WP/12/121

# IMF Working Paper

# Interest Rate Rules, Endogenous Cycles, and Chaotic Dynamics in Open Economies

*Marco Airaudo and Luis-Felipe Zanna*

INTERNATIONAL MONETARY FUND

 WP/12/121

**IMF Working Paper**

Research Department

**Interest Rate Rules, Endogenous Cycles, and Chaotic Dynamics in Open Economies**

**Prepared by Marco Airaudo and Luis-Felipe Zanna***

Authorized for distribution by Andrew Berg

May 2012



Abstract

We present an extensive analysis of the consequences for global equilibrium determinacy in flexible-price open economies of implementing active interest rate rules, i.e., monetary rules where the nominal interest rate responds more than proportionally to inflation. We show that conditions under which these rules generate aggregate instability by inducing liquidity traps, endogenous cycles, and chaotic dynamics depend on specific characteristics of open economies. In particular, rules that respond to expected future inflation are more prone to induce endogenous cyclical and chaotic dynamics the more open the economy to trade.

JEL Classification Numbers: E32, E52, F41

Keywords: Small Open Economy; Interest Rate Rules; Taylor Rules; Multiple Equilibria; Chaos; Cycles and Endogenous Fluctuations.

Author's E-Mail Addresses: marco.airaudo@drexel.edu; fzanna@imf.org

* The authors would like to thank the following people for comments and suggestions: Gian-Italo Bischi, David Bowman, Chris Erceg, Jon Faust, Laura Gardini, Dale Henderson, two anonymous referees, the associate editor, and the seminar participants at the Federal Reserve Board, LUISS, the University of Urbino, the Society for Non-Linear Dynamics and Econometrics and the Latin American Meeting of the Econometric Society. Part of this work was completed while Airaudo was Junior Fellow and then Visiting Fellow at Collegio Carlo Alberto. He is grateful to the Collegio for all the hospitality and research support. The views expressed in this paper are solely the responsibility of the authors and should not be interpreted as reflecting those of the International Monetary Fund.

**Table of Contents** Page

# I. Introduction

In a recent paper, Bullard (2010) argues that the extremely low interest rates observed in the U.S. and in major developed economies, after the financial crisis, are consistent with the theoretical findings by Benhabib et al. (2001b). The latter show how, because of the zero lower bound (ZLB) on the nominal interest rate, simple Taylor rules can induce a liquidity trap—a situation where the nominal interest rate drifts away from its target towards an unintended low steady state—even if the rule satisfies the celebrated Taylor principle by responding more than proportionally to inflation (an active rule).

The liquidity trap in Benhabib et al. (2001b) is an equilibrium that is entirely driven by people's self-fulfilling expectations. It is in fact the natural outcome from a *non-linear* analysis of the prototype micro-founded dynamic model that has become the workhorse for macroeconomic policy discussions. However, as Bullard (2010) points out, this outcome has surprisingly received scarce attention in the policy debate. Central banks often organize their discussions by looking at the impulse responses of *linearized* versions of the truly non-linear models. By doing so, they tend to rule out *a priori,* as a viable equilibrium, all the non-linear trajectories that move away from their targets, including liquidity traps.

Benhabib et al. (2001b, 2002a) also show that a global rather than a local analysis unveils the existence of other Taylor-rule-induced rational expectations equilibria, such as endogenous limit cycles and chaotic dynamics.[1] This questions policy recommendations based on linearized models, since Taylor rules that ensure a unique local equilibrium may actually generate aggregate instability by inducing global cyclical fluctuations. The works by Benhabib et al., however, are based on *closed* economy models raising the question of whether similar results hold in the context of open economies. To this date, this still remains a valid question since most of the studies in open economies have been restricted exclusively to local analyses. De Fiore and Liu (2005), Llosa and Tuesta (2008), and Zanna (2003), for instance, pursue local analysis of Taylor rules in the context of a small open economy set-up; while Benigno and Benigno (2008), Bullard and Singh (2008) and Leith and Wren-Lewis (2009), among others, present similar analyses in open economy models with more than one country.

The purpose of our paper is to assess how the existence of these self-fulfilling cyclical and chaotic fluctuations is affected by opening the economy to international trade in goods. We pursue a global non-linear equilibrium analysis of a traditional flexible-price small-open-economy model with traded and non-traded goods, where monetary policy takes the form of an active interest rate rule responding to expected future CPI inflation, i.e., forward-looking rules. By doing this, we bridge the gap between the *closed* economy literature on *global* analysis of Taylor rules and the *open* economy literature on *local* analysis of these rules.

Our main result is that forward-looking Taylor rules are more prone to induce endogenous cycles and chaos the more open the economy to trade. In contrast to the money-in-the-production-function (MIPF) set-up of Benhabib et al. (2002a), we use a money-in-the-utility-function (MIUF) set-up,

[1] See also Alstadheim and Henderson, (2006), Eusepi (2007), and Evans, Guse and Honkapohja (2008).

whereby consumption and real money balances are non-separable in utility. As a result, these self-fulfilling complex fluctuations, whose existence depends on openness, can occur around either the target interest rate or the unintended low steady state, depending on whether consumption and money are Edgeworth substitutes or complements, respectively. Interestingly, the complements case allows for liquidity traps that converge non-monotonically to a limit cycle around the unintended steady state.

The global equilibrium dynamics in our model are driven by the interaction of an open-economy version of the Fisher equation and the non-linear Taylor rule. The modified Fisher equation is obtained from the combination of the uncovered interest parity condition and the definition of the consumer-price-index (CPI) inflation. In this modified equation, the dynamics of the ex-post real interest rate has to be consistent with future changes in the real exchange rate. Key to the existence of cycles and chaos is the elasticity of the real exchange rate to the policy interest rate. This elasticity is affected by the complementarity/substitutability between consumption and money and, more importantly for our purposes, by the degree of trade openness.

Relative to the works on global analysis in closed economies, we find that trade openness significantly enlarges the risk aversion parameter range under which forward-looking rules induce cycles and chaos. As in Benhabib et al.(2002a), the presence of a ZLB on nominal interest rates is a *necessary* condition for the existence of cyclical fluctuations. However, while these complex dynamics coexist in a MIUF closed economy model for a very *restricted* range of values of the risk aversion parameter, the same dynamics may occur for a much *wider* range of this parameter, as long as the economy is sufficiently open. By enlarging this range, openness plays a key qualitative role in determining the global determinacy properties of Taylor rules.

From a policy-making point of view, our results suggest that to avoid destabilizing endogenous cycles and chaos in *open* economies, the design of interest rate rules should take into account not only the interest response coefficient to inflation, but also specific structural characteristics, such as the degrees of openness. This complements the results of the aforementioned open economy literature on local analysis with one important caveat: as we show below, and in line with Benhabib et al. (2001b, 2002a), policy prescriptions that are derived from local analyses to ensure macroeconomic stability in an open economy can still lead to cycles and chaos (global indeterminacy), where the extent of disagreement between the local and global analyses depends on the degree of openness.

The remainder of this paper is organized as follows. In Section II, we present a flexible-price model with its main assumptions. We define the open economy equilibrium and derive some basic steady state results. In Section III, we pursue local and global equilibrium analyses for an interest rate rule that responds to expected future CPI inflation, focusing on the role played by the degree of openness. Section IV discusses the robustness of our main results under different extensions, including imperfect exchange rate pass-through, incomplete markets, alternative timings for the policy rule and money in utility, and alternative preferences and technologies. Finally, Section V concludes.

# II. A Flexible-Price Model

## A. The Household-Firm Unit

Consider a small open economy (SOE) populated by a large number of identical and infinitely lived household-firm units. Each unit derives utility from consumption ($c_t$), real money balances ($m_t^d$), and not working $(1 - h_t^T - h_t^N)$ according to

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\left[(c_t)^\gamma \left(m_t^d\right)^{1-\gamma}\right]^{1-\sigma} - 1}{1-\sigma} + \psi(1 - h_t^T - h_t^N) \right\} \tag{1}$$

and

$$c_t = (c_t^T)^\alpha (c_t^N)^{(1-\alpha)}, \tag{2}$$

where $\beta$, $\gamma \in (0,1)$, and $\psi, \sigma > 0$ but $\sigma \neq 1$; $E_0$ is the expectations operator conditional on the set of information available at time 0; $c_t^T$ and $c_t^N$ denote the consumption of traded and non-traded goods; $m_t^d = \frac{M_t^d}{p_t}$ are real money balances defined in terms of the CPI $p_t$; $h_t^T$ and $h_t^N$ stand for labor supplied to the production of traded and non-traded goods, respectively, and $\alpha \in (0,1)$ is the share of traded goods in the consumption aggregator (2). We interpret this share as a measure of the degree of trade openness of the economy. As $\alpha$ goes to zero, domestic agents do not value internationally traded goods and the economy is basically closed. Whereas if $\alpha$ goes to one, non-traded goods are negligible in consumption and the economy becomes completely open.

The utility specification in (1) is general enough to show *analytically* how Taylor-rule-induced cyclical dynamics depend on the degree of openness $\alpha$. By defining $U = \frac{\left(c_t^\gamma m_t^{1-\gamma}\right)^{1-\sigma} - 1}{1-\sigma}$ note that the cross derivative of utility with respect to consumption and money $U_{cm}$ satisfies $sign\{U_{cm}\} = sign\{1-\sigma\}$. So we can distinguish between the case of Edgeworth substitutes for which $U_{cm} < 0$, when $\sigma > 1$, and the case of complements for which $U_{cm} > 0$, when $\sigma < 1$.[2]

The representative unit produces traded and non-traded goods by employing labor and fixed stocks of capital, according to the technologies

$$y_t^T = z_t \left(k^T\right)^{1-\theta_T} \left(h_t^T\right)^{\theta_T} \qquad \text{and} \qquad y_t^N = z_t \left(k^N\right)^{1-\theta N} \left(h_t^N\right)^{\theta_N}, \tag{3}$$

where $\theta_T, \theta_N \in (0,1)$ are the labor shares, and $z_t$ is an aggregate productivity shock following a stationary AR(1) stochastic process. We assume that $z_t$ is the sole source of fundamental uncertainty in the economy.

The law of one price holds for traded goods and their foreign price, $P_t^{Tw}$, is normalized to one. Hence $P_t^T = \mathcal{E}_t$, where $P_t^T$ is their domestic currency price and $\mathcal{E}_t$ is the nominal exchange rate. In

[2] The case of $\sigma = 1$ corresponds to the case of separability between consumption and money in the utility function. It can be easily shown that no equilibrium cycles occur in this case.

this regard, the model features complete exchange rate pass-through. This and equation (2) imply the following expression for the CPI:

$$p_t \equiv \frac{(\mathcal{E}_t)^{\alpha}\left(P_t^N\right)^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}, \tag{4}$$

which can be used to derive the gross CPI-inflation rate

$$\pi_t \equiv \frac{p_t}{p_{t-1}} = \epsilon_t^{\alpha}(\pi_t^N)^{(1-\alpha)}. \tag{5}$$

As can be seen, the CPI inflation is just a weighted average of the (gross) nominal devaluation rate or traded goods inflation, $\epsilon_t \equiv \frac{\mathcal{E}_t}{\mathcal{E}_{t-1}}$, and the (gross) non-traded goods inflation, $\pi_t^N \equiv \frac{P_t^N}{P_{t-1}^N}$, with weights related to the degree of openness, $\alpha$. The real exchange rate, in turn, is defined as the ratio of the price of traded goods and the price of non-traded goods

$$e_t \equiv \frac{\mathcal{E}_t}{P_t^N}, \tag{6}$$

and evolves according to

$$\frac{e_t}{e_{t-1}} = \frac{\epsilon_t}{\pi_t^N}. \tag{7}$$

The household-firm units have access to a complete set of internationally traded claims, as in Galí and Monacelli (2005).[3] In each period $t \geq 0$, the agents can purchase two types of financial assets: fiat money $M_t^d$ and nominal state contingent claims, $D_{t+1}$. The latter pay one unit of foreign currency for a specific realization of the fundamental shocks in $t+1$. Under complete markets the representative agent's flow constraint can be written as

$$M_t^d + E_t Q_{t,t+1} D_{t+1} \leq W_t + \mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t \tau_t - \mathcal{E}_t c_t^T - P_t^N c_t^N, \tag{8}$$

where $E_t Q_{t,t+1} D_{t+1}$ denotes the cost of all contingent claims bought at the beginning of period $t$ and $Q_{t,t+1}$ refers to the period-$t$ price of a claim to one unit of currency delivered in a particular state of period $t+1$, divided by the probability of occurrence of that state and conditional on information available in period $t$. Constraint (8) says that the total end-of-period nominal value of the financial assets can be worth no more than the value of the financial wealth brought into the period, $W_t = M_{t-1}^d + D_t$, plus non-financial income net of the value of taxes, $\mathcal{E}_t \tau_t$, and the value of consumption spending. Since the expected period-$t$ price of a claim is equal to the inverse of the risk-free gross nominal interest rate, that is $E_t Q_{t,t+1} = \frac{1}{R_t}$, we can write constraint (8) as the following period-by-period constraint:

$$E_t Q_{t,t+1} W_{t+1} \leq W_t + \mathcal{E}_t y_t^T + P_t^N y_t^N - \mathcal{E}_t \tau_t - \frac{R_t - 1}{R_t} M_t^d - \mathcal{E}_t c_t^T - P_t^N c_t^N. \tag{9}$$

The representative unit is also subject to a Non-Ponzi game condition

$$\lim_{j\to\infty} E_t q_{t+j} W_{t+j} \geq 0 \tag{10}$$

[3] From the technical point of view, complete markets serve the purpose of ruling out the unit root problem of the small open economy (see Schmitt-Grohé and Uribe, 2003). This allows us to derive meaningful local determinacy of equilibrium results and compare them to the ones derived under our global equilibrium analysis.

at all dates and under all contingencies, where $q_{t+j} = Q_{t,t+1}Q_{t+1,t+2}.....Q_{t+j-1,t+j}$.

