very illiquid, and the supply of liquidity was left in the hands of the monetary authorities. The unconventional actions of central banks, including virtually unlimited supply of amounts of cash, contributed to restoring market liquidity and asset prices. This episode reminds us that market liquidity should not be taken for granted and that money supply has many more effects beyond the narrow scope of the determination of short-term policy rates.

We argue in this paper that since liquidity affects trade activity, marginal utilities and state prices, asset prices are higher in liquidity-constrained states of nature. Equivalently, states of nature with a 'tighter' liquidity constraint receive a greater weight under the risk-neutral measure. As a result, the correlation between transaction costs — which in our monetary model are related to the short-term interest rate (we call these monetary costs) — and asset payoffs creates risk-premia in asset prices. Unlike in representative agent models, this premium exists even in absence of aggregate uncertainty (*i.e.*, when the aggregate endowment is not state-dependent), and therefore causes the term structure to lie above levels predicted using the stochastic discount factor (as done by e.g. Backus *et al.* (1989) or Grossman *et al.* (1987)). In our model, the risk-premia are due to the funding transaction costs that affect individuals' marginal utilities, though not the aggregated representative agent's utility. We therefore provide a potential explanation for the Term Premium Puzzle.

Unlike in the precautionary savings model of Weil (1992) and Aiyagari (1994), the additional risk premium exists for any additively separable utility function. In particular, this risk premium does not depend on the third derivative; it is due to the correlation between the source of market imperfection and the asset prices. Unlike in Elul (1997), who showed that unspecified market incompleteness can either increase or decrease interest rates in a generic Radner's (1972) equilibrium with incomplete markets, we restrict the type of market imperfection to the one that we think is likely to matter for interest rates, namely monetary transaction costs. Our focus is therefore on the interaction between the monetary structure and asset prices.

We set out a monetary equilibrium with cash-in-advance constraints built along the lines of Dubey and Geanakoplos (1992, 2003). We assume asset markets are complete with respect to real uncertainty, but monetary transaction costs cannot be hedged. The model transcends representative agent models in several ways. First, trade activity is endogenous and is affected by liquidity. The monetary costs therefore generate a strong correlation between the spot interest rates and trade (and therefore asset prices). Bansal and Coleman II (1996) also produced a general equilibrium where transaction costs had an effect on bond prices, however, in their representative agent model, trade was forced, since the representative agent sold all of her endowments and subsequently bought it back. Furthermore, the transaction technology was exogenously specified and transaction services were generated only from bond holdings and not from asset holdings. Lagos (2010) extends Lagos and Wright (2005)'s search model of money demand to endogenously derive the usefulness of equity shares and government bonds as means of payment — this usefulness becomes a function of the intrinsic properties of the two assets such as the frequency of trade or the terms of trade. Lagos (2010) then shows that this concept of liquidity is helpful to explain the equity premium and risk-free rate puzzles. Other papers such as Fan (2006) or Chabi-Yo *et al.* (2007) show how un-insurable risk and agent heterogeneity add an additional factor in pricing models. Our research can be related to these papers because these models rely on transaction costs which add a un-insurable risk that affects asset prices.

We also argue that liquidity, because it is a broad concept, can not be fully described by the supply of money from the Central Bank. The word 'liquidity' corresponds to two different concepts in this paper. First, there is the supply of money by the Central Bank, or by the banking system in general. Second, liquidity refers to the easiness with which commodities or assets are traded. Extending Grandmont and Younes (1972), we assume in the cash-in-advance constraints that a share λ of the receipts of an agent's sales can be used simultaneously to purchase assets or commodities.

The cash-in-advance constraint and the liquidity parameters are not micro-founded and do not attempt to mimic precisely the functioning of markets. However, their only objective is to replicate intermediate results that need to be in place in any monetary model to discuss monetary transaction costs and asset prices. Our main propositions hold generally, independently of the model specificities that lead to these intermediate steps.

The first intermediate result is nominal determinacy. As we will discuss, the cash-in-advance constraint and the inclusion of outside money as endowments to households generate a demand for money and ensure nominal determinacy. Money demand is downward sloping and a function of the relative liquidity of assets and commodities, or the relative illiquidity of money with respect to assets and commodities, as in Keynes' money demand.

The second intermediate result is that higher monetary costs worsen the consequences of market imperfections even in pure exchange economies. In our model, the wedge between buying and selling prices proxies for inefficiencies in the functioning of markets. In markets that are dysfunctional, sellers are unable to find enough buyers at the equilibrium price. For instance, asymmetric information, sales costs, regulations, result in unsold houses and unemployment. The liquidity parameters and the cash in advance constraint summarize the extent to which a commodity or an asset is difficult to sell. For instance, one interpretation of the liquidity parameter is that it captures the average period needed to sell a house or to find a job. This parameter is exogenous in our model, but it is state-contingent as well as commodity/asset-specific. With this interpretation, the extent of the inefficiency is an increasing function of the time period for which the house is unsold or the worker unemployed, as well as an increasing function of the cost of time (the interest rate). When these two intermediate results hold, we show that asset prices are higher for assets that pay in states of nature in which imperfections are larger and monetary costs higher. This result implies that the yield curve contains a term premium, even in absence of aggregate uncertainty.

In our model the Central Bank policy has no effect on the liquidity of endowments, although it does affect the spot interest rate and therefore the funding cost. This assumption allows us to clearly separate the effects of liquidity that are channeled through interest rates from the direct effects of liquidity on the way trade is conducted. Demand for money becomes a function of two parameters: the spot interest rate and the liquidity of endowments. Consequently, the model is able to show how trade and asset prices depend on liquidity. This result would hold in any monetary economy where short-run liquidity effects exist, and monetary costs act as transaction costs. Furthermore, the model suggests that monetary aggregates do provide additional information, not included in interest rates, on economic activity, inflation and asset prices. Hence, our economy has a distinct monetary flavour that cannot be reproduced in representative agent models.

II. The Baseline Model

The model is an infinite horizon exchange economy with money. Real uncertainty can be hedged, but monetary transactions costs are un-insurable. Exchanges take place between two agents who want to trade across periods (to smooth consumption) and across states (for insurance purposes).

A. Structure of the Model

The model consists of two types of infinitely-lived representative agents, α and β. At time 0, agents α and β maximise

$$U_i(c^i) \equiv \mathbb{E}_0\left[\sum_{t=0}^{\infty} \delta^t u_t^i(c_t^i)\right] \ , \ i \in \{\alpha, \beta\} \tag{1}$$

with $(u_t^i)' > 0$ and $(u_t^i)'' < 0$. For any time period t, the set of possible states of nature is described by the filtration $\mathcal{F}_t$. The probabilities of the state s occurring at time t are $\pi_{t,s}$. Initially, we assume these probabilities are the same for agent α and β, but this assumption is relaxed later on (i.e. our results hold even when the 'subjective' probabilities used by α and β differ). Without loss of generality, we set $\delta = 1$. The two agents consume and trade only one commodity (this assumption will be relaxed in section A). Agents are endowed with $e_{t,s}^i \ , \ s \in \mathcal{F}_t, \ i \in \{\alpha, \beta\}$ and with monetary endowments $m_t^i, \ i \in \{\alpha, \beta\}$, the meaning of which will be discussed in section III.[2]

Since transactions are costly (because of funding costs) and since both agents only consume one commodity, α and β never buy and sell at the same time. As a result, only one agent pays the transaction cost. It is thus important to know who is buying and selling in each period and each state of nature. For the sake of clarity, we assume that, at any given time indexed by an even number ($t = 2z$, $z \geq 0$), agent β holds positive endowments of the commodity in all the states of nature and agent α's endowment is null in all states, whilst at any time$t = 2z + 1$,

[2] Without loss of generality, we assume that these monetary endowments are non-stochastic.

agent α is rich and and agent β has no endowment. This assumption implies that we only need to focus on consecutive time periods when the selling/buying roles of agent α and β are changing . If α is buying in period t, she will be selling in period $t+1$, unless she wants to roll over her loans. If she were to roll over her loans (i.e. her position in the Arrow securities market), the cash-in-advance constraint would not be binding, and the model would collapse to a model without liquidity constraint, in which standard asset pricing formulas apply. In other words, we defined the periods t as those intervals of time at which α and β change roles and the liquidity constraint is binding. A period could, for example, correspond to the years of the housing crisis (in which housing sales collapsed, if the focus is on commodity trade) or to the months immediately following the Lehman collapse (if the focus is on asset illiquidity).

The quantity traded by agent i in time t and state s is denoted by $q^i_{t,s}$. Hence, during periods $t = 2z$ when α is poor and β rich, consumption of agent α is $c^\alpha_{t,s} = q^\alpha_{t,s}$ and consumption of agent β is $c^\beta_{t,s} = e^\beta_{t,s} - q^\beta_{t,s}$. The reverse is true for periods $t = 2z + 1$ when α is rich and β poor. $p_{t,s}$ is the commodity price at time t, state s.

The number of Arrow-Debreu (AD) securities is equal to the *maximum* number of states succeeding any node in the state space. This ensures that all real uncertainty can be hedged. Figure 1 summarizes the endowments and financial markets available for any two periods t, $t+1$, where we consider state k of period t as the initial node. Each period is divided into sub-periods during which the commodity and money markets meet, as pictured in Figure 2. This is the core of the cash-in-advance model. In a nutshell, agents cannot use the receipts of their sales to purchase commodities and securities, because of illiquidity in the goods market (represented by the fact that these same receipts arrive late, at the end of the period). Hence, in anticipation of receipts from sales, agents borrow in the short-term money market (intra-period market) and repay their debt when the cash from sales arrives at then end of each period. As they take this into account in their inter-temporal decisions, the state prices will be a function of their expected funding costs. We will explain below exactly what each agent does in the different sub-periods.