The problem of the representative household-firm unit reduces then to choose the sequences $\{c_t^T$, $c_t^N$, $h_t^T$, $h_t^N$, $M_t^d$, $W_{t+1}\}_{t=0}^{\infty}$ in order to maximize (1) subject to (2), (3), (9), and (10), given $W_0$ and the time paths of $R_t$, $\mathcal{E}_t$, $P_t^N$, $Q_{t,t+1}$ , and $\tau_t$. Since the utility function specified in (1) implies that the preferences of the agent display non-satiation, both constraints (9) and (10) hold with equality. The first order conditions correspond to these constraints and

$$\alpha\gamma \left(c_t^T\right)^{\alpha\gamma(1-\sigma)-1} \left(c_t^N\right)^{(1-\alpha)\gamma(1-\sigma)} \left(m_t^d\right)^{(1-\gamma)(1-\sigma)} = \lambda_t \tag{11}$$

$$\frac{\alpha c_t^N}{(1-\alpha)c_t^T} = e_t \tag{12}$$

$$\frac{\lambda_t}{e_t}\theta_N \left(h_t^N\right)^{(\theta_N-1)} = \psi = \lambda_t\theta_T \left(h_t^T\right)^{(\theta_T-1)} \tag{13}$$

$$m_t^d = \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\left(\frac{R_t}{R_t-1}\right) c_t^N e_t^{-\alpha} \tag{14}$$

$$\frac{\lambda_t}{\mathcal{E}_t}Q_{t,t+1} = \beta\frac{\lambda_{t+1}}{\mathcal{E}_{t+1}} \tag{15}$$

where $\frac{\lambda_t}{\mathcal{E}_t}$ is the Lagrange multiplier of the budget constraint.

The interpretation of conditions (11)-(15) is well-known. Equation (11) is the intertemporal envelope condition that makes the marginal utility of consumption of traded goods equal to the marginal utility of wealth measured in terms of traded goods ($\lambda_t$). Condition (12) implies that the marginal rate of substitution between traded and non-traded goods must be equal to the real exchange rate, while condition (13) equalizes the value of the marginal products of labor in both sectors. Equation (14) represents the demand for real money balances, and condition (15) describes a standard pricing equation for one-step-ahead nominal contingent claims for each period $t$ and for each possible state of nature.

## B. The Government

The government issues money, $M_t^s$, and a one period risk-free domestic bond, $B_t^s$, which pays a gross risk-free nominal interest rate $R_t$. It cannot issue or hold state contingent claims. It chooses the path of taxes to satisfy its intertemporal budget constraint in conjunction with a transversality condition at all times. As a result, we can ignore its budget constraint in the analysis to follow.

Monetary policy, on the other hand, is described as a forward-looking interest rate rule, which sets the nominal interest rate, $R_t$, as a continuous and increasing function of the deviation of the expected future CPI inflation rate, $E_t\pi_{t+1}$, from a target, $\pi^*$. Our interest in this rule is motivated by the empirical evidence of Clarida et al. (1998), as well as Cogley and Sargent (2005), supporting the view that central banks are forward-looking in their monetary policy setting. For analytical purposes, we use the same functional form as that in Benhabib et al. (2002a) and Eusepi (2007):

$$R_t = \rho(E_t\pi_{t+1}) \equiv 1 + (R^* - 1)\left(\frac{E_t\pi_{t+1}}{\pi^*}\right)^{\frac{A}{R^*-1}}, \tag{16}$$

where $R^* = \frac{\pi^*}{\beta} > 1$ corresponds to the interest rate target, and assume that $R_t$ always satisfies the zero bound, i.e., $R_t > 1$. We focus on rules that satisfy the Taylor principle, also known as active rules, whose elasticity to inflation at the target steady state, $\frac{\rho'(\pi^*)\pi^*}{\rho(\pi^*)} = \frac{A}{R^*}$, is strictly bigger than 1.

**Assumption 0:** *The rule is active:* $\xi \equiv \frac{A}{R^*} > 1$.

From now on we will refer to $\xi$ as the degree of activism towards inflation.

## C. International Capital Markets

We assume free international capital mobility. Then the no-arbitrage condition $Q^w_{t,t+1} = Q_{t,t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}$ holds, where $Q^w_{t,t+1}$ refers to the period-$t$ foreign currency price of a claim to one unit of foreign currency delivered in a particular state of period $t+1$ divided by the probability of occurrence of that state and conditional on information available in period $t$. Furthermore, $\frac{\lambda^w_t}{P^{Tw}_t}Q^w_{t,t+1} = \frac{\lambda^w_{t+1}}{P^{Tw}_{t+1}}\beta^w$ holds for the representative agent in the rest of the world (ROW), where $\lambda^w_t$ and $\beta^w$ denote, respectively, the marginal utility of nominal wealth and the subjective discount rate of the ROW. Combining this with condition (15) and assumptions $P^{Tw}_t = 1$, $\beta^w = \beta$, and $P^T_t = \mathcal{E}_t$ yields $\frac{\lambda_{t+1}}{\lambda_t} = \frac{\lambda^w_{t+1}}{\lambda^w_t}$, which holds at all dates and under all contingencies. This condition implies that the domestic marginal utility of wealth is proportional to its foreign counterpart: $\lambda_t = \Lambda\lambda^w_t$, where $\Lambda$ is a constant that determines the wealth difference between the SOE and the ROW. As in Schmitt-Grohé and Uribe (2003), we assume that $\lambda^w_t$—and hence $\lambda_t$—is constant overtime

$$\lambda_t = \lambda = \Lambda\lambda^w. \tag{17}$$

Equations (15) and (17), together with $E_tQ_{t,t+1} = \frac{1}{R_t}$, imply the following expression:

$$R_t = \frac{1}{\beta E_t\frac{1}{\epsilon_{t+1}}}, \tag{18}$$

which is reminiscent of an uncovered interest parity (UIP) condition.

## D. Equilibrium

We will focus on perfect foresight equilibria, where agents in the economy forecast correctly all the anticipated variables. Given our focus on endogenous fluctuations, we set $z_t = 1$ for $t \geq 0$. Therefore, for any variable $x_t$, we have that $E_t x_{t+j} = x_{t+j}$ for $j \geq 0$, implying that we can drop the expectation operator in the previous equations. For instance, condition (18) becomes

$$R_t = \frac{\epsilon_{t+1}}{\beta}, \tag{19}$$

which corresponds to the UIP condition with $\frac{1}{\beta}$ representing the foreign international interest rate.[4]

To provide a definition of the equilibrium dynamics, we derive a reduced non-linear form of the model. By combining conditions (11)-(14) with (17), and the market clearing conditions for money and the non-traded good—i.e., $M_t^d = M_t^s = M_t$ and $y_t^N = \left(k^N\right)^{1-\theta N}\left(h_t^N\right)^{\theta_N} = c_t^N$—we can express the real exchange rate $e_t$ as a function of the nominal interest rate $R_t$ :

$$e_t \equiv e(R_t) = \kappa \left(\frac{R_t}{R_t - 1}\right)^{\nu}, \tag{20}$$

where $\kappa$ and $\nu$ depend on structural parameters. In particular $\nu \equiv \frac{(\sigma-1)(1-\gamma)(1-\theta_N)}{\sigma[\theta_N+\alpha(1-\theta_N)]+(1-\alpha)(1-\theta_N)}$. For the discussion below, it is useful to note that[5]

$$e'\left(R_t\right) \gtreqless 0 \quad \text{for} \quad \sigma \lesseqgtr 1. \tag{21}$$

We can now combine equations (5), (7), and (19), to express the future real depreciation $\frac{e_{t+1}}{e_t}$ in terms of the ex-post real interest rate $\frac{R_t}{\pi_{t+1}}$:

$$\left(\frac{e_{t+1}}{e_t}\right)^{1-\alpha} = \beta \frac{R_t}{\pi_{t+1}}. \tag{22}$$

Equation (22) is an open economy version of the Fisher equation involving a term related to the real exchange rate depreciation, $\left(\frac{e_{t+1}}{e_t}\right)^{1-\alpha}$. It can be used with the rule (16), the definition $R^* = \frac{\pi^*}{\beta}$, and real exchange rate equation (20) to derive the following first order difference equation, which represents the reduced non-linear form of the model and will be used to pursue both the local and global equilibrium analyses:

$$\left(\frac{R_{t+1}}{R_{t+1} - 1}\right)^{\chi} = \frac{R_t}{R^*}\left(\frac{R^* - 1}{R_t - 1}\right)^{\frac{R^*-1}{A}}\left(\frac{R_t}{R_t - 1}\right)^{\chi}, \tag{23}$$

[4] This holds by the previous analysis since $E_t Q_{t,t+1}^w = \frac{1}{R_t^w} = \beta$.

[5] To see why, consider the case of $\sigma < 1$ (Edgeworth complements) and, without loss of generality, assume that the economy has a fixed endowment of non-traded goods, such that $y_t^N = c_t^N = \overline{y}^N$. This together with (12), (14) and (17) allow us to write condition (11) as follows:

$$U_{c^T}\left(c^T\left(e_t, \overline{y}^N\right), \overline{y}^N, m^d\left(\overline{y}^N, e_t, R_t,\right)\right) = \lambda,$$

implying that the marginal utility from traded-goods consumption must be constant. By the definition of $c_t$ in (2) and the fact that $sign\left\{U_{cm}\right\} = sign\left\{1-\sigma\right\}$, the cross-derivative $U_{c^T m}$ is positive. Now suppose the nominal interest rate $R_t$ shifts upwards. By the money demand equation (14), $m_t^d$ decreases, pushing down the left-hand side of $U_{c^T}(.,.) = \lambda$. To restore the equilibrium, a real exchange rate depreciation is necessary, since a higher $e_t$ would imply lower traded-good consumption, and, by concavity, push up marginal utility. Hence, when $\sigma < 1$, $e_t$ and $R_t$ are positively related. By a similar argument, it is possible to motivate why $e'\left(R_t\right) < 0$ when $\sigma > 1$. If $\sigma = 1$, movements in $R_t$ do not affect $e_t$, and the real exchange rate is constant along the equilibrium path.

where

$$\chi \equiv \frac{(\sigma-1)(1-\alpha)(1-\gamma)(1-\theta_N)}{\sigma[\theta_N+\alpha(1-\theta_N)]+(1-\alpha)(1-\theta_N)} \gtreqless 0 \text{ for } \sigma \gtreqless 1. \tag{24}$$

Using (23), we can provide the following definition of a perfect foresight equilibrium in our model.

**Definition 1** *Given the target $R^*$ and the initial condition $R_0$, a perfect foresight equilibrium (PFE), under a forward-looking interest rate rule, is a deterministic process $\{R_t\}_{t=0}^{\infty}$, with $R_t > 1$ for any $t$, satisfying equation (23).*

Although this definition is stated exclusively in terms of the nominal interest rate ($R_t$), multiple PFE solutions to (23) imply *real* indeterminacy of all the endogenous variables, as a result of the non-separability between money and consumption in preferences.[6] In fact, for any given $R_t$, equations (5), (7), (11)-(14), (17), (19), and the market clearing conditions for money and the non-traded good, can be used to obtain all the remaining real endogenous variables.

For the equilibrium analysis, we first identify the steady state(s) of the economy. Setting $R_t = R_{t+1} = R^{ss}$ in (23), we obtain:

$$(R^*-1)^{\frac{R^*-1}{A}} R^{ss} = R^* (R^{ss}-1)^{\frac{R^*-1}{A}}. \tag{25}$$

Clearly the target $R^*$ of the central bank is a solution to (25) and therefore a feasible steady state. But since the rule is active at $R^*$ (Assumption 0), then a second unintended steady state $R^L \in (1, R^*)$ exists. At this state, the elasticity of the rule to inflation is given by $\frac{A}{R^L} < 1$. The following proposition formalizes the existence of this passive steady state. From now on, we are going to refer to $R^*$ as the *active or target* steady state and to $R^L$ as the *passive or unintended* steady state.

**Proposition 1** *If $\frac{A}{R^*} > 1$ (an active rule) and $R^{ss} > 1$ (the zero lower bound) then, besides the solution $R^{ss} = R^*$, there exists a value $R^L \in (1, R^*)$ solving (25).*

**Proof.** See the Appendix. ■

As in Benhabib et al. (2002a), the existence of two steady states will play a crucial role in the derivation of our results. Note that it depends exclusively on policy parameters. However, as we will show below, this does not preclude the possibility that the presence of *global* dynamics *out of steady state* may be driven by other structural parameters, including the degree of openness.

[6] By indeterminacy we refer to a situation where one or more real variables are not pinned down by the model. We use the following terms interchangeably: (i) "indeterminacy" and "multiple equlibria", and (ii) "determinacy" and a "unique equilibrium."

# III. Local and Global Dynamics

In the main analysis, we will keep constant the structural parameters $\beta$, $\gamma$ and $\theta_N$, as well as the policy parameters $A$ and $R^*$. This will allow us to compare the global dynamics of economies that implement the same monetary rule but differ in the degree of openness $\alpha$ and the risk aversion parameter $\sigma$. We will use the following two assumptions.

**Assumption 1:** $1-\gamma > \frac{\theta_N}{1-\theta_N}\frac{1}{2}(R^*-1)\left(1-\frac{R^*}{A}\right)$ and **Assumption 2:** $R^*-1 > A\left(R^L-1\right)$.

Assumption 1 is expressed in terms of the share of real money balances in utility $1-\gamma$. It requires money to be sufficiently important in providing transaction services and will be used in the proofs of local determinacy and existence of period-2 cycles when $\sigma > 1$ (Edgeworth substitutability). As in the closed economy models of Benhabib et al. (2002a) and Eusepi (2007), money is "essential" for the existence of cycles.[7] In fact, a similar assumption would be needed in any closed or open economy money-in-the-utility-function (MIUF) model with endogenous labor choice to derive *analytical* results. Given the values of the share of labor $\theta_N$ and the share of money $1-\gamma$ that are commonly used in the literature, Assumption 1 holds for a very wide range of parametrizations of the interest rate target $R^*$ and the degree of policy activism $\xi \equiv \frac{A}{R^*}$.[8]

Assumption 2 requires a sufficiently positive spread between the active and the passive steady state and will be used in the proof of endogenous cycles around the unintended low steady state, when $\sigma < 1$ (Edgeworth complementarity). Models with money in the production function (MIPF) do not require this assumption, since cycles appear *only* around the active steady state. But a similar assumption would be needed in the context of a MIUF closed economy model, to prove *analytically* the existence of cycles around the low steady state.[9] Moreover, our numerical simulations show that this assumption generally holds for any rule which is active at the target, i.e. $\frac{A}{R^*} > 1$.[10]

[7] Benhabib et al. (2002a) use a flexible-price money-in-the-production-function (MIPF) set-up and rely on numerical simulations to show the existence of cycles. They do not have to state explicitly this type of assumptions, since they do not derive analytical results for cycles. But clearly money is "essential", given that it is used to some degree in productive activities. Similar considerations apply to the part of the paper by Eusepi (2007) that studies global determinacy under a MIPF set-up. Eusepi (2005) also looks at an endowment MIUF set-up. And although he relies on numerical simulations, it is possible to see that in his endowment set-up, Assumption 1 reduces to the inequality $1-\gamma > 0$, which holds for any positive share of money in utility. In fact for $\theta_N = 0$ in (3), our economy would be equivalent to one with a non-traded good endowment and the same inequality $1-\gamma > 0$ would hold.

[8] Based on empirical evidence, in Table 2 below, we assign a value of 0.03 to $1-\gamma$ and of 0.56 to $\theta_N$. A simple numerical evaluation shows that the right-hand side of the inequality of Assumption 1 is always smaller than 0.015 for any combination of $R^* \in [1.0025, 1.025]$ and $\xi \in [1.01, 10]$. This corresponds to an annual interest rate target between 1% and 10% and spans the universe of active rules from weakly to extremely active.

[9] As mentioned before, Benhabib et al. (2002a) and Eusepi (2007) focus on cycles in MIPF models. Eusepi (2005) studies an endowment MIUF model of a closed economy and resorts to numerical simulations to investigate the existence of cycles around the passive steady state.

[10] In fact, the assumption holds for an approximation. Let $RHS\,(R)$ and $LHS(R)$ be, respectively, the right- and the left-hand sides of (25). Then Assumption 2 written as $R^L < 1 + \frac{R^*-1}{A}$ requires $RHS\,(R) > LHS\,(R)$ for $R = 1 + \frac{R^*-1}{A}$. Taking logs and using the approximation $\ln(1+x) \approx x$ for $x$ small, Assumption 2 reduces to $\ln A < A - 1$, which holds for any $A > 1$. This is always valid in our analysis given that the rule is active, i.e., $A > R^*$.

## A. Local Determinacy

The local determinacy analysis focuses on PFE which remain bounded within a small neighborhood of the target steady state. By log-linearizing equation (23) around $R^*$, we obtain the following linear difference equation:

$$\hat{R}_{t+1} = \left(1 + \frac{\frac{R^*}{A} - 1}{\frac{\chi}{R^* - 1}}\right)\hat{R}_t. \tag{26}$$

Since $R_t$ is a non-predetermined variable, local determinacy requires the previous equation to generate explosive dynamics.

**Proposition 2** *Suppose the government follows an active forward-looking rule like (16). Then, there exist threshold values $\sigma^d > 1$ and $\alpha^d \in (0, 1)$, for the risk aversion parameter $\sigma$ and the degree of openness $\alpha$, such that the active steady state is a locally determinate equilibrium for any $\alpha \geq 0$ when $\sigma \in \left(0, \sigma^d\right)$, but only for $\alpha > \alpha^d$ when $\sigma \geq \sigma^d$.*

**Proof.** See the Appendix. ■

This proposition shows the importance of $\alpha$ and $\sigma$ in the local equilibrium characterization. In a nutshell, active forward-looking rules guarantee local determinacy in the following cases: when, regardless of the degree of openness $\alpha$, the risk aversion parameter $\sigma$ is sufficiently low; and when, for higher values of $\sigma$, the economy is sufficiently open.[11] Note that Proposition 2 also implies that, when consumption and money are Edgeworth substitutes ($\sigma > 1$), an active rule might induce local sunspot-driven fluctuations—i.e., multiple local equilibria—if the economy is sufficiently open ($\alpha < \alpha^d$). In contrast, for any degree of trade openness, an active rule will always ensure local determinacy if consumption and money are complements ($\sigma < 1$). However, as we show next, local determinacy does not rule out global indeterminacy, which may be affected by the degree of openness.

## B. Cycles and Chaos

### B.1. The Main Results from a Global Non-linear Analysis

For the global analysis, we rewrite equation (23) as $R_{t+1} = f\left(R_t\right)$, where

$$f\left(R_t\right) \equiv \frac{1}{1 - J\left(R_t\right)^{\frac{1}{\chi}}} \tag{27}$$

and

$$J\left(R_t\right) \equiv \frac{R^*}{\left(R^* - 1\right)^{\frac{R^*-1}{A}}} \frac{\left(R_t - 1\right)^{\chi + \frac{R^*-1}{A}}}{R_t^{1+\chi}}. \tag{28}$$

[11] A quick inspection of Proposition 2 also suggests that if the economy is very open, an active rule always leads to local equilibrium determinacy.

We study the global PFE that satisfies $R_{t+1} = f(R_t)$ for a given initial condition $R_0 > 1$ and subject to the zero lower bound (ZLB) $f^n(R_0) > 1$ for any $n \geq 1$. The economy displays a globally indeterminate PFE, if there exists a continuum of initial conditions $R_0$ for which (27) satisfies the ZLB at any point in time. We are interested in global indeterminacy taking the form of cyclical and chaotic dynamics conforming to the following definitions.

**Definition 2** ***Period-n cycle.*** *A value "R" is a point of a period-n cycle if it is a fixed point of the n-th iterate of the mapping $f(.)$, i.e., $R = f^n(R)$, but not a fixed point of an iterate of any lower order. If "R" is such, we call the sequence $\left\{R, f(R), f^2(R), ..., f^{n-1}(R)\right\}$ a period-n cycle.*

**Definition 3** ***Topological chaos.*** *The mapping $f(.)$ is topologically chaotic if there exists a set "S" of uncountable many initial points, belonging to its domain, such that no orbit that starts in "S" will converge to one another or to any existing period orbit.*

The global analysis requires the characterization of the function $f$ in (27) over its entire domain. It is straightforward to prove that its analytical properties mainly depend on the composite parameter $\chi$ defined in (24). More specifically, $f$ can be single-peaked (respectively, single-troughed) for $\chi$ positive (respectively, negative). This implies the possibility of a negative derivative of $f$ at either one of the two steady states, which is a necessary condition for the existence of endogenous cycles in continuously differentiable maps.[12] More precisely, by (24) and the equivalence $sign\{U_{cm}\} = sign\{1-\sigma\}$, $f$ is negatively sloped at the active steady state if $\sigma > 1$ (Edgeworth substitutability) and at the passive steady state if $\sigma < 1$ (Edgeworth complementarity).

The following Proposition contains our main result by stating the exact analytical conditions, with respect to the degree of trade openness $\alpha$, under which endogenous period-2 cycles occur in our economy.

**Proposition 3** *Suppose the government follows an active forward-looking rule like (16). Then there exist threshold values $\sigma^f \in (0,1)$ and $\sigma^d > 1$ for the risk aversion parameter $\sigma$, together with thresholds $\alpha^f \in (0,1)$ and $\alpha^d \in (0,1)$ for trade openness $\alpha$, such that*

1. *period-2 equilibrium cycles occur around the* ***passive steady state,*** *for $\alpha > \alpha^f$ when $\sigma \in \left(0, \sigma^f\right]$ and for any $\alpha \geq 0$ when $\sigma \in \left(\sigma^f, 1\right)$.*
2. *period-2 equilibrium cycles occur around the* ***active steady state,*** *for $\alpha > \alpha^d$ when $\sigma \geq \sigma^d$ and for any $\alpha \geq 0$ when $\sigma \in \left(1, \sigma^d\right)$.*

**Proof.** See the Appendix. ■

[12] See Lorenz (1993) for a general discussion on limit cycles in non-linear maps.

To simplify the statement of Proposition 3, we have abused of notation, to some extent, since the thresholds $\alpha^f$ and $\alpha^d$ can be in fact seen as functions $\alpha^f(\sigma)$ and $\alpha^d(\sigma)$ of the risk aversion parameter $\sigma$. Table 1 and Figure 1, which summarize and provide a graphical representation of Proposition 3, make these functions explicit. The table and figure also present the results of Proposition 2, allowing for a comparison of the results of both the local and the global determinacy analyses.

Using Table 1 and Figure 1, it is easy to grasp the following key messages from the global analysis.

**Table 1: Local versus Global Analysis**

| | Global Analysis | | Local Analysis |
|---|---|---|---|
| | Cycles around $R^L$ | Cycles around $R^*$ | $R^*$ locally determinate |
| $\sigma \in (0, \sigma^f)$ | $\alpha > \alpha^f(\sigma)$ | x | $\alpha \in (0,1)$ |
| $\sigma \in (\sigma^f, 1)$ | $\alpha \in (0,1)$ | x | $\alpha \in (0,1)$ |
| $\sigma \in (1, \sigma^d)$ | x | $\alpha \in (0,1)$ | $\alpha \in (0,1)$ |
| $\sigma > \sigma^d$ | x | $\alpha > \alpha^d(\sigma)$ | $\alpha > \alpha^d(\sigma)$ |

Note: $\sigma^f \in (0,1)$ and $\sigma^d > 1$. "x" stands for no cycles of any periodicity around the steady state. $R^*$ and $R^L$ stand for the target and the unintended low steady states, respectively.

First, for $\sigma$ falling outside the range $(\sigma^f, \sigma^d)$, endogenous period-2 cycles exist as long as the economy is sufficiently open to trade. That is, openness is a *necessary* condition for endogenous cycles. In this sense, a rule that responds actively to expected future inflation is more prone to induce endogenous cyclical dynamics the more open the economy.

Second, depending on whether consumption and real money balances are Edgeworth complements ($\sigma < 1$) or substitutes ($\sigma > 1$), equilibrium cycles can appear either around the passive or the active steady state, respectively. In this regard, our MIUF approach is more general than the MIPF set-up of Benhabib et al. (2002a). Were the MIPF adapted to a small open economy, it would *only* display cycles around the active steady state.[13] To the best of our knowledge, our paper is the first one to derive *analytical* conditions for the existence of cycles around the passive steady state, which depends on openness. These dynamics are rather interesting as one could think of them as cyclical liquidity traps, that is, a situation where the nominal interest rate is trapped in a small neighborhood around the low steady state.

Third, a comparison of the global and local analyses reveals some crucial differences with important policy implications. As displayed in Figure 1, global indeterminacy in the form of equilibrium cycles occurs in regions where, by local analysis, one concludes in favor of a unique equilibrium (white area).

[13] Benhabib et al. (2001a) show that, in closed economies, a MIUF model with Edgeworth substitutability ($U_{cm} < 0$) and a MIPF model are isomorphic for what concerns the *local* stability properties of Taylor rules. The literature has not proved *analytically* this isomorphism in the context of *global* analysis of *open* economies. In the Appendix we establish this global isomorphism.

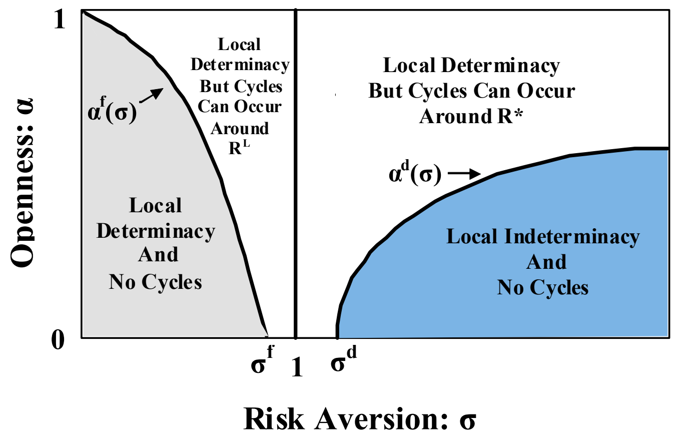


Figure 1: **Local and global equilibrium analyses for active forward-looking rules**, as the degree of openness $\alpha$ and the risk aversion coefficient $\sigma$ vary. $R^*$ and $R^L$ stand for the target and the unintended low steady state, respectively.

Even more strikingly, for $\sigma > \sigma^d$, the condition for local equilibrium determinacy is identical to the sufficient condition for the existence of period-2 cycles. Therefore, while for $\sigma > \sigma^d$ the local analysis concludes that higher openness ($\alpha$) makes active Taylor rules more prone to stabilize the economy around the target, the global analysis instead identifies openness as a source of cycles around it (global indeterminacy). This contrast is due to the fact that, by log-linearizing around the target steady state, local analysis implicitly assumes that any path starting arbitrarily close to it and diverging cannot be part of an equilibrium, since it will eventually explode violating the transversality condition. This is not the case in the global analysis, where there may exist limit cycles towards which equilibrium paths—which start arbitrarily close to the target steady state—may converge to.[14]

**Table 2: Parametrization**

| $\theta_N$ | $\beta$ | $\pi^*$ | $R^*$ | $1-\gamma$ | $\frac{A}{R^*}$ |
|---|---|---|---|---|---|
| 0.56 | 0.99 | $1.031^{\frac{1}{4}}$ | $1.072^{\frac{1}{4}}$ | 0.03 | 2.25 |

Given the complicated functional form of the map $R_{t+1} = f(R_t)$ in (27), we cannot derive simple analytical conditions for the existence of higher order cycles and chaotic dynamics. We then resort to numerical simulations for a calibrated version of our economy. This analysis also allows us to assess the quantitative relevance of the analytical results of Proposition 3.

[14] We say "may converge" rather than "will converge" because, due to the complicated functional form of $R_{t+1} = f(R_t)$ in (27), it is not possible to obtain meaningful analytical conditions for the stability of cycles. However, as the numerical analysis will show, our economy features stable limit cycles of various periodicity.