The Central Bank provides money in all money markets: $\forall s \in \mathcal{F}_t,\ t \geq 0$, in the short-term money market in period t and state s, money supply is $M_{t,s}$ and the short-term interest rate is $r_{t,s}$. The money supplies in the different states of nature are exogenous. Although the interest rate in each of these money markets will be determined by demand and supply, the form of the model ensures that the interest rates are inversely proportional to the Central Bank money supply. In addition to these money markets, the two agents can trade Arrow-Debreu (AD) securities defined at time t state k to cover all states of nature in the next period: $(AD_{t,k,s})_{s\in\mathcal{F}_{t+1}}$. The Arrow securities are nominal : the security AD_s pays 1 unit of account in state s and 0 in all other states $s' \neq s$. All Arrow securities are in zero net supply.

Figure 1. Time and uncertainty structure of the model

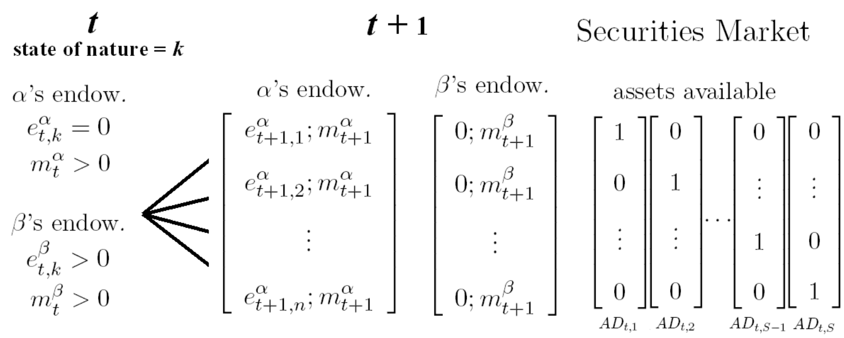
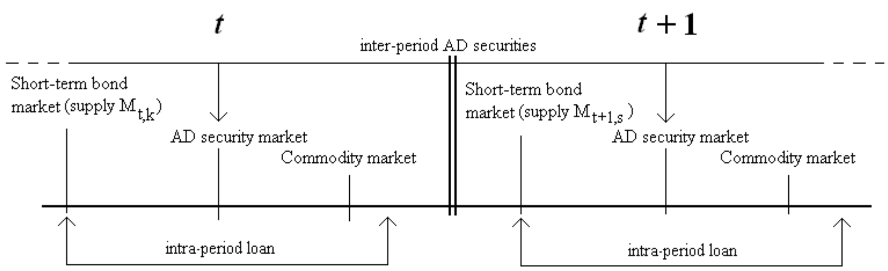


Figure 2. Timing of commodity, asset and money markets



B. Budget Set for Agent α

To minimise notation, when time subscripts are suppressed, subscript k will always mean states of nature at times t where α buys goods (for instance time periods indexed by an even number), while subscript s will refer to states of nature at period $t+1$ where she sells goods (time periods indexed by an odd number). Subscripts j for time $t-1$ and subscripts l for time $t+2$ will also be needed. We assume that at period t and state k, agent α is endowed with goods $e^{\alpha}_{t,k}=0$, while at period $t+1$, $e^{\alpha}_{t+1,s}>0$.

In period t **state** k, agent α 'borrows' with repayment conditional on the future states to finance consumption at time t, i.e. she sells $h^{\alpha}_{t,s}$ Arrow-Debreu securities at price θ_s, and hence receive income $\theta_s h^{\alpha}_{t,s}$. She also receives income from her previous holdings of AD securities, h^{α}_{k}, and she is endowed with money m^{α}_{t}. Note also that agent α will not borrow in period t money market since she will not have any income to repay her loan.

In period $t+1$ **state** s, agent α's issues of AD security reach maturity and agent α has to service $h^{\alpha}_{t,s}$ to agent β who — as we will see later — had bought the AD securities. She may also want to invest again for the next period, by buying new AD securities $\sum_{l\in\mathfrak{F}_{t+2}} h^{\alpha}_{l}$. Since α cannot yet use the receipts of goods sold in state s because the goods market meets after financial markets, she has to borrow short-term (*i.e.* roll over her debt), at cost $1/(1+r_s)$. She thus borrows $\mu^{\alpha}_{s}/(1+r_s)$, and uses this amount with her money endowment m^{α}_{s} to pay agent β on the AD_s market and to invest for the next period. She then uses the receipts of her state-s sales ($p_s q^{\alpha}_{s}$) to repay state-s loan μ^{α}_{s}.

Agent α's maximisation programme is therefore (in brackets are the Lagrangian multipliers):[3]

$$\max_{q^{\alpha}_{k},q^{\alpha}_{s},\mu^{\alpha}_{s},h^{\alpha}_{s},h_{l^{\alpha}}} \quad \sum_{t=0}^{\infty}\left[\sum_{k\in\mathfrak{F}_t}\pi_{t,k}u_{\alpha}\left(q^{\alpha}_{t,k}\right)+\sum_{s\in\mathfrak{F}_{t+1}}\pi_{t+1,s}u_{\alpha}(e^{\alpha}_{t+1,s}-q^{\alpha}_{t+1,s})\right] \quad s.t. \tag{2}$$

$$\forall t,\forall k\in\mathfrak{F}_t,\quad p_{t,k}q^{\alpha}_{t,k}=m^{\alpha}_{t}+h^{\alpha}_{k}+\sum_{s\in\mathfrak{F}_{t+1}}\theta_s h^{\alpha}_{s} \qquad (\varphi^{\alpha}_{t,k}) \tag{3}$$

(purchases of goods $=$ outside money + asset paying off + asset sales)

$$\forall t,\forall s\in\mathfrak{F}_{t+1},\quad h^{\alpha}_{s}+\sum_{l\in\mathfrak{F}_{t+2}}\theta_l h^{\alpha}_{l}=\frac{\mu^{\alpha}_{s}}{1+r_s}+m^{\alpha}_{t+1} \qquad (\Psi^{\alpha}_{t+1,s}) \tag{4}$$

(servicing AD security+asset purchases $=$ short-term borrowing + outside money)

$$\mu^{\alpha}_{s}=p_s q^{\alpha}_{s} \qquad (\chi^{\alpha}_{t+1,s}) \tag{5}$$

(short-term debt repayment $=$ income from selling goods)

[3] We do not write explicitly in the budget constraints that she can carry money over from period t to period $t+1$ because we focus on interior equilibria where liquidity constraints are binding. Otherwise the value for money is zero, which implies that transaction costs become null and the standard pricing formulas apply.

C. Budget Set for Agent β

We assume that agent β is endowed with goods at times t indexed by even numbers $e^{\beta}_{t,k} > 0$ while she receives no endowment of goods at times $t+1 : \forall s,\ e^{\beta}_{s} = 0$. The maximisation problem follows the same logic than for agent α:

$$\max_{q^{\beta}_{k}, q^{\beta}_{s}, \mu^{\beta}_{k}, h^{\beta}_{s}, h^{\beta}_{l}} \sum_{t=0}^{\infty} \left[\sum_{k \in \mathfrak{F}_t} \pi_{t,k} u_{\beta}(e^{\beta}_{t,k} - q^{\beta}_{t,k}) + \sum_{s \in \mathfrak{F}_{t+1}} \pi_{t+1,s} u_{\beta}\left(q^{\beta}_{s}\right) \right] \quad s.t. \tag{6}$$

$$\forall t, \forall k \in \mathfrak{F}_t,\ h^{\beta}_{t,k} + \sum_{s \in \mathfrak{F}_{t+1}} \theta_s h^{\beta}_{s} = m^{\beta}_{t} + \frac{\mu^{\beta}_{t,k}}{1 + r_{t,k}} \qquad (\Psi^{\beta}_{t,k}) \tag{7}$$

(servicing AD security + asset purchases $=$ outside money + short-term borrowing)

$$\mu^{\beta}_{k} = p_k q^{\beta}_{k} \qquad (\chi^{\beta}_{t,k}) \tag{8}$$

(short-term debt repayment $=$ income from selling goods)

$$\forall t, \forall s \in \mathfrak{F}_{t+1}, \qquad p_s q^{\beta}_{s} = m^{\beta}_{s+1} + h^{\beta}_{s} + \sum_{l \in \mathfrak{F}_{t+2}} \theta_l h^{\beta}_{l} \qquad (\varphi^{\beta}_{t+1,s}) \tag{9}$$

(purchases of goods $=$ outside money + servicing asset + asset purchases)

III. Monetary Equilibrium

There are many versions of cash-in-advance models in the monetary theory literature (e.g. Lucas and Stokey, 1983 and 1987; Svensson, 1985; Bloise, Dreze and Polemarchakis, 2005). We follow in this paper the model of Dubey and Geanakoplos (1992) in which, to ensure a positive nominal interest rate, a sufficient requirement is that agents hold some exogenous monetary endowment (sometimes called 'outside money'). If the aggregate private monetary endowment is m, then in a one-period version of the model the short-term nominal interest rate is $r = \frac{m}{M}$, where M is the supply of money by the Central Bank. Hence a short-term liquidity effect is obtained in a very simple way since r is decreasing in M. Dubey and Geanakoplos (2006), following Gurley and Shaw's (1960) initial statement of the difference between inside and outside money, argue that outside money is indeed a reality: for instance, when money is printed by the government to purchase real assets, commodities, or to pay for labour, it gives money to the private sector free and clear of any liability, and independently from Central Bank lending. Thinking about outside money m as government expenditure and seeing interest payments rM as revenue of the Central Bank (and therefore of the general government) makes also clearer the link with the non-Ricardian fiscal theory of prices. It is well known that non-Ricardian fiscal policy suffices to ensure nominal determinacy (Woodford, 1996). The inside-outside money formulation, indeed, captures this. Although the budget is balanced in equilibrium ($m = rM$), outside of equilibrium, with an interest rate r and prices p off the equilibrium values, $m \neq rM$ and the budget is not balanced. One can also think of outside money as a compact simplification for a more general nominal friction that pins down the price of money. In any case, as mentioned in the introduction, what is important for our model is the existence of a liquidity effect, not its origin.

Positive interest rates generate both nominal determinacy and money non-neutrality.[4] Indeed, any model with a monetary transaction technology removes nominal indeterminacy because scaling prices up or down necessarily requires some monetary injections or withdrawals which will, in turn, alter interest rates.[5] Hence, for a given interest rate there can be only one price vector. Second, nominal determinacy implies monetary non-neutrality because any change in the money supply requires a change in the interest rate, which will change equilibrium allocations if money is needed for transactions.