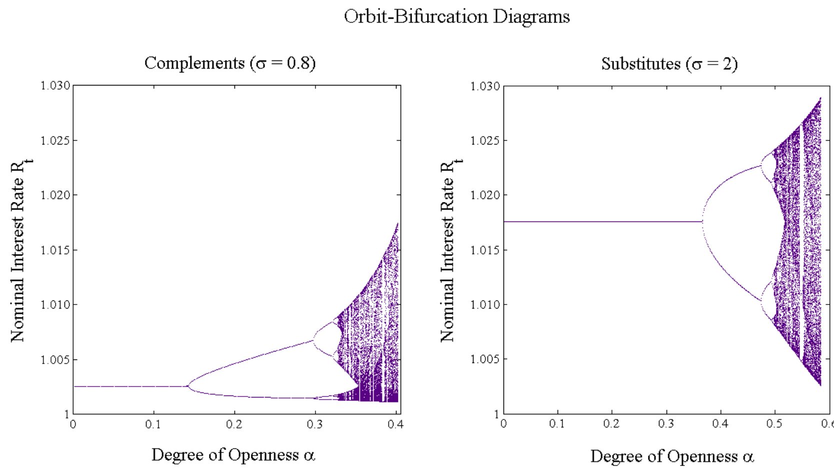


Figure 2: **Orbit-bifurcation diagrams for the degree of openness.** They show the set of limit points for the nominal interest rate $R_t$, as a function of the degree of openness $\alpha$, under Edgeworth complementarity ($\sigma = 0.8$) and sustitutability ($\sigma = 2$).

Table 2 summarizes the benchmark parametrization, where Canada is mostly used as the representative economy, and the time unit is a quarter. From Mendoza (1995) we borrow the labor income share for the non-traded sector $\theta_N = 0.56$. In line with empirical evidence by Holman (1998), we set the share of money in utility $1 - \gamma$ equal to 0.03.[15] The steady-state inflation and nominal interest rate are the averages of the CPI inflation and the central bank discount rate, during the last two decades. This yields $\pi^* = 1.031^{\frac{1}{4}}$ and $R^* = 1.072^{\frac{1}{4}}$, implying $\beta = 0.99$. For illustrative purposes we use an elasticity $\frac{A}{R^*}$ of 2.25, which satisfies the Taylor principle and reflects a moderate degree of policy activism towards inflation, as suggested by Boivin (2006) and Cogley and Sargent (2005).

The simulations presented in Figure 2 show that cycles of various periodicities and chaos exist as long as the economy is sufficiently open. We vary openness $\alpha$ over the continuum $(0, 1)$, while setting $\sigma$ equal to either 0.8 or 2.0, which correspond to consumption and money being Edgeworth complements or substitutes, respectively. The solid lines represent the stable period-$n$ limit cycles of the mapping $R_{t+1} = f(R_t)$ in (27). The PFE dynamics settle on a steady state (flat solid lines), as long as $\alpha$ is sufficiently small. This attractive steady state is either the unintended steady state $R^L \approx 1.01^{\frac{1}{4}}$, under complementarity ($\sigma = 0.8$), or the target steady state $R^* = 1.072^{\frac{1}{4}}$, under substitutability ($\sigma = 2.0$). Once $\alpha$ passes a certain threshold—about 0.15 for $\sigma = 0.8$ and 0.37 for $\sigma = 2.0$—a stable period-2 cycle appears, as displayed by the first split into two branches. As we continue increasing $\alpha$ the branches split again and a cascade of further period doubling bifurcations occurs, yielding cycles of period 4, 8, 16, and so on. Finally for sufficiently high values of $\alpha$, the rule produces cycles of odd periodicity as well as aperiodic chaotic dynamics, where the attractor of the mapping changes from a finite to an infinite set of points.

[15] This is roughly the average of the GMM estimates using U.S. data. Because of the similar degree of financial development, we take this as a reasonable value for Canada.

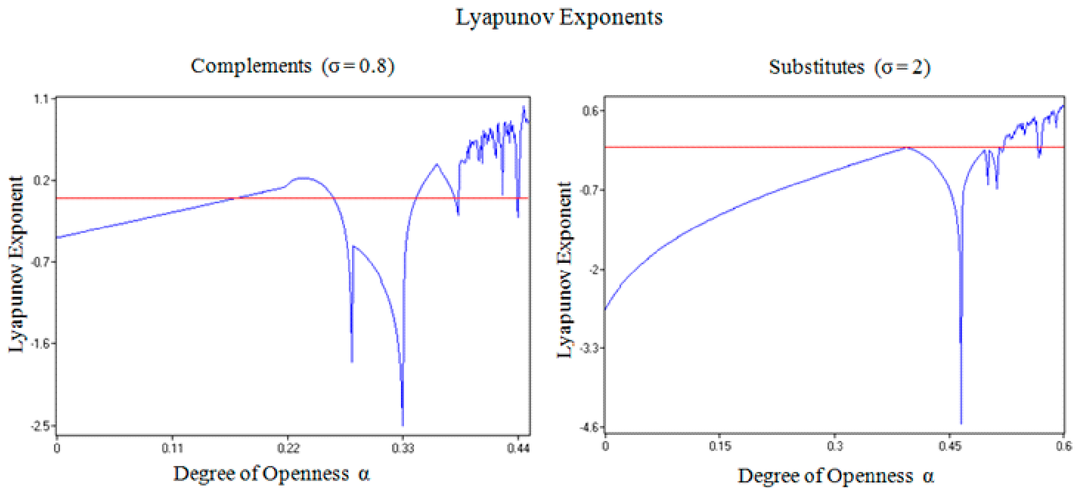


Figure 3: **Lyapunov exponents for forward-looking rules**, as a function of the degree of openness ($\alpha$) and under Edgeworth complementarity ($\sigma = 0.8$) and sustitutability ($\sigma = 2$).

Figure 3 confirms the existence of chaos for sufficiently open economies by looking at the Lyapunov exponent. A positive exponent measures the degree of exponential divergence between two series generated by the same map and starting from nearby, but not identical, initial conditions. This characterizes chaotic systems. Consistently with Figure 2, there is evidence of chaos for $\alpha > 0.38$, when $\sigma = 0.8$, and for $\alpha > 0.5$, when $\sigma = 2.0$.[16]

Openness significantly enlarges the risk aversion parameter range under which forward-looking rules induce cycles. To see this, we look at our results from another angle. While Proposition 3 defines the bifurcation thresholds with respect to trade openness, it is possible to show that, for given $\alpha$, equilibrium cycles require $\sigma$ to belong to an interval $\left(\sigma^F, \sigma^D\right)$. Formally $\sigma^F$ and $\sigma^D$ are functions of $\alpha$ and correspond to the inverse of the functions $\alpha^f(\sigma)$ and $\alpha^d(\sigma)$ depicted in Figure 1. These functions $\sigma^F(\alpha)$ and $\sigma^D(\alpha)$ are, respectively, strictly decreasing and strictly increasing in $\alpha$. As a result, the interval $\left(\sigma^F, \sigma^D\right)$ under which cycles can occur is strictly *increasing* in openness $\alpha$, implying that it is much wider for an open economy ($\alpha > 0$) than for a fully closed economy ($\alpha = 0$). In the latter case $\left(\sigma^F, \sigma^D\right) = \left(\sigma^f, \sigma^d\right)$, where $\sigma^f$and $\sigma^d$ are depicted in Figure 1. We formalize all this analysis in Corollary 1.

**Corollary 1** *Assume $\sigma \neq 1$ and $\alpha \in [0,1)$. For given $\alpha$, period-2 cycles occur only for $\sigma$ belonging to the interval $\left(\sigma^F, \sigma^D\right)$, where $\sigma^F$ and $\sigma^D$ are, respectively, strictly decreasing and strictly increasing functions in $\alpha$. Therefore, as $\alpha$ increases—i.e., openness increases—period-2 cycles occur for a much wider range of $\sigma$. The cycles are around the active steady state for $\sigma > 1$ and around the passive steady state for $\sigma < 1$.*

[16] Figure 3 is constructed using the *E&F Chaos* software package by Diks, Hommes, Panchenko, and Van der Weide developed at the Center for Non-Linear Dynamics in Economics and Finance (CeNDEF), University of Amsterdam. This software quickly iterates the non-linear equilibrium mapping $f$ and computes the set of limit points. Diks et al. (2008) provide a detailed description of the *E&F Chaos* software and its functionalities.

**Proof.** The proof follows from the proof to Proposition 3 and is available upon request. ■

Table 3 highlights the quantitative importance of Corollary 1. Using the calibration of Table 2, it allows us to compare quantitatively the case of a completely closed economy, like in Benhabib et al. (2002a) and Eusepi (2007), with the case of an open economy. For a fully closed economy, i.e., $\alpha = 0$, period-2 cycles exist for a very restricted range of values of the risk aversion, namely, $\left(\sigma^F, \sigma^D\right) = (0.82, 1.47)$.[17] As openness increases, this range also increases. When the economy is very open, e.g., $\alpha = 0.65$, this range becomes $\left(\sigma^F, \sigma^D\right) = (0.62, 10.87)$, which is substantially much wider.

**Table 3**: **Openness, Risk Aversion, and Cycles**

| | $\alpha = 0$ | $\alpha = 0.15$ | $\alpha = 0.35$ | $\alpha = 0.65$ |
|---|---|---|---|---|
| $\left(\sigma^F, \sigma^D\right)$ | $(0.82, 1.47)$ | $(0.80, 1.60)$ | $(0.75, 1.95)$ | $(0.62, 10.87)$ |

Summarizing, the results in this subsection suggest that openness is an important factor for global indeterminacy under standard Taylor rules.

### B.2. An Intuition of the Results

To build an intuition of our results, we use the rule (16) and equation (20) to rewrite the equilibrium condition (22) as:

$$\left[e\left(\rho(\pi_{t+2})\right)\right]^{1-\alpha} = \left[e\left(\rho(\pi_{t+1})\right)\right]^{1-\alpha} \beta \frac{\rho(\pi_{t+1})}{\pi_{t+1}}, \tag{29}$$

which has to be satisfied by any sequence of inflation rates in order to be a PFE. In addition, we focus on the case of $\sigma < \sigma^f < 1$. In this case we know, from Proposition 3 and Figure 1, that cyclical and chaotic dynamics exist around the passive steady state if the economy is sufficiently open. This is due to a flip bifurcation threshold $\alpha^f$, below which the passive steady state is dynamically stable and above which stability is lost. The switch from stability to instability depends on the elasticity of the real exchange rate to the interest rate factor $\frac{R_t}{R_t - 1}$, which by equation (20), is given by $\nu \equiv \frac{(\sigma-1)(1-\gamma)(1-\theta_N)}{\sigma[\theta_N + \alpha(1-\theta_N)] + (1-\alpha)(1-\theta_N)}$. For $\sigma < \sigma^f < 1$, the elasticity $\nu$ is always negative and decreasing in absolute value as $\alpha$ approaches 1. That is, the more open the economy, the smaller the impact of the nominal interest rate on the real exchange rate.

[17] By numerical analysis, we have also found that, when $\alpha = 0$, the range $\left(\sigma^F, \sigma^D\right)$, i.e., $\left(\sigma^f, \sigma^d\right)$ shrinks—and hence openness plays a bigger role to some extent—if we assume a) a lower interest rate target; b) a lower degree of activism towards inflation; c) a larger role for real money balances in preferences; or d) a lower labor share in production. For instance, other things equal, setting $\frac{A}{R^*} = 1.5$ (the standard Taylor coefficient) gives $\left(\sigma^f, \sigma^d\right) = (0.84, 1.25)$. Meanwhile, setting $R^* = 1.042^{\frac{1}{4}}$ (around 4% per year), gives $\left(\sigma^f, \sigma^d\right) = (0.89, 1.24)$.

Suppose now the economy is at the passive steady state $\pi^L \equiv \beta R^L$, where the rule is passive, i.e., it responds less than proportionally to inflation with $\frac{A}{R^L} < 1$. Assume that suddenly the private sector expects a higher inflation in the future. Since the rule is passive, the real interest rate $\frac{\rho(\pi_{t+1})}{\pi_{t+1}}$ declines. If openness $\alpha$ is high, the elasticity $\nu$ is small and the increase in $\rho(\pi_{t+1})$ has a small positive effect on $e_t$ and even less on $e_t^{1-\alpha}$.[18] As a result, the right-hand side of (29) decreases. But because of the small elasticity, to have a lower left-hand side it is necessary to have a substantial drop in $\pi_{t+2}$, i.e., $|\pi_{t+2}| > |\pi_{t+1}|$. Hence, when $\alpha$ is high, the passive steady state is dynamically unstable and the equilibrium series of inflation rates starts diverging from it.

The convergence to a cycle can be explained as follows. Suppose $\pi_{t+1} = \pi^h > \pi^L$, that is, agents expect an inflation level above the passive steady state $\pi^L$. Since the rule is passive we have that $\frac{\rho(\pi^h)}{\pi^h} < \frac{\rho(\pi^L)}{\pi^L}$; and because of the high openness—hence, a low $\nu$—the term $[e(\rho(\pi))]^{1-\alpha}$ is only slightly affected by changes in inflation, implying that the right-hand side of (29) is then lower than what it is at the steady state. But then, by (29), next period inflation must be below the steady state, i.e., $\pi_{t+2} = \pi^l < \pi^L$. Following the same argument, in the next period it must be that $\frac{\rho(\pi^l)}{\pi^l} > \frac{\rho(\pi^L)}{\pi^L}$, pushing the right-hand side of (29) above the steady state, thus requiring next period inflation to be at $\pi^h > \pi^L$, and so on.

Similar arguments can be invoked to construct a cyclical PFE around the active steady state for the case of $\sigma > \sigma^d > 1$.

### B.3. Policy Activism and the Degree of Openness

To underscore the quantitative importance of our results, we investigate their implications in terms of policy activism. We use basins of attractions to unveil the interaction between the degree of openness $\alpha$ and the rule's elasticity to inflation $\xi$. The left-hand side of Figure 4 shows these basins when $\sigma = 0.8$. The light grey area corresponds to trajectories converging to the passive steady state. Moving rightward, the darker grey area corresponds to period-2 cycles and the black area to period-4 cycles. The dotted area represents period-3 as well as higher order cycles and chaos. In the white area, the economy diverges. The areas of the right-hand side of the figure, when $\sigma = 2$, have the same interpretation with two caveats: the left-hand light grey area, associated with low $\alpha$'s, corresponds to trajectories converging to the active steady state; while the right-hand light grey area represents trajectories violating the restriction $R_t > 1$ for any $t$.