Lemma 1 shows that short-term interest rates are a declining function of the money supply (the liquidity effect). A positive value of money ensures that the demand for money is determined and this pins down the nominal value of trade (Lemma 2). Because monetary costs matter for transactions, the allocations are affected by the short-term interest rate (Proposition 1). As a result, marginal utilities, and therefore asset prices, are a function of liquidity (Proposition 2). The proofs are relegated to Appendix 1.

Lemma 1: Value of Money
In an interior equilibrium,

$$\forall t,\ \forall k \in \mathfrak{F}_t, \quad r_k M_k = m_t^\alpha + m_t^\beta$$

Lemma 2: Quantity Theory of Money

$$\forall t = 2z,\ \forall k \in \mathfrak{F}_t\ \ p_k q_k^\beta = p_k q_k^\alpha = m_t^\alpha + m_t^\beta + M_k$$

$$\text{and}\ \ \forall s \in \mathfrak{F}_{t+1}\ \ p_s q_s^\alpha = p_s q_s^\beta = m_{t+1}^\alpha + m_{t+1}^\beta + M_s$$

Agent α is the only agent who needs cash in period t, state k, and she only needs it to buy $q_{t,k}$. Therefore, she will use all available cash (her cash and the cash supplied by agent β, $m_t^\beta + M_{t,k}$, via the AD securities) to buy $q_{t,k}^\alpha$. This implies the quantity theory of money for state k, since q_k^α represents all trade in this state of nature. It should be noted that this is a non-trivial quantity theory of money since nominal changes affect both rices and quantities, as we show now.

The propositions below are proved for times $t = 2z + 1$ where agent α is selling goods, but since the model is symmetric, the proof is identical for times $t = 2z$ where agent β is selling.

[4] The existence of an equilibrium with positive value for money is guaranteed in our model since the endowment configuration makes it very important to trade, and therefore money demand is positive even with positive interest rates (see Dubey and Geanakoplos, 1992, 2006).

[5] The only exception is when interest rates are equal to zero. Then changing prices and money supply is tantamount to changing units of account while maintaining zero interest rates.

q_s^α should therefore be understood more generally as the volume of trade.

Proposition 1: Non-Neutrality of Money

In an interior equilibrium, without aggregate uncertainty, states with higher interest rates are those where trade is lower (*i.e.* $r_s > r_{s'} \Longrightarrow q_s^\alpha < q_{s'}^\alpha \quad \forall s, s' \in \mathcal{F}_{t+1},\ s \neq s'$).

What are the consequences of this result for asset prices? Proposition 2 shows that if agent β's relative risk-aversion (RRA) is greater than or equal to 1, states with higher interest rates see their state prices biased upwards.

Proposition 2: Asset Prices

Assume $\forall t = 2z, \forall x \in \mathcal{F}_{t+1}$,
and $\forall q > 0, \quad RRA^\beta(q) = \frac{-u''_\beta(q)q}{u'_\beta(q)} \geq 1$.
Then $\forall s, s' \in \mathcal{F}_{t+1}, \quad s \neq s',\ r_{s'} > r_s \quad \Longrightarrow \frac{\theta_{s'}}{\pi_{s'}} > \frac{\theta_s}{\pi_s}$.

We can also restate this result in terms of risk-neutral probabilities $\hat{\pi}_s = \frac{\theta_s}{\sum_{j \in \mathcal{F}_{t+1}} \theta_j}$

$$\forall s \neq s'\ r_{s'} > r_s \Longleftrightarrow \frac{\hat{\pi}_{s'}}{\pi_{s'}} > \frac{\hat{\pi}_s}{\pi_s}$$

This is the first important result of our model. Even in absence of aggregate uncertainty, states with higher interest rates are given higher weights. This provides a possible explanation for the Term Premium Puzzle. As in Aiyagari (1994) or Weil (1992), the risk premium is due to the existence of un-insurable risk. However, the term premium we propose exists for any concave utility function, independently of its third derivative. The term premium is here a consequence of the correlation between short-term interest rates and allocations (which are affected by the monetary transaction costs). The propositions have been written assuming no aggregate uncertainty and assuming the probabilities π of the borrower and the lender are equal. Nevertheless, applying a continuity argument, we can show that these general results hold locally, *i.e.* if the 'subjective' probabilities or the endowments of α and β differ by an infinitesimal quantity (Proposition 5 in Appendix 1). We can also carry the results with non-infinitesimal differences assuming that preferences are given by CRRA utility with the same coefficient of risk-aversion for both agents (Example 1).

Example 1: CRRA Example

If agents α and β preferences are given by a CRRA utility function with the same constant coefficient of risk-aversion ρ, then $\forall s, s' \in \mathcal{F}_{t+1}$

$$\frac{q_s^\alpha / e_s^\alpha}{q_{s'}^\alpha / e_{s'}^\alpha} = \frac{(\frac{\pi_{s'}^\alpha}{\pi_{s'}^\beta}\frac{1+r_{s'}}{1+r_k})^{1/\rho} + \frac{e_k^\beta}{q_k^\beta} - 1}{(\frac{\pi_s^\alpha}{\pi_s^\beta}\frac{1+r_s}{1+r_k})^{1/\rho} + \frac{e_k^\beta}{q_k^\beta} - 1} \tag{10}$$

and

$$\frac{\theta_s}{\theta_{s'}} = \frac{\pi_s^\beta}{\pi_s^\beta}\left(\frac{e_{s'}^\alpha}{e_s^\alpha}\right)^{\rho-1}\frac{M_{s'} + m_{t+1}^\alpha + m_{t+1}^\beta}{M_s + m_{t+1}^\alpha + m_{t+1}^\beta}\left(\frac{\left(\frac{\pi_s^\alpha}{\pi_s^\beta}\frac{1+r_s}{1+r_k}\right)^{\frac{1}{\rho}} + \frac{e_k^\beta}{q_k^\beta} - 1}{\left(\frac{\pi_{s'}^\alpha}{\pi_{s'}^\beta}\frac{1+r_s}{1+r_k}\right)^{\frac{1}{\rho}} + \frac{e_k^\beta}{q_k^\beta} - 1}\right)^{\rho-1} \tag{11}$$

Example 1 allows us to summarize the core of the model in only two equations, however, at the expense of forcing risk aversion to be the same for the two agents. The first result tells us that the proportion of endowments traded is greater the lower the interest rate. This result is therefore akin to Proposition 1 in the case of aggregate uncertainty. However, in this general framework with different subjective probabilities, the relative importance of state s for agent α and β also matters. For example, if β (who is buying the AD securities) gives more weight (relatively to α) to state s, there will be more trade in state s.

The second equation provides the intuition behind Proposition 2. The first two terms are characteristic of any asset pricing Euler equation. If subjective probabilities (of the asset buyer) are higher for state s, this will increase the state price. Furthermore, if aggregate endowment is lower in state s, this will also increase the state price because consumption is also lower. The last two terms are, however, special to our model. The ratio of money supplies matters because the state prices are prices of assets the payoffs of which are set in nominal terms. Since a higher money supply implies a higher price level (Lemma 2), the value of any asset today is a decreasing function of next period's price level. Finally, the last term is the trade effect. A higher spot interest rate r_s tends to lower trade activity and therefore lowers consumption of the agent who buys in the future (*i.e.* agent β). But agent β is also the one who is buying the AD securities, and since she is willing to pay a higher price for assets which pay off when her consumption is low, she is therefore ready to pay more for AD-securities that pay off when interest rates are high. Hence, states with higher interest rates also have higher state prices.

This result is not an application of the risk-premium found in pure exchange general equilibrium models with heterogeneous agents or in a representative agent model (Lucas, 1978; Breeden, 1979 ; Backus *et al.*, 1989). Indeed, even when the endowment risk-premium has been removed as in Proposition 1, state prices in Proposition 2 are still a function of money. The CRRA example clarifies that the additional risk-premium comes from the effect of money on trade and therefore on β's marginal utilities.

The upshot of our argument is that uncertainty in aggregate production or in aggregate consumption is only one part of uncertainty in agents' marginal utilities. Transaction costs also generate variability of marginal utilities (and thus of asset demands) in the future. Therefore, any model of risk-premium that attempts to proxy welfare by production or consumption will underestimate the risk-premium. This is especially important for the term-structure risk premium since the spot interest rate has an effect on both the asset price and the transaction cost, implying, in this simplified model, a perfect correlation between the marginal utilities and asset prices.

IV. Liquidity and the Term Structure of Interest Rates

The model can be applied to show the existence of a *liquidity-risk* term premium. The liquidity premium is due to the additional costs incurred by investors that an uncertain money supply generates when liquidity is constrained.[6] We give an example in a fully solved one-commodity, three-period economy with logarithmic utility, and the time structure depicted in Figure 3.

There is no uncertainty in the first two periods and only in the third period there are S possible money supplies. The model is solved for the prices of the 1-period and 2 period-bonds ${}_0P_1$ and ${}_1P_2$, and hence for the implied rates ${}_0r_1$ and ${}_1r_2$. We note $card(\mathcal{F}_2) = S$ and assume subjective probabilities across both agents are equal to $1/S$ for each state of nature. Example 2 presents the solution of the model (see Appendix 1 for a proof).