Two results stand out from Figure 4. When the economy is closed ($\alpha = 0$) and either $\sigma = 0.8$ or $\sigma = 2.0$, cycles and chaos are ruled out for *any* degree of policy activism ($\xi$). As the economy opens up to trade, these complex dynamics can be induced by *quantitatively realistic* degrees of policy activism. For instance, when $\sigma = 0.8$ and $\alpha \approx 0.3$, period-2 cycles are possible for degrees of activism $\xi$ that are between the famous Taylor coefficient of 1.5 and 3.0. Of course, judging the feasibility of these degrees of activism also depends on feasible values for the degree of openness and other structural parameters. In this regard, consider Table 4 where, given some calibrated values for these structural characteristics

[18] Recall the results in (21): $e'(R_t) > 0$ if $\sigma < 1$.

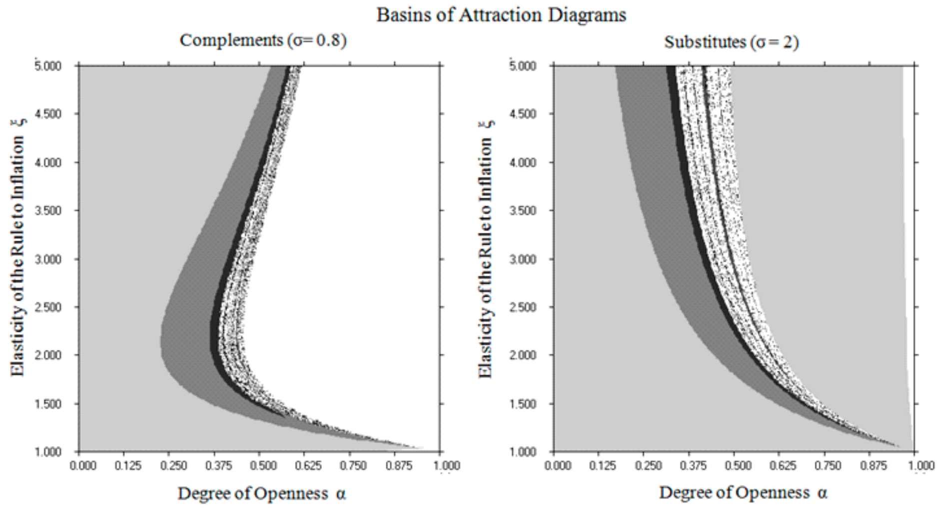


Figure 4: **Basis of attraction for forward-looking rules** for different combinations of the degree of openness $\alpha$ and the elasticity of the forward-looking rule to inflation $\xi$, under Edgeworth complementarity ($\sigma = 0.8$) and substitutability ($\sigma = 2$). See the main text for the interpretation of the colored areas.

of several inflation targeting countries, we calculate the thresholds for the elasticity $\xi$ above which, according to our model, complex global dynamics occur. Clearly these thresholds are to some extent possible in practice, in particular for Norway and Sweden. Interestingly, Bask (2002) and Gogas and Serletis (2000) suggest that the nominal and real exchange rates in the Scandinavian countries may have followed chaotic dynamics in the 1980s and 1990s.

**Table 4: Activism, Cycles, and Chaos**

| Country | Parametrization | 2-Period Cycles 1/ | Chaos 1/ |
|---|---|---|---|
| Norway | $\sigma = 1.58$ $\alpha = 0.34$ $\pi^{*}= 0.025$ $\theta_N= 0.57$ | $\xi > 1.75$ | $\xi > 2.26$ |
| Sweden | $\sigma = 1.58$ $\alpha = 0.26$ $\pi^{*}= 0.02$ $\theta_N= 0.65$ | $\xi > 1.7$ | $\xi > 2.17$ |
| United Kingdom | $\sigma = 1.59$ $\alpha = 0.34$ $\pi^{*}= 0.02$ $\theta_N= 0.45$ | $\xi > 2.61$ | $\xi > 5.47$ |
| Australia | $\sigma = 1.58$ $\alpha = 0.31$ $\pi^{*}= 0.025$ $\theta_N= 0.48$ | $\xi > 2.26$ | $\xi > 3.80$ |
| New Zealand | $\sigma = 1.58$ $\alpha = 0.38$ $\pi^{*}= 0.015$ $\theta_N= 0.56$ | $\xi > 1.94$ | $\xi > 2.75$ |

Sources: Ogaki et al. (1996) for $\sigma$; Petursson (2004) for $\pi^{*}$; Bentolilla and Saint Paul (2003) for $\theta_N$; and Campa and Goldberg (2006) for $\alpha$.

1/ The thresholds for $\xi$ are for a forward-looking rule and assume $1 - \gamma = 0.03$ for all countries.

# IV. Extensions and Robustness

This Section discusses the robustness of our results to alternative variations of our framework, including incomplete exchange rate pass-through, incomplete markets, different timings for the policy rule and for money in utility, and different preferences and technologies.[19] Details of some of these extensions can be found in the Appendix.

## A. Incomplete Exchange Rate Pass-Through

We have assumed perfect exchange rate pass-through into import prices. This, however, is at odds with empirical evidence. Campa and Goldberg (2006), for instance, suggest that the exchange rate pass-through into import prices varies across countries and has changed over time. We then introduce imperfect pass-through by assuming distribution costs as in Burnstein, Neves, and Rebelo (2003). More specifically, we assume that to consume one unit of the traded good, it is required $\eta$ units of the non-traded good as a result of some distribution services.[20] Then the consumer price of the traded good is

$$P_t^T = \tilde{P}_t^T + \eta P_t^N, \tag{30}$$

where $\tilde{P}_t^T$ is the producer price in domestic currency. To simplify the analysis we assume that the law of one price holds for traded goods at the production level and normalize the foreign price of the traded good to one ($\tilde{P}_t^{Tw} = 1$). Thus $\tilde{P}_t^T = \mathcal{E}_t \tilde{P}_t^{Tw} = \mathcal{E}_t$.

The presence of distribution services leads to imperfect exchange rate pass-through into import prices. To see this we combine (30), $\tilde{P}_t^T = \mathcal{E}_t$, and $e_t = \mathcal{E}_t / P_t^N$ to express the gross inflation of import prices $\pi_t^T$ as

$$\pi_t^T = \left( \frac{e_{t-1}}{e_{t-1} + \eta} \right) \epsilon_t + \left( \frac{\eta}{e_{t-1} + \eta} \right) \pi_t^N,$$

where $\epsilon_t \equiv \frac{\mathcal{E}_t}{\mathcal{E}_{t-1}}$ and $\pi_t^N \equiv \frac{P_t^N}{P_{t-1}^N}$ correspond to the gross nominal depreciation and non-traded inflation, respectively. Clearly when $\eta = 0$—the case already studied—then there is perfect pass-through of the nominal depreciation rate into the inflation of import prices, since $\frac{\partial \pi_t^T}{\partial \epsilon_t} = 1$. But if $\eta > 0$ then we obtain imperfect pass-through, given that $\frac{\partial \pi_t^T}{\partial \epsilon_t} = \left( \frac{e_{t-1}}{e_{t-1} + \eta} \right) \in (0, 1)$. As the parameter of distribution costs $\eta$ increases, the degree of pass-through decreases.

Increasing $\eta$—decreasing the exchange rate pass-through—has a non-trivial impact on the global dynamics. Using the parametrization of Table 2 and setting $\alpha = 0.4$, Figure 5 displays the related orbit-bifurcation diagrams for $\eta$. When $\eta = 0$—perfect pass-through—chaos and period-2 cycles occur when $\sigma = 0.8$ and $\sigma = 2.0$, respectively. But these complex dynamics disappear once $\eta$ passes a

[19] An additional extension is the inclusion of nominal price rigidities (see the Appendix). Similar to Eusepi (2005), we find that under sticky prices, liquidity traps that *converge* to the "unintended" low steady state are the only type of global PFE dynamics.

[20] This is certainly not the only approach to model imperfect exchange rate pass-through. For other approaches see Monacelli (2005), among others.

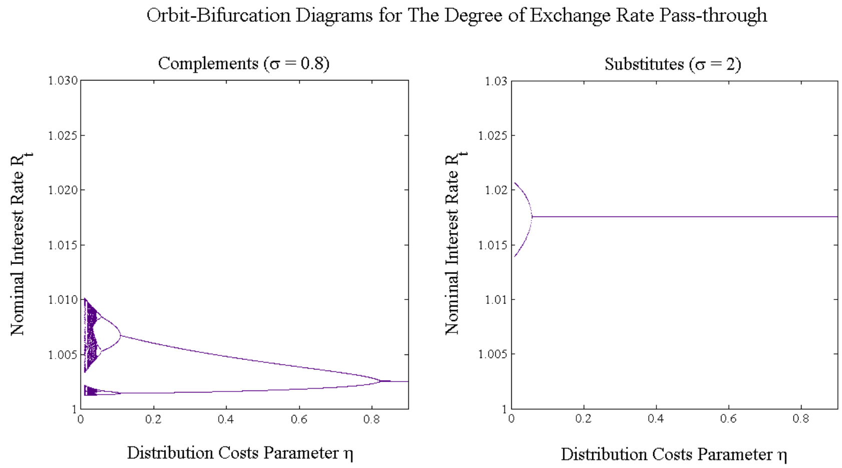


Figure 5: **Orbit-bifurcation diagrams for the degree of exchange rate pass-through.** They show the set of limit points for the nominal interest rate $R_t$ as a function of the distribution costs parameter $\eta$, under Edgeworth complementarity ($\sigma = 0.8$) and substitutability ($\sigma = 2$). A lower $\eta$ means a higher exchange rate pass-through into import prices.

certain threshold, as the economy eventually settles on either the passive or the active steady state. Such thresholds for $\eta$ are about 0.84 for $\sigma = 0.8$ and 0.07 for $\sigma = 2$. In this regard, openness plays a more fundamental role for cyclical and chaotic dynamics, the lower $\eta$ (the higher the exchange rate pass-through).

**Proposition 4** *Given openness, forward-looking rules are more prone to induce cyclical and chaotic dynamics the higher the degree of exchange rate pass-through.*

## B. Incomplete Markets

The assumption about complete markets was made in order to allow a meaningful local determinacy analysis. However, it is not essential for the existence of cyclical and chaotic dynamics. To see this, suppose that the agent has access to only one international bond $b_t^w$, paying a constant risk-free rate $R^w$, and a domestic bond $B_t$, paying the time-varying risk-free interest rate $R_t$. By the first-order conditions and the standard assumption $\beta R^w = 1$, we would still obtain $\lambda_t = \lambda_{t+1}$ together with the UIP condition $\epsilon_t = \beta R_t$. By the same steps highlighted in Section D, it is immediate to show that the global PFE dynamics are entirely described by the same non-linear difference equation (23). Hence, under incomplete international markets, the results on the existence of cycles/chaos would be exactly identical to those of the sub-Section B.1. In addition to that, one could also show that cycles and chaos may occur in the accumulation of foreign bonds, $b_t^w$, and therefore in the current account, $ca_t = b_t^w - b_{t-1}^w$.

## C. Alternative Timings for the Policy Rule

Our focus on forward-looking interest rate rules has been mostly motivated by their tractability and the strong empirical support in the data. Alternative timings include contemporaneous rules of which Lubik and Schorfheide (2007) provide evidence, as well as backward-looking rules.

### Contemporaneous Rules

Suppose the central bank sets the nominal interest rate in response to *current* CPI inflation according to $R_t = 1 + (R^* - 1)\left(\frac{\pi_t}{\pi^*}\right)^{\frac{A}{R^*-1}}$ with $R^* = \frac{\pi^*}{\beta}$ and $\xi = \frac{A}{R^*} > 1$ (active rule). We find that the degree of openness affects the existence of complex dynamics, but only for the case of $\sigma > 1$ (Edgeworth substitutability). However, in contrast to the previous results, openness *also* determines whether the cycles are centered around the passive or active steady state.[21] More specifically, cycles and chaos occur around the active (respectively, passive) steady state for low (respectively, intermediate) degrees of openness.[22]

### Backward-Looking Rules

For a policy rule that responds to *past* CPI inflation, we find that regardless the degree of openness, the PFE dynamics always converge to either the active or the passive steady state, depending on whether consumption and money are Edgeworth substitutes or complements, respectively. Thus the only long-run equilibrium is a liquidity trap. The non-existence of cycles is in line with the findings by Eusepi (2007).[23]

## D. Cash-in-Advance Timing of Money in Utility

Our set-up follows the "Cash-When-I'm-Done" (CWID) timing for real money balances, traditionally adopted in the literature. Carlstrom and Fuerst (2001) have promoted the alternative "Cash-In-Advance" (CIA) timing, where the real money balances entering the agent's utility are those left after leaving the bond market, but before entering the goods market. As the next proposition shows, openness still plays a key role for the presence of period-2 cycles under this CIA timing.

**Proposition 5** *Suppose that the government follows the active forward-looking rule (16), and let $\sigma > 1$ (consumption and money are Edgeworth substitutes). Then, under the CIA timing of money in utility, there exists thresholds $\sigma^{cia} > 1$ and $\alpha^{cia} \in (0,1)$, such that period-2 equilibrium cycles occur around the active steady state for any $\alpha \in (0,1)$ if $\sigma \in \left(1, \sigma^{cia}\right)$ and for $\alpha > \alpha^{cia}$ if $\sigma \geq \sigma^{cia}$.*

**Proof.** See the Appendix. ■

[21] Recall that under a forward-looking rule whether cycles/chaos occur around the passive or active steady state depend entirely on $\sigma$, while $\alpha$ determines whether they exist or not.

[22] See the Appendix.

[23] See the Appendix.