Example 2: Three-Period Example

Assume there are only three periods, with $\mathcal{F}_0 = \{0\}, \mathcal{F}_1 = \{1\}$ and $\mathcal{F}_2 = \{1 \dots S\}$. Assume furthermore that $u_\alpha = u_\beta = ln$, and that $\pi_s^\alpha = \pi_s^\beta = 1/S, \forall s \in \mathcal{F}_2$. In this case, the solution of the model is:

$$r_0 = \frac{m_0}{M_0}, \;\; r_1 = \frac{m_1}{M_1}, \;\; \forall s \;\; r_s = \frac{m_2}{M_s}$$

$$\theta_1 = \frac{2(M_0 + m_0) - \bar{M}}{M_1 + m_1}$$

$$\theta_s = \frac{M_1 + m_1}{S(m_2 + M_s)(2 - \bar{M}/(M_0 + m_0))}$$

where $\bar{M}$ is an exogenous inherited liability for agent α.[7]

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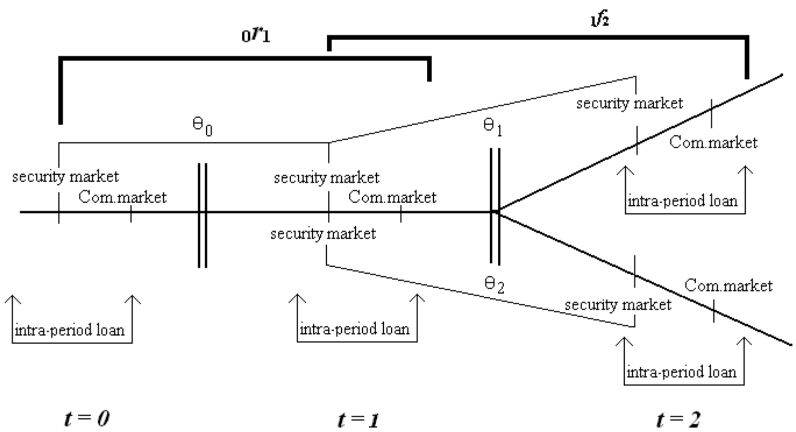
[6] Note that the level of money supply is irrelevant in the long run. If prices adjust to money supply, constraints on liquidity do not have real effects - although this is not captured in the cash-in-advance constraint where the optimal supply of money would be infinite. However, the variance (or risk) of liquidity still has effects. This is exactly what is captured in this model, where we show that larger liquidity risks generate higher long-term interest rates. *Stricto sensu*, this is a model of the liquidity-risk premium.

[7] We need one agent to be in debt initially so that the first period bond market be active for reasonable stable values of the money supply

Figure 3. Three period example



Using a no-arbitrage argument, the first inter-period bond has a price ${}_0P_1$ higher than $\theta_1 \frac{1}{1+r_1}$, since, if its price were lower, an agent in the model could buy such a bond, and sell bonds h_1 and borrow in the short-term money market μ_1. This way the agent would make safe profits. The rate of return ${}_0r_1$ is thus lower than the combined rate $\bar{r}_1 + r_1$, where $\bar{r}_1 = \frac{1}{\theta_1} - 1$. Had we allowed for an external speculator who is not limited by the liquidity constraints (*i.e.* does not consume in each period) and who can lend to the Central Bank, then using again a no-arbitrage argument, one could show that the rate ${}_0r_1$ is equal to the combined rate $\bar{r_1} + r_1$. Similarly, the price of a safe inter-temporal bond in the second period would be

$$ {}_1P_2 = \sum_{s \in \mathcal{F}_2} \theta_s \frac{1}{1+r_s} = \frac{1}{1+\bar{r_2}} \sum_{s \in \mathcal{F}_2} \hat{\pi}_s \frac{1}{1+r_s}, \text{ where } \frac{1}{1+\bar{r_2}} = \sum_{s \in \mathcal{F}_2} \theta_s. $$

A simple approximation yield ${}_1f_2 \approx \bar{r_2} + \sum_{s \in \mathcal{F}_2} \hat{\pi}_s(r_s) r_s$.

Since $\hat{\pi}_s$ is an increasing function of r_s, the model generates a risk premium even in absence of aggregate real uncertainty. Indeed, the forward interest rate $\bar{r}_2 + \mathbb{E}_{\hat{\pi}}[r_f]$ is above the expected future spot rate $\bar{r}_2 + \mathbb{E}_0[r_f]$. This is what we call a liquidity-risk term premium, since it is generated from uncertainty in the supply of liquidity from the Central Bank. The second implication of this result is that a larger variance in expected spot rates will generate a higher term premium. Consequently, stability of monetary policy matters in determining the equilibrium value of long-term interest rates.

V. EXTENSIONS

A. Multiple Commodities

An extension with multiple goods for additively separable utility functions is also possible. The main result of the paper obtains, assuming that the direction of trade of all goods is independent of the state of nature (in fact, a sufficient condition is that there exists one good that is always bought, either by agent α or β, in all states of nature, the price of which decreases when money supply decreases). The propositions and the proofs are left to Appendix 2.

B. An Extension of Cash-In-Advance Constraints

A common criticism of the cash-in-advance models is that the constraints used are *ad hoc* and do not adequately capture liquidity or collateral requirements. General specifications of cash-in-advance constraints can, however, partly answer this critique. If x_1 and x_2 are consumption levels in commodities 1 and 2, p_1 and p_2 are commodity prices, a general form of the cash-in-advance budget constraint is

$$ p_1x_1 + p_2x_2 \leq p_1\Lambda_1e_1 + p_2\Lambda_2e_2 $$

where e_1 and e_2 are commodity endowments and Λ_1 and Λ_2 liquidity parameters. For instance, if good 1 is the illiquid consumption good, and cannot serve for transaction, $\Lambda_1 = 0$. If good 2 is fiat money (*i.e.* a liquid commodity that cannot be consumed) then $\Lambda_2 = 1$ and $x_2 = 0$ (this is the specification we used until now). The main intuition of the generalised form of the cash-in-advance constraint is that the different commodities of the economy are not equally liquid, *i.e.* not all receipts from sales can be contemporaneously used for other purchases. In other words, trade is affected by the requirement of financial intermediation, the cost of which is proxied by interest rates. As long as some liquidity parameters for the commodity endowments are less than 1, money (or credit) demand is positive in order to bridge the gap between expenditures and receipts. In the previous sections of the paper, we set $\Lambda_1 = 0$ and $\Lambda_2 = 1$, following the modern treatment. In this section, however, we extend the framework in two ways: first, we follow Grandmont and Younes (1972) in their definition of liquidity of commodities; second we add a parameter λ capturing the illiquidity of assets.

C. Budget Set for Agent α

For ease of exposition, we restrict here our presentation to a two-period model. The maximisation problem faced by agent α is as follows.

$$\max_{q_0^\alpha,(q_s^\alpha,\mu_s^\alpha,h_s^\alpha)_{s\in\mathcal{F}}} u_\alpha(q_0^\alpha) + \sum_{s\in\mathcal{F}} \pi_s^\alpha u_\alpha(e_s^\alpha - q_s^\alpha) \tag{12}$$

$$\begin{aligned}
&\textit{s.t.} & p_0 q_0^\alpha &= \sum_{s\in\mathcal{F}} \theta_s h_s^\alpha & (\varphi_0^\alpha) \qquad (13)\\
&\forall s \in \mathcal{F} & h_s^\alpha &= \Lambda_s p_s q_s^\alpha + \frac{\mu_s^\alpha}{1+r_s} & (\Psi_s^\alpha) \qquad (14)\\
& & \mu_s^\alpha &= (1-\Lambda_s) p_s q_s^\alpha & (\chi_s^\alpha) \qquad (15)
\end{aligned}$$

The budget constraints, and in particular equations (14) and (15), imply that a share $1 - \Lambda_s$ of commodity q_s sales receipts cannot be used to clear asset payments due to β. As a result, α has to borrow to the Central Bank $\mu_s^\alpha/(1 + r_s)$.

D. Budget Set for Agent β

The maximisation problem is very similar for agent β at time 0. Moreover, in the next period, agent β cannot use immediately the totality of her asset payments to buy commodities, and therefore needs to borrow to the Central Bank $\mu_s^\beta/(1 + r_s)$.

$$\max_{q_0^\beta,(q_s^\beta,h_s^\beta,\mu_0^\beta)_{s\in\mathcal{F}}} u_\beta(e_0^\beta - q_0^\beta) + \sum_{s\in\mathcal{F}} \pi_s^\beta u_\beta(q_s^\beta) \tag{16}$$

$$s.t. \quad \sum_{s\in\mathcal{F}} \theta_s h_s^\beta = \Lambda_0 p_0 q_0^\beta + \frac{\mu_0^\beta}{1+r_0} + m_0^\beta \qquad (\varphi_0^\beta) \tag{17}$$

$$\mu_0^\beta = (1-\Lambda_0) p_0 q_0^\beta \qquad (\chi_0^\beta) \tag{18}$$

$$\forall s \in \mathcal{F} \;\; p_s q_s^\beta = \frac{\mu_s^\beta}{1+r_s} + \lambda_s h_s^\beta + m_s^\beta \qquad (\Psi_s^\beta) \tag{19}$$

$$\mu_s^\beta = (1-\lambda_s) h_s^\beta \qquad (\chi_s^\beta) \tag{20}$$

The illiquidity of assets is modeled using the parameter λ_s, which indicates in equation (19) that only a fraction λ_s of the assets payoff h_s^β can be used to pay for the commodity in state s (the cost of which is $p_s q_s^\beta$). The definition of a Monetary Equilibrium is as usual. The following lemmas (see Appendix 3 for the proofs), show how the liquidity parameters matter for the effective quantity of money and asset prices, although they have no effect on the short-term interest rate. This characteristic of the model allows us to disentangle the effect of 'liquidity' (money supply) due to Central Bank policy from the effect of the intrinsic liquidity of assets/commodities.

Lemma 3: Short-Term Interest Rates

$$\forall s \in \mathcal{F} \cup \{0\} \;\; r_s = \frac{m_s^\beta}{M_s}$$

Lemma 4: Quantity theory of money and Liquidity

$$p_0 q_0 = \frac{M_0 + m_0^\beta}{1-\Lambda_0}$$

$$\forall s \in \mathcal{F} \quad p_s q_s = \frac{M_s + m_s^\beta}{1 - \Lambda_s + (1-\lambda_s)(\Lambda_s + \frac{1-\Lambda_s}{1+r_s})}$$

Lemma 4 mirrors the canonical Quantity Theory of Money $MV = PQ$. In its canonical formulation due to Irving Fisher (2011), the velocity of money V denotes the number of times that money is used for transactions within the period in consideration. In our model, V is implicitly set to one, as only one trade in commodities or asset is allowed per period. However, the parameters λ and Λ capture the fact that money is used in an economy where assets or commodities cannot be sold/liquidated instantaneously. The relative illiquidity of commodities and assets magnifies the importance of financing costs (indeed, more liquid assets and commodities require less of the (costly) use of money). The following two propositions, that generalise the results from section III, illustrate that, when the risk that asset and commodities become illiquid is high (for a given distribution of interest rate), trade quantities and therefore state prices are volatile. This implies that the liquidity-risk term premium is a function of the risk that assets or commodities become illiquid.

Proposition 3: Non-Neutrality of Money and Liquidity

If $\forall s \in \mathcal{F} e_s^\alpha = e^\alpha$, then, $\forall s \neq s'$,

$$r_s(2 - \lambda_s - \Lambda_s) > r_{s'}(2 - \lambda_{s'} - \Lambda_{s'}) \Longrightarrow q_s < q_{s'}$$

Proposition 4: Asset Prices and Liquidity

Assume $\forall q, RRA(q) \geq 1$. *Ceteris paribus*,

$$r_s > r_{s'} \Longrightarrow \theta_s > \theta_{s'}$$

$$\lambda_s > \lambda_{s'} \Longrightarrow \theta_s < \theta_{s'}$$

$$\Lambda_s > \Lambda_{s'} \Longrightarrow \theta_s < \theta_{s'}$$

The previous propositions have a distinct monetary flavour: the path of the short-term interest rate by itself does not convey enough information to determine future trade and asset prices. The evolution of monetary aggregates (nominal income and money supply) is also important to assess the markets' need for liquidity (*i.e.* λs and Λs) and therefore the transmission of monetary policy to activity, inflation and asset prices. For instance, Λ_s, which is a determinant of trade activity, can be deduced from $p_s q_s$ and $M_s + m_s^\beta$ (Lemma 4) for a given λ_s.

VI. CONCLUSION

In states of nature with high interest rates, market imperfections that lead to unsold commodities or assets lower trade, and this affects state prices. In particular, risk-neutral probabilities are higher in states with higher interest rates (and in general with low liquidity). It is important to stress that this result is explained by the interaction of the monetary technology with the exchange economy and therefore cannot be found in a general equilibrium without money. Ultimately, it is the risk of variations in the supply of money and in the extent of market imperfections that matters to determine the risk in trade values. Liquidity shocks are crucial to understanding the upward sloping term structure because two phenomena push in the same direction: first, the futures spot interest rates are affected ; second, the risk-neutral probabilities are modified. The interaction of these two effects pushes long-term rates above the historical average of future spot rates, even in absence of aggregate real risk. That is, the more uncertainty in the future spot rates, the higher the long-term rates. Stability of monetary policy is therefore, required to maintain flat yield curves.

This connects to another subject of discussion in the current yield curve literature: the fact that the changes in the term structure can not be related systematically to changes in the fundamentals that matter according to the standard theory (inflation, inflation risk, macroeconomic volatility, risk-aversion, fiscal policy). Interpretations relating to the development of financial markets and liquidity risks are also needed. We argued in this paper that risk in the functioning of markets and in the liquidity of assets may generate term premia, and that these premia could explain changes in the yield curve of a larger magnitude than has been explained by non-monetary models. Proposition 4 also relates to a monetary view of the transmission mechanism. Short-term interest rates do not convey enough information to determine activity and asset prices. Monetary aggregates are also important to assess the markets' need for liquidity and therefore the prospects for trade activity, inflation and asset prices.

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APPENDIX 1

Proof of Lemma 1

For a period $t = 2z$ where β is borrowing in the short-term money market,

$$\forall k,\ \ (1+r_k)M_k = \mu_k^\beta = p_k q_k^\beta = p_k q_k^\alpha = m_t^\alpha + h_k^\alpha + \sum_{s\in\mathcal{F}_{t+1}} \theta_s h_s^\alpha$$
$$= m_t^\alpha + m_t^\beta + \frac{\mu_k^\beta}{1+r_k}$$

$$(1+r_k)M_k = m_t^\alpha + m_t^\beta + M_k$$

Note that no agent would carry money over from t to $t+1$ since she could have reduced her money demand, saved the interest payments and, hence, sold less of her endowment to repay her loan. The proof is similar for periods $t = 2z+1$ when agent α is borrowing on the short-term money market.

□

Proof of Lemma 2

We know, for $t = 2z$

$$p_k q_k^\alpha = m_t^\alpha + h_{t,k}^\alpha + \sum_{s\in\mathcal{F}_{t+1}} \theta_s h_s^\alpha \tag{21}$$

and, from the market clearing conditions

$$h_{t,k}^\alpha + \sum_{s\in\mathcal{F}_{t+1}} \theta_s h_s^\alpha = h_{t,k}^\beta + \sum_{s\in\mathcal{F}_{t+1}} \theta_s h_s^\beta = m_t^\beta + \frac{\mu_{t,k}^\beta}{1+r_{t,k}} \tag{22}$$

The result then follows, since $\frac{\mu_{t,k}^\beta}{1+r_{t,k}} = M_{t,k}$. The same proof applies for $t = 2z+1$.

□

Proof of Proposition 1

From the first-order conditions of agent α, $\forall s$

$$\pi_s u_\alpha'(e_s^\alpha - q_s^\alpha) = \frac{1}{1+r_s}\theta_s p_s \pi_k \frac{u_\alpha'(q_k^\alpha)}{p_k}$$

Hence, $\forall s \neq s' \in \mathcal{F}_t$

$$\pi_s \frac{u'_\alpha(e^\alpha_s - q^\alpha_s)}{p_s \frac{1}{1+r_s}\theta_s} = \pi_{s'} \frac{u'_\alpha(e^\alpha_{s'} - q^\alpha_{s'})}{p_{s'} \frac{1}{1+r'_s}\theta_{s'}} \tag{23}$$

Similarly, from the first-order conditions of agent β, $\forall s \in \mathcal{F}_{t+1}$

$$\pi_s \frac{u'_\beta(q^\beta_s)}{p_s} = \pi_k \frac{u'_\beta(e^\beta_k - q^\beta_k)/p_k}{1/(1+r_k)} \theta_s$$

Hence, $\forall s \neq s'$

$$\pi_s \frac{u'_\beta(q^\beta_s)}{p_s} \frac{1}{\theta_s} = \pi_{s'} \frac{u'_\beta(q^\beta_{s'})}{p_s} \frac{1}{\theta_{s'}} \tag{24}$$

Dividing equation (23) by equation (24), we find, $\forall s, s'$

$$\frac{u'_\alpha(e^\alpha_s - q^\alpha_s)}{u'_\beta(q^\beta_s)}(1+r_s) = \frac{u'_\alpha(e^\alpha_{s'} - q^\alpha_{s'})}{u'_\beta(q^\beta_{s'})}(1+r'_s)$$

or equivalently, using the market clearing conditions $\forall s,\ q^\alpha_s = q^\beta_s$

$$\frac{u'_\alpha(e^\alpha_s - q^\alpha_s)}{u'_\beta(q^\alpha_s)}(1+r_s) = \frac{u'_\alpha(e^\alpha_{s'} - q^\alpha_{s'})}{u'_\beta(q^\alpha_{s'})}(1+r_{s'}) \tag{25}$$

Assuming $\forall s\ \ e^\alpha_s = e$, (25) becomes

$$\frac{u'_\alpha(e - q^\alpha_s)}{u'_\beta(q^\alpha_s)}(1+r_s) = \frac{u'_\alpha(e - q^\alpha_{s'})}{u'_\beta(q^\alpha_{s'})}(1+r_{s'}) \tag{26}$$

The function

$$f(q^\alpha_s) = \frac{u'_\alpha(e - q^\alpha_s)}{u'_\beta(q^\alpha_s)}$$

is increasing in q^α_s. Therefore, in (26), if $r_s > r_{s'}$, $f(q^\alpha_s) < f(q^\alpha_{s'}) \Longrightarrow q^\alpha_s < q^\alpha_{s'}$.

$\square$

Proof of Proposition 2

From (24) and the market clearing conditions

$$\frac{\pi_s}{\pi_{s'}} \frac{u'_\beta(q^\alpha_s)}{u'_\beta(q_{s'})^\alpha} = \frac{\theta_s p_s}{\theta_{s'} p_{s'}} \Longrightarrow \frac{\theta_s}{\theta_{s'}} = \frac{\pi_s}{\pi_{s'}} \frac{u'_\beta(q^\alpha_s)/p_s}{u'_\beta(q^\alpha_{s'})/p_{s'}}$$

Using the quantity theory of money proposition,

$$\frac{\theta_s}{\theta_{s'}} = \frac{\pi_s}{\pi_{s'}} \frac{u'_\beta(q^\alpha_s)q^\alpha_s/(M_s + m^\alpha_{t+1} + m^\beta_{t+1})}{u'_\beta(q^\alpha_{s'})q^\alpha_{s'}/(M_{s'} + m^\alpha_{t+1} + m^\beta_{t+1})} \tag{27}$$

When liquidity is higher in state s, from Proposition 1, activity q_s^α is higher. If relative risk aversion is greater than 1, $u'_\beta(q)q$ is decreasing[8] and therefore $u'_\beta(q_s^\alpha)q_s^\alpha$ is lower. Hence $u'_\beta(q_s^\alpha)q_s^\alpha/(M_s + m_{t+1}^\alpha + m_{t+1}^\beta)$ is also lower. This shows that θ_s is lower when r_s is lower. Since r_s is an inverse function of M_s we deduce the final result

$$\forall s \neq s' \quad r_{s'} > r_s \Longleftrightarrow \frac{\theta_{s'}}{\pi_{s'}} > \frac{\theta_s}{\pi_s}$$

$\square$

Example 2 : Three-Period Example

Assume there are only three periods, with $\mathcal{F}_0 = \{0\}, \mathcal{F}_1 = \{1\}$ and $\mathcal{F}_2 = \{1 \ldots S\}$. Assume furthermore that $u_\alpha = u_\beta = ln$, and that $\pi_s^\alpha = \pi_s^\beta = 1/S, \forall s \in \mathcal{F}_2$. Then, the solution of the model is:

$$r_0 = \frac{m_0}{M_0}, \;\; r_1 = \frac{m_1}{M_1}, \;\; \forall s \;\; r_s = \frac{m_2}{M_s}$$

$$\theta_1 = \frac{2(M_0 + m_0) - \bar{M}}{M_1 + m_1}$$

$$\theta_s = \frac{M_1 + m_1}{S(m_2 + M_s)(2 - \bar{M}/(M_0 + m_0))}$$

where $\bar{M}$ is an exogenous inherited liability for agent α.[9]

Proof

The maximisation problem of agent α is:

$$\max \;\; ln(e_0 - q_0) + ln(q_1) + \sum_{s \in \mathcal{F}_2} \frac{1}{S} ln(e_s - q_s)$$

[8]Since $g(q) = u'_\beta(q)q$ is decreasing in q if and only if $u''_\beta(q)q + u'_\beta(q) \leq 0 \Leftrightarrow RRA(q) = \frac{-u''_\beta(q)q}{u'_\beta(q)} \geq 1$.