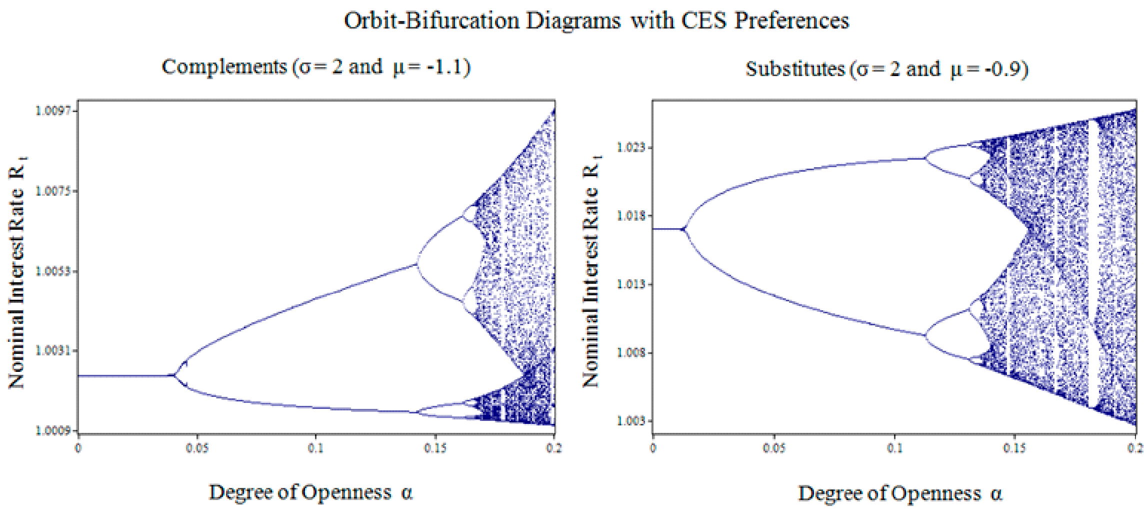


Figure 6: **Orbit-bifurcation diagrams with CES preferences.** They show the set of limit points for the nominal interest rate $R_t$, as a function of the degree of openness ($\alpha$), for Edgeworth complementarity ($\sigma = 2$ and $\mu = -1.1$) and substitutability ($\sigma = 2$ and $\mu = -0.9$).

## E. Constant-Elasticity-of-Substitution (CES) Preferences

We replace the Cobb-Douglas specification in (1) with the CES specification $\left[\gamma c_t^{\mu} + (1-\gamma)\left(m_t^d\right)^{\mu}\right]^{\frac{1}{\mu}}$, where $\mu < 1$ and $(1-\mu)^{-1}$ is the intratemporal elasticity of substitution between $c_t$ and $m_t^d$. Note that for $\mu = 0$, this elasticity is equal to 1, and the model collapses to the Cobb-Douglas case. Moreover, we now have that $sign\left\{U_{cm}\right\} = sign\left\{1 - \frac{\sigma}{1-\mu}\right\}$. Then consumption and money are Edgeworth complements for $\mu < 1-\sigma$ and substitutes for $\mu > 1-\sigma$. Under these CES preferences, the model is less tractable than under the Cobb-Douglas specification, preventing us from deriving simple analytical conditions for the existence of cycles as we did in Proposition 3.[24] Nevertheless, numerical simulations are sufficient to confirm that our results are robust to this change in specification. Figure 6 presents the orbit bifurcation diagrams with respect to the degree of openness $\alpha$. It considers the cases of Edgeworth substitutability ($\sigma = 2$ and $\mu = -0.9$) and complementarity ($\sigma = 2$ and $\mu = -1.1$).

Three results clearly emerge from Figure 6. First, despite the CES preferences, more open economies are more prone to endogenous cycles/chaos under active forward-looking rules. Second, whether the cycles are around the passive or the active steady state still depends on whether consumption and money are Edgeworth complements or substitutes. And third, under a CES specification, cycles and chaos seem to start appearing at much lower degrees of openness relative to the Cobb-Douglas case.

[24] Keeping all the remaining structure unchanged, extensive algebra shows that the PFE dynamics of our economy are described by a non linear difference equation $R_{t+1} = \left[1 - J\left(R_t\right)\right]^{-1}$, similar to equation (27) in the main text, but with the function $J\left(R_t\right)$ taking a more complicated form than that of (28) and depending on the parameter $\mu$.

### F. Constant-Return-to-Scale (CRS) Technologies

The production functions in (3) imply decreasing returns to scale with respect to labor. Our analysis and the main messages would not change under CRS technologies. However, to introduce these technologies, it is not sufficient to set $\theta_N = \theta_T = 1$ in (3). The reason is that to ensure an interior solution in the model, it is necessary to assume some convexity on the disutility of labor. We then modify (1) by assuming that the disutility from working takes the form $-\psi\left[\frac{\left(h_t^T\right)^{1+\varphi}}{1+\varphi}+\frac{\left(h_t^N\right)^{1+\varphi}}{1+\varphi}\right]$, where $\varphi$ is the inverse of the Frisch elasticity of labor.[25] From this, and the CRS technologies $y_t^T = z_t h_t^T$ and $y_t^N = z_t h_t^N$, the global PFE dynamics would still be defined by the mapping $R_{t+1} = f(R_t)$ in (27), but with $\chi \equiv \frac{(1-\alpha)\varphi(\sigma-1)(1-\gamma)}{\sigma+\varphi+(\sigma-1)\alpha\varphi}$. As a result, the model is *isomorphic* to that extensively studied in the sub-Section B.1. Hence, it is possible to derive analytical conditions for the existence of period-2 cycles similar to those of Proposition 3, which depend on openness, as well as to show numerically the occurrence of higher order cycles and chaos.[26]

## V. Conclusions

In this paper, we fully characterize the set of *global* perfect foresight equilibria of a standard two-good small-open-economy model, where monetary policy takes the form of an active forward-looking Taylor-type interest rate rule. We show that a higher degree of trade openness—measured by the share of traded goods in aggregate consumption—makes the rule more prone to induce endogenous complex dynamics, such as equilibrium cycles of various periodicities and chaos. Interestingly, the results on chaos should not be perceived as theoretical curiosities. They are to some extent consistent with the evidence reported by Bask (2002) and Goga and Serletis (2000) on chaotic dynamics for the nominal and real exchange rates of some inflation targeting countries.[27]

To the best of our knowledge, our paper is the first to bridge the gap between the *closed* economy literature on *global* analysis of Taylor rules and the *open* economy literature on *local* analysis of these rules. And by doing this, we unveil some crucial differences. On one hand, relative to the global analyses on closed economy models, we show that trade openness considerable widens the range of values of the risk aversion parameter under which cycles and chaos can arise. In this regard, our results are in the same spirit as those from Weder (2001) and Meng and Velasco (2003) of the RBC literature on production externalities: openness makes it easier to obtain expectations-driven fluctuations. On the other hand, relative to the local analysis on open economy models, we show that rules that imply local determinacy can still lead to global indeterminacy by inducing cyclical and chaotic dynamics.

[25] This functional form is quite common in multi-sector DSGE models. See, for instance, Horvath (2000) and Carlstrom et al. (2006).

[26] Note that under CRS, one would have to make an assumption similar to Assumption 1 of the main text. For the functional form used in this section this would be $1-\gamma > \frac{1}{2\varphi}(R^*-1)\left(1-\frac{R^*}{A}\right)$, i.e., we would still need a sufficient role for money in providing transaction services. However, this inequality would hold for a wide range of parametrizations.

[27] As hinted above, in our model, a chaotic nominal interest rate implies chaotic dynamics for other variables including the real exchange rate.

This is certainly in line with Benhabib et al (2001b, 2002a), but the key difference here is that the extent of disagreement between the local and global analyses depends on the degree of openness.

Our model can be extended in different directions. Because of tractability, our analytical results are restricted to the case of a flexible-price and perfectly-competitive economy. Using a continuous time framework, in Airaudo and Zanna (2012), we extend the current model to allow for monopolistic competition and nominal price rigidities in one sector (e.g., the non-traded goods sector). We prove analytically that cycles appear through a Hopf bifurcation with respect to the weight on the inflation of the flexible-price sector (e.g., the traded sector), which in the small open economy set-up is related to the degree of trade openness. Furthermore, in this paper we have assumed the government follows a fixed monetary policy rule. Many authors, including Benhabib et al. (2002b), have advocated for an explicit commitment to switch to a different monetary rule (exchange-rate-based or money-growth-based rules) to escape from or rule out liquidity traps. It would be interesting to assess whether cyclical and chaotic dynamics still occur in such regime-switching environments. We leave this extension for future research.

# A Appendix

This appendix presents the proofs of the Propositions in the paper and further elaborates and shows the results of some extensions of the model, including contemporaneous rules, backward-looking rules, nominal prince rigidities and money in the production function set-ups.

## A. Proofs of the Propositions

### A.1. Proof of Proposition 1

**Proof.** Consider the left-hand side and the right-hand side of equation (25) as functions of the steady state interest rate $R^{ss}$, and denote them, respectively, as $LHS(R^{ss})$ and $RHS(R^{ss})$. Notice that, for any $R^{ss} > 1$, both of them are strictly increasing in $R^{ss}$, satisfying:

$$\begin{aligned}
\frac{\partial LHS(R^{ss})}{\partial R^{ss}} &= (R^{*}-1)^{\frac{R^{*}-1}{A}} > 0 \\
\frac{\partial RHS(R^{ss})}{\partial R^{ss}} &= R^{*}\frac{R^{*}-1}{A}\left(R^{ss}-1\right)^{\frac{R^{*}-1}{A}-1} > 0
\end{aligned}$$

Moreover:

$$\begin{aligned}
\lim_{R^{ss}\to 1} LHS(R^{ss}) &= (R^{*}-1)^{\frac{R^{*}-1}{A}} > 0 \\
\lim_{R^{ss}\to 1} RHS(R^{ss}) &= 0
\end{aligned}$$

Recall that $R^{ss} = R^{*} > 1$ is a viable solution to (25). It then follows that a sufficient condition for a second solution $R^{L} < R^{*}$ to exist is that $\frac{\partial LHS(R^{ss})}{\partial R^{ss}} > \frac{\partial RHS(R^{ss})}{\partial R^{ss}}$ at $R^{ss} = R^{*}$. By continuity of both functions, this guarantees that $LHS(R^{ss})$ and $RHS(R^{ss})$ also cross at some $R^L \in (1, R^*)$. Since $\left.\frac{\partial RHS(R^{ss})}{\partial R^{ss}}\right|_{R^{ss}=R^*} = \frac{R^*}{A}\left(R^*-1\right)^{\frac{R^*-1}{A}}$, the sufficient condition is in fact $\frac{A}{R^*} > 1$.

Furthermore, taking a second derivative, $\frac{A}{R^*} > 1$ is also necessary and sufficient for $\frac{\partial^2 RHS(R^{ss})}{\partial (R^{ss})^2} = R^* \frac{R^*-1}{A}\left(R^{ss}-1\right)^{\frac{R^*-1}{A}-2}\left(\frac{R^*-1}{A}-1\right) < 0$ : that is, $RHS(R^{ss})$ is strictly concave. Therefore no other solution to (25), other than $R^*$ and $R^L$, exists. ■

## A.2. Proof of Proposition 2

**Proof.** Let $\phi \equiv \left(1 + \frac{\frac{R^*}{A}-1}{\frac{\chi}{R^*-1}}\right)$, which satisfies $\phi > 0$. Since $\hat{R}_t$ is non-predetermined, the necessary and sufficient condition for local determinacy is $\phi \in (0,1)$. By simple algebra, this requires $\chi < \frac{1}{2}(R^*-1)\left(1-\frac{R^*}{A}\right)$. To simplify the notation, let $\Upsilon^d \equiv \frac{1}{2}(R^*-1)\left(1-\frac{R^*}{A}\right)$ and notice that $\Upsilon^d$ is always positive since $A > R^* > 1$ (by Assumption 0 and the zero-lower-bound condition).

From the definition of $\chi$ in (24), it follows that the condition $\chi < \Upsilon^d$ always hold when $\sigma \in (0,1)$, independently of $\alpha$. Hence, the equilibrium is always locally determinate when $\sigma \in (0,1)$, i.e., when consumption and money are Edgeworth complements.

Now consider the case of $\sigma > 1$. By the definition of $\chi$, the inequality $\chi < \Upsilon^d$ is equivalent to:

$$\alpha > \frac{(1-\sigma)(1-\gamma)(1-\theta_N) + \Upsilon^d(\sigma\theta_N + 1 - \theta_N)}{(1-\sigma)(1-\theta_N)(1-\gamma+\Upsilon^d)} \equiv \alpha^d$$

Simple algebra shows that:

$$\begin{aligned} \alpha^d &< 1 \text{ always and} \\ \alpha^d &\geq 0 \text{ iff } \sigma \geq \frac{(1-\theta_N)\left(1-\gamma+\Upsilon^d\right)}{(1-\gamma)(1-\theta_N) - \Upsilon^d\theta_N} \equiv \sigma^d, \end{aligned}$$

where $\sigma^d > 1$ by Assumption 1.[28] Hence, the equilibrium is locally determinate for any $\alpha \geq 0$ when $\sigma \in \left(1, \sigma^d\right)$, and for any $\alpha > \alpha^d$ when $\sigma \geq \sigma^d$. ■

## A.3. Proof of Proposition 3

**Proof.** The proof involves the analytical characterization of the non-linear mapping $R_{t+1} = f(R_t)$ defined in (27)-(28). We identify cycles by looking for a flip bifurcation with respect to the degree of openness $\alpha$, i.e., a critical value that determines a change in stability for the steady state where $f$ is negatively sloped. If the steady state is stable, any equilibrium orbit that starts in a map invariant set around it will asymptotically converge to the steady state itself, monotonically or spirally: that is, endogenous cycles cannot occur. On the contrary, if the steady state is unstable, the orbit will

[28] If Assumption 1 did not hold, then $\alpha^d \leq 0$ and the equilibrium would be locally determinate for any degree of trade openness. However, Assumption 1 holds for any realistic calibration of the model.