[9]We need one agent to be in debt initially so that the first period bond market be active for reasonnable stable values of the money supply

$$\theta_1 h_1 + \bar{M} \leq \frac{\mu_0}{1+r_0} + m_0 \qquad (\varphi_1^\alpha)$$

$$\mu_0 \leq p_0 q_0 \qquad (\chi_1^\alpha)$$

$$p_1 q_1 \leq h_1 + \sum_{s\in\mathcal{F}_2} \theta_s h_s \qquad (\xi_1^\alpha)$$

$$\forall s \in \mathcal{F}_2 \quad h_s \leq \frac{\mu_s}{1+r_s} + m_2 \qquad (\Psi_s^\alpha)$$

$$\forall s \in \mathcal{F}_2 \quad \mu_s \leq p_s q_s \qquad (\chi_s^\alpha)$$

The maximisation problem of agent β is :

$$\max \;\; ln(q_0) + ln(e_1 - q_1) + \sum_{s\in\mathcal{F}_2} \frac{1}{S} ln(q_s)$$

$$p_0 q_0 \leq \theta_1 h_1 + \bar{M} \qquad (\varphi_1^\beta)$$

$$h_1 + \sum_{s\in\mathcal{F}_2} \theta_s h_s \leq \frac{\mu_1}{1+r_1} + m_1 \qquad (\chi_1^\beta)$$

$$\mu_1 \leq p_1 q_1 \qquad (\xi_1^\beta)$$

$$\forall s \in \mathcal{F}_2 \quad p_s q_s \leq h_s \qquad (\Psi_s^\beta)$$

From Propositions 1 and 2, we know that:

$$p_0 q_0 = M_0 + m_0, \;\; p_1 q_1 = M_1 + m_1, \;\; \forall s \in \mathcal{F}_2 \;\; p_s q_s = M_s + m2$$

$$r_0 = \frac{m_0}{M_0}, \;\; r_1 = \frac{m_1}{M_1}, \;\; \forall s \;\; r_s = \frac{m_2}{M_s}$$

To solve for θ_s, we first note that, from β's first-order conditions, namely (24),

$$\forall s, s' \in \mathcal{F}_2 \quad \frac{1}{p_s q_s \theta_s} = \frac{1}{p_{s'} q_{s'} \theta_{s'}}$$

Since $\forall s, \;\; p_s q_s = h_s$, we deduce that

$$\forall s, s' \in \mathcal{F}_2 \;\; \theta_s h_s = \theta_{s'} h_{s'}$$

Since

$$p_1 q_1 \leq h_1 + \sum_{s\in\mathcal{F}_2} \theta_s h_s$$

holds as an equality in an interior equilibrium,

$$\forall s_0 \in \mathcal{F}_2 \;\; \sum_{s\in\mathcal{F}_2} \theta_s h_s = S\theta_{s_0} h_{s_0} = M_1 + m_1 - h_1$$

Since $h_{s_0} = p_{s_0} q_{s_0} = M_{s_0} + m_2$, we deduce that

$$\theta_{s_0} = \frac{M_1 + m_1 - h_1}{M_{s_0} + m_2} \tag{28}$$

which confirms that θ_{s_0} is decreasing in M_{s_0}, and hence increasing in r_{s_0}. Finally, we need to solve for θ_1 and h_1. From β's first- order conditions, we have $\theta_1 \varphi_1^\beta = \chi_1^\beta$, and $\forall s \;\; \chi_1^\beta = \frac{\Psi_s^\beta}{\theta_s}$. Hence,

$$\theta_1 \varphi_1^\beta = \frac{\Psi_s^\beta}{\theta_s}$$

Since $\varphi^\beta = \frac{1}{p_0 q_0} = \frac{1}{M_0 + m_0}$ and since $\Psi_s^\beta = \frac{1}{S(p_s q_s)} = \frac{1}{S(M_s + m_2)}$, we deduce that

$$\frac{\theta_1}{M_0 + m_0} = \frac{1}{S(M_s + m_2)\theta_s}$$

Substituting in agent α's first budget constraint, we find that

$$h_1 = \frac{M_0 + m_0 - \bar{M}}{\theta_1} = \frac{(M_0 + m_0 - \bar{M})(M_s + m_2) S \theta_s}{M_0 + m_0} \tag{29}$$

Replacing (29) in (28), we deduce

$$\theta_s = \frac{M_1 + m_1}{S(m_2 + M_s)(2 - \bar{M}/(M_0 + m_0))}$$

$$\theta_1 = \frac{2(M_0 + m_0) - \bar{M}}{M_1 + m_1}$$

□

Proposition 5: Local Properties

Suppose that the endowments or the subjective probabilities of state s and s' differ by an infinitesimal quantity.

$$r_s > r_{s'} \Longrightarrow q_s^\alpha < q_{s'}^\alpha \;\text{ and }\; \frac{\theta_s}{\pi_s^\beta} > \frac{\theta_{s'}}{\pi_{s'}^\beta}$$

Proof

We show here how the continuity argument works for different subjective probabilities, but the proof follows *mutatis mutandis* for different endowments. Let $1 + \epsilon = \frac{\pi_s^\alpha}{\pi_{s'}^\alpha} / \frac{\pi_s^\beta}{\pi_{s'}^\beta}$. Because monotonic transformations of the intertemporal utility functions do not affect the maximisation problems, one can normalise the subjective probabilities $\frac{\pi_s^\alpha}{\pi_{s'}^\alpha}$ and $\frac{\pi_s^\beta}{\pi_{s'}^\beta}$.

Therefore, without loss of generality, ϵ can represent any difference between α's and β's subjective probabilities. We show here the dependence of q_s^α and $q_{s'}^\alpha$ on the subjective probabilities of states s and s' (keeping in mind that other variables matter as well) : $q_s^\alpha = q_s^\alpha(\pi_s^\alpha, \pi_{s'}^\alpha, \pi_s^\beta, \pi_{s'}^\beta)$ and $q_{s'}^\alpha = q_{s'}^\alpha(\pi_s^\alpha, \pi_{s'}^\alpha, \pi_s^\beta, \pi_{s'}^\beta)$. Monotonic transformations of the intertemporal utility functions do not affect the maximisation problems, therefore

$$q_s^\alpha = q_s^\alpha\left(\frac{\pi_s^\alpha}{\pi_{s'}^\alpha}, \frac{\pi_s^\beta}{\pi_{s'}^\beta}\right) = q_s^\alpha\left(\frac{\pi_s^\beta}{\pi_{s'}^\beta}(1+\epsilon), \frac{\pi_s^\beta}{\pi_{s'}^\beta}\right)$$

$$q_{s'}^\alpha = q_{s'}^\alpha\left(\frac{\pi_s^\alpha}{\pi_{s'}^\alpha}, \frac{\pi_s^\beta}{\pi_{s'}^\beta}\right) = q_{s'}^\alpha\left(\frac{\pi_s^\beta}{\pi_{s'}^\beta}(1+\epsilon), \frac{\pi_s^\beta}{\pi_{s'}^\beta}\right)$$

Keeping the ratio $\frac{\pi_s^\beta}{\pi_{s'}^\beta}$ constant, only ϵ is variable here, and, simplifying, $q_s^\alpha = q_s^\alpha(\epsilon)$ and $q_{s'}^\alpha = q_{s'}^\alpha(\epsilon)$, equation (25) becomes

$$(1+\epsilon)\frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} = \frac{1+r_{s'}}{1+r_s} < 1.$$

Note first that

$$\lim_{\epsilon\to 0}(1+\epsilon)\frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} = \lim_{\epsilon\to 0}\frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} + \lim_{\epsilon\to 0}\epsilon\frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} = \frac{1+r_{s'}}{1+r_s} < 1$$

Furthermore,

$$\lim_{\epsilon\to 0}\epsilon\frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} = 0$$

because $0 < \frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} < \frac{1}{1+\epsilon} < 1$ and hence $\frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))}$ is bounded. Hence, we have in the limit that $\frac{f(q_s^\alpha(\epsilon))}{f(q_{s'}^\alpha(\epsilon))} < 1$, and, since f is continuous and increasing, $q_s^\alpha(\epsilon) < q_{s'}^\alpha(\epsilon)$ as $\epsilon \to 0$. The proof of the second part then follows the proof of Proposition 4, in particular equation (27).

$\square$

APPENDIX 2

To simplify the presentation, we restrict the model with multiple commodities to two periods, although the extension to multiple periods is straightforward.