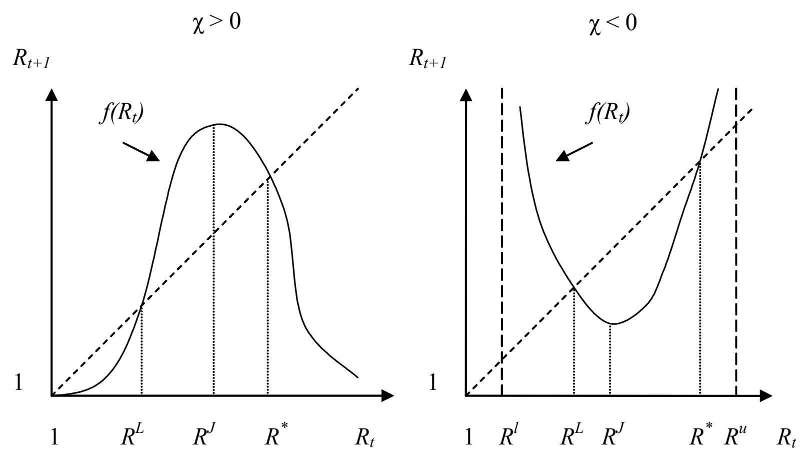


Figure 7: **The mapping $R_{t+1} = f(R_t)$ for forward-looking rules**. Two cases are considered (i) $\chi > 0$, and (ii) $\chi < 0$ but $\chi + \frac{R^*-1}{A} > 0$. $R_t$ denotes the nominal interest rate. The diagonal dotted line corresponds to the $45^o$ line.

keep oscillating, and either will converge to a stable $n$-period cycle, or not converge at all, displaying aperiodic but bounded dynamics, i.e., chaos. We are going to consider the case of $\sigma < 1$ (point 1) and the case of $\sigma > 1$ (point 2), separately. To facilitate the exposition, we are going to define the conditions for the existence of cycle first in terms of the composite parameter $\chi$, and then in terms of the degree of openness $\alpha$ and the risk aversion parameter $\sigma$. Figure 7 presents a graphical representation of the non-linear mapping $R_{t+1} = f(R_t)$ depending on $\chi$.

1. Assume that $\sigma < 1$. Recall that, by the definition (24), this implies $\chi < 0$. Consider the two following cases: (a) $\chi \leq -\frac{R^*-1}{A}$ and (b) $-\frac{R^*-1}{A} < \chi < 0$.

   (a) For $\chi \leq -\frac{R^*-1}{A}$, the mapping $f$ is monotonically increasing for any $R_t > 1$, with $f'\left(R^L\right) > 0$ and $f'(R^*) > 1$. This rules out cycles of any periodicity, since a necessary (but not sufficient) condition for their existence is a negative slope at the steady state. We can then conclude that if $\chi \leq -\frac{R^*-1}{A}$ holds, the only PFE dynamics are paths that monotonically converge to the passive steady state.

   (b) For $-\frac{R^*-1}{A} < \chi < 0$, the mapping $f$ has the following properties: a) $f : \left(R^l, R^u\right) \rightarrow (1, +\infty)$, where $R^l$ and $R^u$ are the solutions to $J(R_t) = 1$ with $R^l \in \left(1, R^L\right)$ and $R^u > R^*$; b) $f(.)$ is continuously differentiable over the domain $\left(R^l, R^u\right)$; c) it has a unique minimum at $R^J$; and d) $\lim_{R_t \rightarrow R^{l+}} f(R_t) = \lim_{R_t \rightarrow R^{u-}} f(R_t) = +\infty$ ; e) $f'(R^*) > 1$ always while $f'\left(R^L\right) \gtreqless 0$. In other words, the mapping $f$ is well-defined and U-shaped within the interval $\left(R^l, R^u\right)$.
   
   Now, suppose that $f'\left(R^L\right) < 0$—we already know that if $f'\left(R^L\right) \geq 0$ there cannot be cycles. If this holds, $f$ attains its minimum at $R^J \in \left(R^L, R^*\right)$.
   
   To prove the existence of cycles, define the auxiliary function $g(R) \equiv R - f^2(R)$. By Definition 2, a period-2 cycle is the solution to $g(R) = 0$. Although the mapping $f$ is

continuously differentiable on its domain $\left(R^l, R^u\right)$, this is not necessarily true for $g$. To see this, let $\tilde{R} \equiv f^{-1}\left(R^*\right)$ and note that $\tilde{R} \in \left(R^l, R^L\right)$. By a simple graphical representation of $f$, one can see that if $f\left(R^J\right)$—which is the minimum— falls below $\tilde{R}$, there exist sequences generated by the mapping $f$ that will leave the domain $\left(R^l, R^u\right)$ after a finite number of iterations. We have to consider three possible alternatives: i. $f\left(R^J\right) \geq \tilde{R}$; ii. $f\left(R^J\right) \in \left(R^l, \tilde{R}\right)$ and iii. $f\left(R^J\right) \leq R^l$.

i. Suppose that $f\left(R^J\right) \geq \tilde{R}$. Then, the set $\left[\tilde{R}, R^*\right]$ is invariant under the mapping $f$, i.e., $f : \left[\tilde{R}, R^*\right] \to \left[\tilde{R}, R^*\right]$, and given any $R_0 \in \left[\tilde{R}, R^*\right]$, we have that $f^t\left(R_0\right) \in \left[\tilde{R}, R^*\right]$ for any $t \geq 0$. Hence, the auxiliary function $g$ is continuously differentiable over $\left[\tilde{R}, R^*\right]$. Moreover, by the definition of a steady state: $g\left(R^L\right) = g\left(R^*\right) = 0$; while by the chain rule: $g'\left(R^{ss}\right) = 1 - \left[f'\left(R^{ss}\right)\right]^2$ for $R^{ss} = R^L, R^*$. By Assumption 0—the rule is active at the target—and simple algebra, we have that $f'\left(R^*\right) = \frac{A}{R^*} > 1$, which implies $g'\left(R^*\right) < 0$. The Intermediate Value Theorem guarantees that there is a value $R^c \in \left(R^L, R^*\right)$ where $g\left(R\right) = 0$ if $g'\left(R^L\right) = 1 - \left[f'\left(R^L\right)\right]^2 < 0$, i.e, if $f'\left(R^L\right) < -1$. In other words, there exist period-2 cycles as long as the passive steady state is dynamically unstable. Simple differentiation of the mapping $f$ shows that:

$$f'\left(R^L\right) \lesseqqgtr -1 \text{ iff } \chi \gtreqqless \Upsilon^f, \tag{31}$$

where $\Upsilon^f \equiv \frac{1}{2}\left[R^L\left(1 - \frac{R^*-1}{A}\right) - 1\right]$ and $-\frac{R^*-1}{A} < \Upsilon^f < 0$ (by Assumptions 0 and 2). From the result stated in (31), we can conclude the following:

$$\text{if } \chi \in \left(\Upsilon^f, 0\right) : f'\left(R^L\right) < -1 \qquad \Longrightarrow \text{period-2 cycles occur around } R^L \tag{32}$$

ii. Suppose that $f\left(R^J\right) \in \left(R^l, \tilde{R}\right)$. From the properties of the mapping $f$, it follows that $f^n : \left[\tilde{R}, R^*\right] \to \left[R^l, R^u\right]$ for $n = 1, 2$. This implies that the auxiliary function $g$ is still continuously differentiable on $\left[\tilde{R}, R^*\right]$. By the same argument used for i. period-2 cycles around the passive steady state exist if the condition stated in (32) holds.

iii. Suppose that $f\left(R^J\right) \leq R^l$. Then there exist a closed interval $I_1 \equiv \left[R_A, R_B\right]$—where $R_A$ and $R_B$ are the solutions to $f\left(R\right) = R^l$—such that $f\left(R\right) < R^l$ for any $R \in I_1$. Moreover, there exist closed intervals $I_2 \subset \left(\tilde{R}, R^L\right)$ and $I_3 \subset \left(R_B, R^*\right)$—where $I_2$ and $I_3$ are such that $f : I_2 \to I_1$ and $f : I_3 \to I_1$—such that $f^2\left(R\right) < R^l$ for any $R \in \left\{I_2 \cup I_3\right\}$. Hence, the auxiliary function $g$ is continuously differentiable only over the range $\left[\tilde{R}, R^*\right] \setminus \left\{\cup_{i=1}^3 I_i\right\}$. Let $I_2 \equiv \left[R_C, R_D\right]$—where $R_C = f^{-1}\left(R_B\right)$ and $R_D = f^{-1}\left(R_A\right)$—and consider the interval $\left(\tilde{R}, R_C\right)$ over which $g$ is continuously differentiable. From the definition of $\tilde{R}$ we have that:

$$g\left(\tilde{R}\right) = \tilde{R} - f^2\left(\tilde{R}\right) = \tilde{R} - f\left(R^*\right) = \tilde{R} - R^* < 0,$$

while from the definition of $R_C$ and $R_B$ we have that

$$\lim_{R \to R_C} g\left(R\right) = R_C - f^2\left(R_C\right) = R_C - f\left(R_B\right) = R_C - R^l > 0.$$

Then, by the Intermediate Value Theorem, there exists a value $R^c \in \left(\tilde{R}, R_C\right)$ such that $g(R^c) = 0$. Hence, in this case, period-2 cycles always exist.

We can now group the results in i., ii., and iii. into the following single statement: when $\sigma < 1$, $\chi \in \left(\Upsilon^f, 0\right)$ is a sufficient condition for the existence of period-2 cycles around the passive steady state. From the definition of $\chi$ in (24), this condition is equivalent to:

$$\alpha > \frac{(1-\sigma)(1-\theta_N)(1-\gamma) + \Upsilon^f(\sigma\theta_N + 1 - \theta_N)}{(1-\sigma)(1-\theta_N)(1-\gamma+\Upsilon^f)} \equiv \alpha^f,$$

where

$$\begin{aligned} \alpha^f &< 1 \text{ always} \\ \alpha^f &\geq 0 \text{ iff } \sigma \leq \frac{(1-\theta_N)\left(1-\gamma+\Upsilon^f\right)}{(1-\theta_N)(1-\gamma) - \Upsilon^f\theta_N} \equiv \sigma^f \text{ with } \sigma^f < 1. \end{aligned}$$

That is, period-2 cycles around the passive steady state occur for $\alpha > \alpha^f$ when $\sigma \in \left(0, \sigma^f\right]$, and for any $\alpha \geq 0$ when $\sigma \in \left(\sigma^f, 1\right)$.

2. Assume that $\sigma > 1$. By the definition (24), $\chi > 0$. In addition, the mapping $f$ has the following properties: a) $f : (1, +\infty) \to (1, +\infty)$ and is continuously differentiable; b) $f$ has a global maximum at $R^J \equiv (1+\chi)\left(1 - \frac{R^*-1}{A}\right)^{-1}$ where $R^J > R^L$; c) $\lim_{R_t \to 1^+} f(R_t) = \lim_{R_t \to \infty} f(R_t) = 0$; and d) $f'\left(R^L\right) > 1$ while $f'(R^*) \gtreqless 0$. That is, the mapping $f$ is continuously differentiable and single-peaked on $(1, +\infty)$. From d), we can conclude that there cannot be cycles around the passive steady state.

   Now, suppose that $f'(R^*) < 0$ (if $f'(R^*) \geq 0$ there would not be cycles around the active steady state as well). If this holds, then the maximum occurs at $R^J \in \left(R^L, R^*\right)$.

   Define the auxiliary function $g(R) \equiv R - f^2(R)$. The auxiliary function $g$ is clearly continuously differentiable over the entire range $(1, +\infty)$. A period-2 cycles is the solution to $g(R) = 0$. By the definition of a steady state: $g\left(R^L\right) = g(R^*) = 0$; while by the chain rule: $g'\left(R^L\right) = 1 - \left[f'\left(R^L\right)\right]^2 < 0$. The Intermediate Value Theorem guarantees that there is a value $R^c \in \left(R^L, R^*\right)$ where $g(R) = 0$ if $g'(R^*) = 1 - \left[f'\left(R^L\right)\right]^2 < 0$, i.e., if $f'(R^*) < -1$. In other words, there exist period-2 cycles as long as the active steady state is dynamically unstable. Simple differentiation of the map $f$ shows that:

$$f'(R^*) \lesseqgtr -1 \text{ iff } \chi \lesseqgtr \Upsilon^d, \tag{33}$$

   where $\Upsilon^d \equiv \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right) > 0$ (since $\frac{A}{R^*} > 1$ and $R^* > 1$). From the result stated in (33), we can conclude the following:

$$\text{if } \chi \in \left(0, \Upsilon^d\right) : f'(R^*) < -1 \qquad \Longrightarrow \text{period-2 cycles occur around } R^*$$

   But from the definition of $\chi$ in (24), this condition is equivalent to:

$$\alpha > \frac{(1-\sigma)(1-\gamma)(1-\theta_N) + \Upsilon^d(\sigma\theta_N + 1 - \theta_N)}{(1-\sigma)(1-\theta_N)(1-\gamma+\Upsilon^d)} \equiv \alpha^d,$$

where

$$\alpha^d < 1 \text{ always}$$
$$\alpha^d \geq 0 \text{ iff } \sigma \geq \frac{(1-\theta_N)\left(1-\gamma+\Upsilon^d\right)}{(1-\gamma)(1-\theta_N)-\Upsilon^d\theta_N} \equiv \sigma^d,$$

where $\sigma^d > 1$ by Assumption 1. That is, period-2 cycles around the active steady state occur for $\alpha > \alpha^d$ when $\sigma > \sigma^d$, and for any $\alpha \geq 0$ when $\sigma \in \left(1, \sigma^d\right)$.

■

### A.4. Proof of Proposition 5

**Proof.** We modify our model to feature a CIA timing for money in utility following Appendix A in Woodford (2003). Extensive algebra shows that the global PFE dynamics of our economy, under the forward-looking interest rate rule (16) are described by the following non-linear difference equation:

$$\left(\frac{R_{t+1}-1}{R_{t+1}^{\Psi}}\right)^{\chi} = \frac{R^*}{(R^*-1)^{\frac{R^*-1}{A}}} \frac{(R_t-1)^{\chi+\frac{R^*-1}{A}}}{R_t^{1+\Psi\chi}}, \tag{34}$$

The only difference between (34) and its CWID counterpart (23) is the coefficient $\Psi \equiv \frac{1-\theta_N(1-\sigma)}{(1-\sigma)(1-\gamma)(1-\theta_N)}$. This is positive for $\sigma \in (0,1)$ and negative for $\sigma > 1$.[29] The parameter $\chi$ is still defined by (24). Let $K(R_{t+1})$ and $J(R_t)$ be the left and the right hand sides of (34) respectively. First of all notice that for $\sigma > 1$, we have $\Psi < 0$ and $\chi > 0$. Then consider the function $K(.)$. Straightforward algebra shows that $\lim_{R_{t+1}\to 1^+} K(R_{t+1}) = 0$ and $\lim_{R_{t+1}\to+\infty} K(R_{t+1}) = +\infty$, and that $K(.)$ is strictly increasing over the domain $(1,+\infty)$. It then follows that $K(.)$ is globally invertible on $(1,+\infty)$. Hence, there exists a well defined mapping $f(R_t) = K^{-1}(J(R_t))$ describing the forward equilibrium dynamics.