Proposition 6: Non-Neutrality of Money With Multiple Commodities

For any state $s \in \mathcal{F}_t \cup \{0\}$, let (s, L) index a good bought by agent β.[10] and assume $e^{\alpha}_{s,L} > 0, e^{\beta}_{s,L} = 0,\ \forall s \in \mathcal{F}_t$. This implies L is sold by α in all states of nature. Then, in absence of aggregate uncertainty, $\forall s \neq s',\ r_s > r'_s \iff q^{\alpha}_{s,L} < q^{\alpha}_{s',L}$

Proof

Let $l \in L = \{1, \ldots, L\}$ be the index of goods available in any state or period, and $u_{l,i}$ utility of agent i in good l. We note $L^i_s(+)$ the set of goods in state s that agent i will sell, and $L^i_s(-)$ is the set of goods that agent i will buy. With positive transaction costs, Dubey and Geanakoplos (2006) show that there are no wash sales, *i.e.* $L^i_s(+) \cap L^i_s(-) = \emptyset$. Furthermore, since we have only two agents, $L^{\alpha}_s(+) = L^{\beta}_s(-)$ and $L^{\alpha}_s(-) = L^{\beta}_s(+)$.

We focus on a 2-period model here, for the sake of exposition. The first period variables are indexed by 0, while the second period variables are indexed by $s \in \mathcal{F}_t$ (to ensure consistency with previous notations). Each trade and endowment variable is indexed by the state of nature and by the commodity to which it refers. Asset trade, represented by h^i_s, may be positive or negative. The maximisation problem of agent $i \in \{\alpha, \beta\}$ is: (in brackets are the lagrangian multipliers)

$$\max \sum_{l \in L^i_0(+)} u_{i,l}(e^i_{0,l} - q^i_{0,l}) + \sum_{l \in L^i_0(-)} u_{i,l}(q^i_{0,l}) + \sum_{s \in \mathcal{F}_t} \pi_s \left[\sum_{l \in L^i_s(+)} u_{i,l}(e^i_{s,l} - q^i_{s,l}) + \sum_{l \in L^i_s(-)} u(q^i_{s,i}) \right]$$

[10] Similarly, the index $(s, 1)$ denotes a good bought by agent α.

s.t.

$$\sum_{l\in L_0^h(-)} p_{0,l}q_{0,l}^i = \sum \theta_s h_s^i + \frac{\mu_0^i}{1+r_0} \qquad (\varphi^i)$$

$$\mu_0^i = \sum_{l\in L_0^i(+)} p_{0,l}q_{0,l}^i \qquad (\xi^i)$$

$$\forall s \in \mathfrak{F}_t \quad \sum_{l\in L_s^i(-)} p_{s,l}q_{s,l}^i + h_s^i = \frac{\mu_s^i}{1+r_s} + m_s^i \qquad (\Psi_s^i)$$

$$\forall s \in \mathfrak{F}_t, \qquad \mu_s^i = \sum_{l\in L_s^i(+)} p_{s,l}q_{s,l}^i \qquad (\chi_s^i)$$

Value of Money and Quantity Theory of Money

The following lemmas hold with many commodities (the proofs are omitted as they follow *mutatis mutandis* from Lemmas 1 and 2).

Lemma 4: Short-term interest rate
For an interior equilibrium,

$$\forall s \in \mathfrak{F}_t \cup \{0\},\ r_s = \frac{m_s^\alpha + m_s^\beta}{M_s}$$

Lemma 4: Quantity theory of money

$$\begin{aligned}\forall s \in \mathfrak{F}_t \cup \{0\},\ \sum_{l\in L_s^\alpha(-)} p_{s,l}q_{s,l}^\alpha + \sum_{l\in L_s^\beta(-)} p_{s,l}q_{s,l}^\beta &= \sum_{l\in L_s^\beta(+)} p_{s,l}q_{s,l}^\beta + \sum_{l\in L_s^\alpha(+)} p_{s,l}q_{s,l}^\alpha \\ &= M_s + m_s^\alpha + m_s^\beta\end{aligned}$$

Using the First-Order Conditions, for a good in time 0 that is bought by α:

$$u'_{\alpha,1}(q_{0,1}^\alpha) = p_{0,1}\frac{\Psi_s^\alpha}{\theta_s} = p_{0,1}\frac{\pi_s}{\theta_s}u'_{\alpha,1}(q_{s,1}^\alpha)$$

Also

$$u'_{\alpha,1}(q_{0,1}^\alpha) = p_{0,1}\frac{\Psi_s^\alpha}{\theta_s} = p_{0,1}\frac{\chi_s^\alpha}{\theta_s/(1+r_s)} = p_{0,1}\frac{\pi_s(1+r_s)}{\theta_s}\frac{u'_{\alpha,L}(e_{s,L}^\alpha - q_{s,L}^\alpha)}{p_{s,L}}$$

Similarly, for commodity $(0, L)$ in time 0 that is sold by α:

$$\frac{u'_{\alpha,L}(e_{0,L}^\alpha - q_{0,L}^\alpha)}{p_{0,L}} = \frac{1}{1+r_0}\varphi^\alpha p_{0,1} = \frac{u'_{\alpha,1}(q_{0,1}^\alpha)}{(1+r_0)p_{0,1}}$$

Hence, for a good L that is sold by α in states s and s',

$$(1+r_s)\pi_s \frac{u'_{\alpha,L}(e^{\alpha}_{s,L} - q^{\alpha}_{s,L})}{p_{s,L}\theta_s} = (1+r_{s'})\pi_{s'} \frac{u'_{\alpha,L}(e^{\alpha}_{s',L} - q^{\alpha}_{s',L})}{p_{s',L}\theta_{s'}} \tag{30}$$

This same good L is therefore bought by β in both states s and s' and therefore:

$$\frac{\pi_s}{\theta_{s'}} \frac{u'_{\beta,L}(q^{\beta}_{s,L})}{p_{s,L}} = \frac{\pi_{s'}}{\theta_{s'}} \frac{u'_{\beta,L}(q^{\beta}_{s',L})}{p_{s',L}} \tag{31}$$

Dividing equation (30) by equation (31),

$$(1+r_s)\frac{u'_{\alpha,L}(e^{\alpha}_{s,L} - q^{\alpha}_{s,L})}{u'_{\beta,L}(q^{\beta}_{s,L})} = (1+r_{s'})\frac{u'_{\alpha,L}(e^{\alpha}_{s',L} - q^{\alpha}_{s',L})}{u'_{\beta,L}(q^{\beta}_{s',L})}$$

From the market clearing conditions, and assuming $e_{s,L} = e_{s',L} = e_L$,

$$(1+r_s)\frac{u'_{\alpha,L}(e^{\alpha}_{L} - q^{\alpha}_{s,L})}{u'_{\beta,L}(q^{\alpha}_{s,L})} = (1+r_{s'})\frac{u'_{\alpha,L}(e^{\alpha}_{L} - q^{\alpha}_{s',L})}{u'_{\beta,L}(q^{\alpha}_{s',L})}$$

Hence (this replicates the proof for Proposition 3), $r_s > r_{s'} \iff q^{\alpha}_{s,L} < q^{\alpha}_{s',L}$

$\square$

Assume furthermore that the direction of trade of all goods is independent of the state of nature. In that case, the asset prices have again the properties established in Proposition 4.

Proposition 7: Asset Prices with Multiple Commodities

Assume that $\forall i \in \{\alpha, \beta\}, \forall s, s' \in \mathfrak{F}_t,\ s \neq s',\ L^i_s(-) = L^i_{s'}(-)$. Assume furthermore that $RRA \geq 1$ and that there is no aggregate uncertainty. Then, $r_s > r_{s'} \Longrightarrow \theta_s/\pi_s > \theta_{s'}/\pi_{s'}$.

Proof

We use β's first-order conditions for a good L always bought by β (independently of the state of nature), and, from the market clearing condition, $q^{\beta}_{s,L} = q^{\alpha}_{s,L}$. Hence,

$$\frac{\pi_s}{\theta_s} \frac{u'_{\beta,L}(q^{\alpha}_{s,L})}{p_{s,L}} = \frac{\pi_{s'}}{\theta_{s'}} \frac{u'_{\beta,L}(q^{\alpha}_{s',L})}{p_{s',L}}$$

Rearranging,

$$\frac{\theta_s/\pi_s}{\theta_{s'}\pi_{s'}} = \frac{u'_{\beta,L}(q^{\alpha}_{s,L})/p_{s,L}}{u'_{\beta,L}(q^{\alpha}_{s',L})/p_{s',L}}$$

Equivalently,

$$\frac{\theta_s/\pi_s}{\theta_{s'}\pi_{s'}} = \frac{u'_{\beta,L}(q^{\alpha}_{s,L})q^{\alpha}_{s,L}/(p_{s,L}q^{\alpha}_{s,L})}{u'_{\beta,L}(q^{\alpha}_{s',L})q^{\alpha}_{s',L}/(p_{s',L}q^{\alpha}_{s',L})} \tag{32}$$

Let $r_s > r_{s'}$, *i.e.* $M_s < M_{s'}$. From the quantity theory of money proposition,

$$\sum_{l\in L^{\alpha}_s(+)} p_{s,l}q^{\alpha}_{s,l} + \sum_{l\in L^{\beta}_s(+)} p_{s,l}q^{\beta}_{s,l} < \sum_{l\in L^{\alpha}_{s'}(+)} p_{s',l}q^{\alpha}_{s',l} + \sum_{l\in L^{\beta}_{s'}(+)} p_{s',l}q^{\beta}_{s',l}$$

Therefore, there must be one good g for which $p_{s,g}q^i_{s,g} \leq p_{s',g}q^i_{s',g}$. This good is traded in only one direction (like all goods, by assum,ption) , independently of the state of nature. From Proposition 6, we know that $r_s > r_{s'} \Rightarrow q^i_{s,g} < q^i_{s',g}$. With $RRA \geq 1$,

$$u'_{\beta,g}(q^{\alpha}_{s,g})q^{\alpha}_{s,g} > u'_{\beta,g}(q^{\alpha}_{s',g})q^{\alpha}_{s',g}$$

Since $p_{s,g}q^{\alpha}_{s,g} \leq p_{s',g}q^{\alpha}_{s',g}$, from (32), it follows that $\frac{\theta_s}{\pi_s} > \frac{\theta_{s'}}{\pi_{s'}}$

$\square$

APPENDIX 3

Lemma 3 : Short-Term Interest Rates

$$\forall s \in \mathcal{F} \cup \{0\} \;\; r_s = \frac{m_s^\beta}{M_s}$$

Proof

$$\begin{aligned} p_0 q_0^\alpha &= \sum_{s \in \mathcal{F}} \theta_s h_s^\alpha = \Lambda_0 p_0 q_0^\beta + \frac{\mu_0^\beta}{1+r_0} + m_0^\beta \\ \implies (1-\Lambda_0) p_0 q_0^\beta &= \frac{\mu_0^\beta}{1+r_0} + m_0^\beta = M_0 + m_0^\beta \end{aligned} \tag{33}$$

from the money market equilibrium condition. Furthermore,

$$\mu_0^\beta = M_0 + m_0^\beta = M_0(1+r_0)$$

Hence, $r_0 = \frac{m_0^\beta}{M_0}$. Similarly, summing (14) and (19),

$$h_s^\alpha + p_s q_s^\beta = \Lambda_s p_s q_s^\alpha + \frac{\mu_s^\alpha + \mu_s^\beta}{1+r_s} + \lambda_s h_s^\beta + m_s^\beta$$

From the money and asset market equilibrium conditions,

$$(1-\Lambda_s) p_s q_s^\alpha = M_s + m_s^\beta + (\lambda_s - 1) h_s^\beta$$

which implies, using (15) and (20),

$$\mu_s^\alpha = M_s + m_s^\beta - \mu_s^\beta$$

Hence, $M_s(1+r_s) = \mu_s^\alpha + \mu_s^\beta = M_s + m_s^\beta$, which implies $r_s = \frac{m_s^\beta}{M_s}$

$\square$

Lemma 4 : Quantity theory of money and Liquidity

$$p_0 q_0 = \frac{M_0 + m_0^\beta}{1-\Lambda}$$

$$\forall s \in \mathcal{F} \quad p_s q_s = \frac{M_s + m_s^\beta}{1 - \Lambda_s + (1-\lambda_s)(\Lambda_s + \frac{1-\Lambda_s}{1+r_s})}$$

Proof

The first part was proved in equation (33). For the second part of the proposition,

$$\mu_s^\alpha + \mu_s^\beta = M_s + m_s^\alpha = (1-\Lambda_s)p_s q_s^\alpha + (1-\lambda_s)h_s^\alpha$$

$$= (1-\Lambda_s)(p_s q_s^\alpha) + (1-\lambda_s)(\Lambda_s p_s q_s^\alpha + \frac{\mu_s^\alpha}{1+r_s})$$

$$= (1-\Lambda_s)p_s q_s + (1-\lambda_s)(\Lambda_s p_s q_s^\alpha + \frac{1}{1+r_s}(1-\Lambda_s)p_s q_s^\alpha$$

$$= p_s q_s^\alpha(1-\Lambda_s + (1-\lambda_s)(\Lambda_s + \frac{1}{1+r_s}(1-\Lambda_s))$$

□

Proposition 3 : Non-Neutrality of Money and Liquidity

If $\forall s \in \mathcal{F} e_s^\alpha = e^\alpha$, then, $\forall s \neq s'$,

$$r_s(2-\lambda_s-\Lambda_s) > r_{s'}(2-\lambda_{s'}-\Lambda_{s'}) \Longrightarrow q_s < q_{s'}$$

Proof

From α's first-order conditions,

$$u'_\alpha(e_s^\alpha - q_s) = p_s(\Lambda_s + (1-\Lambda_s)\frac{1}{1+r_s})\theta_s \varphi_0^\alpha$$

Hence, $\forall s \neq s'$,

$$\frac{u'_\alpha(e_s^\alpha - q_s)}{p_s\theta_s(\Lambda_s + (1-\Lambda_s)\frac{1}{1+r_s})} = \frac{u'_\alpha(e_{s'}^\alpha - q_{s'})}{p_{s'}\theta_{s'}(\Lambda_{s'} + (1-\Lambda_{s'})\frac{1}{1+r'_s}} \tag{34}$$

Similarly, from β's first-order conditions,

$$\theta_s \varphi_0^\beta = (\lambda_s + (1-\lambda_s)\frac{1}{1+r_s})\frac{u'_\beta(q_s)}{p_s}$$

Hence,$\forall s \neq s'$,

$$\frac{u'_\beta(q_s)}{p_s\theta_s}(\lambda_s + (1-\lambda_s)\frac{1}{1+r_s}) = \frac{u'_\beta(q_{s'})}{p_{s'}\theta_{s'}}(\lambda_{s'} + (1-\lambda_{s'})\frac{1}{1+r'_s}) \tag{35}$$

Furthermore, dividing (34) by (35),

$$\frac{u'_\alpha(e^\alpha_s - q_s)}{u'_\beta(q_s)(\Lambda_s+(1-\Lambda_s)\frac{1}{1+r_s})(\lambda_s+(1-\lambda_s)\frac{1}{1+r_s})}$$

$$= \frac{u'_\alpha(e^\alpha_{s'} - q_{s'})}{u'_\beta(q_{s'})(\Lambda_{s'} + (1-\Lambda_{s'})\frac{1}{1+r'_s})(\lambda_{s'} + (1-\lambda_{s'})\frac{1}{1+r'_s})} \tag{36}$$

For $\frac{1}{1+r_s} \approx 1$ (*i.e.* for small intra-period interest rates), the following approximation holds,

$$\frac{1}{(\Lambda_s + (1-\Lambda_s)\frac{1}{1+r_s})(\lambda_s + (1-\lambda_s)\frac{1}{1+r_s})} \approx 1 + r_s(2 - \Lambda_s - \lambda_s)$$

Furthermore, let us assume that $\forall s, s'\;\; e^\alpha_s = e^\alpha_{s'} = e^\alpha$, then we define

$$f(q^\alpha_s) = \frac{u'_\alpha(e - q^\alpha_s)}{u'_\beta(q^\alpha_s)}$$

f is increasing in q^α_s for all utility functions such that $u'_\alpha > 0$; $u''_\alpha < 0$ and $u'_\beta > 0$; $u''_\beta < 0$.
Indeed

$$f'(q^\alpha_s) = \frac{-u''_\alpha(e - q^\alpha_s)u'_\beta(q^\alpha_s) - u'_\alpha(e - q^\alpha_s)u''_\beta(q^\alpha_s)}{u'_\beta(q^\alpha_s)^2} > 0$$

Hence, (36) can be re-written

$$f(q^\alpha_s)(1 + r_s(2 - \Lambda_s - \lambda_s)) \approx f(q^\alpha_{s'})(1 + r_{s'}(2 - \Lambda_{s'} - \lambda_{s'}))$$

from which the proposition is deduced.

$\square$

Proposition 4 : Asset Prices and Liquidity

Assume $\forall q, RRA(q) \geq 1$. *Ceteris paribus*,

$$r_s > r_{s'} \Longrightarrow \theta_s > \theta_{s'}$$

$$\lambda_s > \lambda_{s'} \Longrightarrow \theta_s < \theta_{s'}$$

$$\Lambda_s > \Lambda_{s'} \Longrightarrow \theta_s < \theta_{s'}$$

Proof

We rearrange β's first-order conditions (35) and multiply and divide by trade q:

$$\frac{\theta_s}{\theta_{s'}} = \frac{\frac{u'_\beta(q_s)q_s}{p_s q_s}(\lambda_s + (1-\lambda_s)\frac{1}{1+r_s})}{\frac{u'_\beta(q_{s'})q_{s'}}{p_{s'} q_{s'}}(\lambda_{s'} + (1-\lambda_{s'})\frac{1}{1+r'_s})} \tag{37}$$

From Proposition 4

$$\frac{\theta_s}{\theta_{s'}} = \frac{\frac{u'_\beta(q_s)q_s}{M_s+m^\beta}(\lambda_s + (1-\lambda_s)\frac{1}{1+r_s})(1-\Lambda_s + (1-\lambda_s)(\Lambda_s + (1-\Lambda_s)\frac{1}{1+r_s}))}{\frac{u'_\beta(q_{s'})q_{s'}}{M_{s'}+m^\beta}(\lambda_{s'} + (1-\lambda_{s'})\frac{1}{1+r'_s})(1-\Lambda_{s'} + (1-\lambda_{s'})(\Lambda_{s'} + (1-\Lambda_{s'})\frac{1}{1+r'_s}))} \tag{38}$$

Let us define, $\forall s$

$$L_s = \frac{1}{M_s+m^\beta}(\lambda_s + (1-\lambda_s)\frac{1}{1+r_s})(1-\Lambda_s + (1-\lambda_s)(\Lambda_s + (1-\Lambda_s)\frac{1}{1+r_s}))$$

Since $\frac{1}{1+r_s} = \frac{M_s}{M_s+m_s^\beta}$,

$L_s = \frac{1}{M_s+m^\beta}\left(\frac{\lambda_s(M_s+m^\beta)+(1-\lambda_s)M_s}{M_s+m^\beta}\right)$

$$\left(1-\Lambda_s + (1-\lambda_s)\left(\frac{\Lambda_s(M_s+m^\beta)+(1-\Lambda_s)M_s}{M_s+m^\beta}\right)\right)$$

$$L_s = \frac{1}{M_s+m^\beta}\left(\frac{\lambda_s m^\beta + M_s}{M_s+m^\beta}\right)\left(1-\Lambda_s + (1-\lambda_s)\left(\frac{\Lambda_s m^\beta + M_s}{M_s+m^\beta}\right)\right)$$

Although the second and third elements depend on M_s, their dependence is of the second order. Hence, because of the preponderance of the first element,

$$\frac{\partial L_s}{\partial M_s} < 0$$

As a result, a higher M_s (*i.e.* a lower r_s) which implies a higher q_s and hence a lower $u'(q_s)q_s$, is associated with lower a state price θ_s. Similarly, the effect of λ on the second element is of the second order, while its effect on the third element is preponderant. Hence,

$$\frac{\partial L_s}{\partial \lambda_s} < 0$$

This implies that a higher λ_s (which also implies higher trade and hence a lower $u'(q_s)q_s$), is associated with a lower state price.

Finally,

$$\frac{\partial L_s}{\partial \Lambda_s} < 0$$

Again, a higher Λ_s (which also implies higher trade and hence a lower $u'(q_s)q_s$), is associated with a lower state price.

□