Now consider the function $J(R_t)$. Since $\chi + \frac{R^*-1}{A} > 0$ for any $\alpha \in (0,1)$, we have that $\lim_{R_t\to 1^+} J(R_t) = 0$; while $\lim_{R_t\to+\infty} J(R_t) = +\infty$ if $\chi > \frac{\left(1-\frac{R^*-1}{A}\right)}{(1-\Psi)}$ and $\lim_{R_t\to+\infty} J(R_t) = 0$ if $\chi < \frac{\left(1-\frac{R^*-1}{A}\right)}{(1-\Psi)}$, where $\frac{\left(1-\frac{R^*-1}{A}\right)}{(1-\Psi)} > 0$.

Since, we only want to show that there exist parametrizations under which cycles are possible under CIA timing, suppose that $\chi < \frac{\left(1-\frac{R^*-1}{A}\right)}{(1-\Psi)}$. Under this condition, $J(.)$ is single-peaked with its critical point at $R^J = \frac{1+\Psi\chi}{\left(1-\frac{R^*-1}{A}\right)} > 1$. This combined with the fact that $K(.)$ is strictly increasing on $(1,+\infty)$ implies that the function $f$ looks like a logistic map, with $f : (1,+\infty) \to (1,+\infty)$. Then, along the lines of the proof to Proposition 3, a sufficient condition for period-2 cycles around the active steady state is $f'(R^*) < -1$. Simple algebra shows that this is equivalent to $\chi < \frac{\left(1-\frac{R^*-1}{A}\right)R^*-1}{2R^*}$. Given $\sigma > 1$

[29] Although, the two set-ups are non-nested, one could think of the CWID model as one where $\Psi = 1$.

and the definition of $\chi$ in (24), it is possible to show that there exists threshold values $\sigma^{cia} > 1$ and $\alpha^{cia} \in (0,1)$ such that $\chi < \frac{\left(1-\frac{R^*-1}{A}\right)R^*-1}{2R^*}$ (hence, period-2 cycles exist) for any $\alpha \in (0,1)$ if $\sigma < \sigma^{cia}$, and for $\alpha > \alpha^{cia}$ if $\sigma > \sigma^{cia}$. ■

## B. Contemporaneous Rules

Motivated by the recent estimations of rules by Lubik and Schorfheide (2007) for the United kingdom, Canada, Australia and New Zealand, we study the determinacy of equilibrium for contemporaneous rules that respond to current CPI inflation: $R_t = 1 + (R^* - 1)\left(\frac{\pi_t}{\pi^*}\right)^{\frac{A}{R^*-1}}$ with $R^* = \frac{\pi^*}{\beta}$ and $R_t > 1$. Under these rules the equilibrium dynamics are described by

$$\left(\frac{R_{t+1}-1}{R_{t+1}}\right)^{\chi}\left(\frac{R^*-1}{R_{t+1}-1}\right)^{\frac{R^*-1}{A}} = \left(\frac{R_t-1}{R_t}\right)^{\chi}\frac{R^*}{R_t} \tag{35}$$

where $\chi$ defined in (24) depends on $\alpha$ and $\sigma$. An explicit representation for either the forward or the backward dynamics of (35) is not available in this case. Although it is feasible to derive some analytical results as before, for reasons of space, we only present some numerical simulations. These are sufficient to make the point that the degree of openness still affects the appearance of complex dynamics.

Using the parametrization of Table 2 and setting $\sigma = 2.5$ (substitute goods), we construct the panel in Figure 8. From this figure it is possible to infer the global dynamics of the model for contemporaneous rules under different degrees of openness. For a given $\alpha \in \{0.01, 0.38, 0.90\}$, each column of the panel plots the map $R_{t+1} = f(R_t)$, implicitly defined in equation (35), its second iterate $R_{t+2} = f^2(R_t)$, and its third iterate $R_{t+3} = f^3(R_t)$, respectively. The straight line corresponds to the $45^o$ line. Note that for $\alpha \in \{0.01, 0.38\}$, namely an almost closed economy and a moderately open economy, the second and third iterates, $R_{t+2} = f^2(R_t)$ and $R_{t+3} = f^3(R_t)$, have fixed points different from the steady state values $R^*$ and $R^L$, implying the existence of cycles of periods 2 and 3. Then by the Sarkovskii's Theorem and the Li and Yorke's Theorem, $f(R_t)$ features cycles of any order, as well as aperiodic cyclical dynamics (topological chaos).[30] However, observe that for $\alpha = 0.01$, cycles and chaos appear around the active steady state, while for $\alpha = 0.38$ they occur around the passive steady state. Finally, for very open economies ($\alpha = 0.90$), no cycles and chaotic dynamics appear at all. The monotonicity of the map $f(R_t)$ implies that standard liquidity traps are the only type of global equilibrium multiplicity in this case.[31]

When consumption and money are complements, endogenous global fluctuations do not exist. This is an important difference with respect to the results derived under forward-looking rules. But a close look at the results under substitutability reveal more differences. First, for contemporaneous rules, very

[30] See Lorenz (1993) for a precise statement of these two theorems.

[31] It is also possible to find the exact numerical values of the $\alpha$ thresholds triggering a qualitative switch in dynamics. We find that period-2 cycles appear around the active steady state when $\alpha \in (0.001, 0.22)$; and period-3 cycles occur around the active steady state when $\alpha \in (0.001, 0.16)$. Pushing $\alpha$ up, period-2 cycles appear around the passive steady state for $\alpha \in (0.25, 0.43)$, whereas period-3 cycles (and therefore chaos) exist for $\alpha \in (0.33, 0.38)$. Finally for $\alpha > 0.39$ only liquidity traps exist.

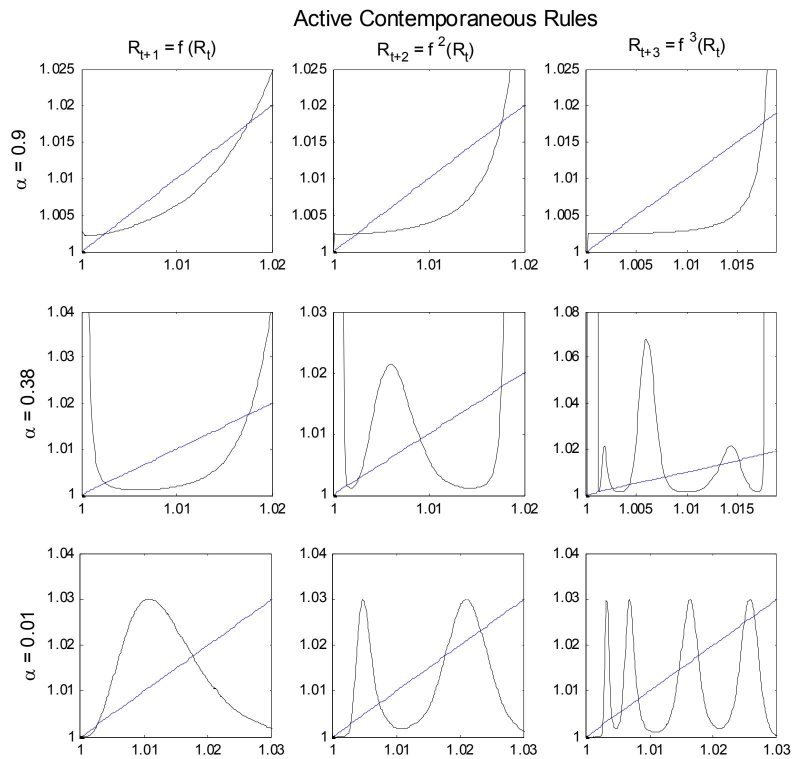


Figure 8: **The implicit mapping $R_{t+1} = f(R_t)$ for contemporaenous rules.** The figure shows the first, second, and third iterates of the mapping, under different degrees of openness of the economy ($\alpha$).

open economies seem to be less prone to endogenous cycles. Second, the qualitative changes induced by a variation in the degree of openness can be even more dramatic, since they might even trigger a switch from one steady state to the other as the focus of fluctuations. The following proposition summarizes these results.

**Proposition 6** *Under active contemporaneous rules:*

1. *if consumption and money are Edgeworth complements, i.e.* $\sigma \in (0,1)$*, there cannot be equilibrium cycles of any periodicity for any degree of openness* $\alpha \in (0,1)$*;*
2. *if consumption and money are Edgeworth substitutes, i.e.* $\sigma > 1$*,*
   (a) *there cannot be equilibrium cycles of any periodicity if the economy is sufficiently open;*
   (b) *cyclical and chaotic dynamics occur around the passive steady state for intermediate degrees of openness;*
   (c) *cyclical and chaotic dynamics around the active steady state occur if the economy is sufficiently closed.*

## C. Backward-Looking Rules

We conclude the analysis of different timings for the rule by studying backward-looking rules defined as $R_t \equiv 1 + (R^* - 1)\left(\frac{\pi_{t-1}}{\pi^*}\right)^{\frac{A}{R^*-1}}$ with $\xi \equiv \frac{A}{R^*} > 1$. This specification in tandem with equation $\pi_{t+1}\left(\frac{R_{t+1}}{R_{t+1}-1}\right)^{\chi} = \left(\frac{R_t}{R_t-1}\right)^{\chi}\beta R_t$, which can be derived by combining equations (20) and (22), conform a system of two first-order difference equations that can be used to pursue the global determinacy analysis. Since it is very difficult to derive analytical results, we rely on simulations in order to assess whether for different values of $\alpha$ and $\sigma$, the system presents cycles or chaos. The simulation results show that these dynamics are not present, regardless of the degree of openness $\alpha$ when $\sigma \in \{0.8, 1.5, 2.0, 2.5\}$. The interest rate converges to either the active or the passive steady state.

## D. Nominal Price Rigidities

We proceed to assess whether cyclical and chaotic dynamics still arise once we introduce sticky prices. To do so, we introduce price stickiness for non-traded goods, following the specification in Benhabib et al. (2001a) of quadratic price adjustment costs in the utility function. In this sense, the representative agent $j$ now maximizes:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\left[(c_{jt})^{\gamma}\left(m_{jt}^d\right)^{1-\gamma}\right]^{1-\sigma} - 1}{1-\sigma} + \psi(1 - h_{jt}^T - h_{jt}^N) - \frac{\phi}{2}\left(\frac{P_{jt}^N}{P_{jt-1}^N} - \pi^*\right)^2 \right\}$$

where $c_{jt}$ is defined as in (2). We assume that the non-traded good is a composite good, and that, because of monopolistic competition, the household-firm unit $j$ can choose the price of the non-traded good it supplies, $P_{jt}^N$, subject to the demand constraint $y_{jt}^N \geq c_t^N \left(\frac{P_{jt}^N}{P_t^N}\right)^{-\mu}$, where $c_t^N = \int_0^1 c_{jt}^N dj$ and $\mu > 1$. The budget constraint for the representative household-firm unit $j$ is identical to (8), apart for the subscript $j$. For computational tractability, we focus on the case of a contemporaneous interest rate rule, as in part B of this Appendix. Under symmetry, the equilibrium dynamics are entirely described by the following conditions:[32]

$$\left(\pi_{t+1}^N - \pi^*\right)\pi_{t+1}^N = \frac{1}{\beta}\left(\pi_t^N - \pi^*\right)\pi_t^N + \frac{\lambda(\mu-1)}{\beta\phi}\frac{c_t^N}{e_t} - \frac{\psi\mu}{\theta^N\beta\phi}\left(c_t^N\right)^{\frac{1}{\theta^N}}$$

$$e_t = e_{t-1}\frac{\epsilon_t}{\pi_t^N}$$

$$\epsilon_{t+1} = \beta\left[1 + (R^* - 1)\left(\frac{\pi_t}{\pi^*}\right)^{\frac{A}{R^*-1}}\right]$$

where $c_t^N = \mathfrak{n} e_t^{\varphi}\left[(R^*-1)\left(\frac{\pi_t}{\pi^*}\right)^{\frac{A}{R^*-1}}\right]^{-\omega}\left[1+(R^*-1)\left(\frac{\pi_t}{\pi^*}\right)^{\frac{A}{R^*-1}}\right]^{\omega}$ and $\pi_t = \epsilon_t^{\alpha}(\pi_t^N)^{(1-\alpha)}$, with $\mathfrak{n} > 0$, $\varphi \equiv \frac{1-\alpha(1-\sigma)}{\sigma}$ and $\omega \equiv \frac{(1-\gamma)(1-\sigma)}{\sigma}$.

First of all, it is straightforward to show that we still have two distinct steady states. Second, it is possible to show that, for any degree of openness and any elasticity $\xi > 1$, the active steady state is dynamically unstable. For the rest, we rely on numerical simulations, varying the degree of nominal price rigidity $\phi$ in the range $[2.8, 44]$, which is in line with Dib's (2003) estimates. We consider different combinations of degree of openness $\alpha$ and the coefficient $\sigma$, while keeping the remaining parameters as in Table 2. Through a wide range of experiments, we have not been able to detect cycles of any periodicity, around neither of the two steady states. Under sticky prices, the only type of global indeterminacy seems to be the liquidity trap. In this sense, our results are in line with those reported by Eusepi (2007).

## E. Money in the Utility Function (MIUF) versus Money in the Production Function (MIPF)

For what concerns *local* equilibrium determinacy in a *closed* economy, Benhabib et al. (2001a) have proved that, in the context of *Taylor rules*, there is an isomorphism between the MIUF set-up when money and consumption are substitutes and the MIPF set-up.[33] To the best of our knowledge, this isomorphism has not been proved *analytically* in the context of *global* equilibrium analysis for *Taylor rules* in an *open* economy. We then proceed to prove this isomorphism for the existence of period-2 cycles in a small open economy. To do this, we consider a MIPF version of our small open

[32] The derivation of these conditions is available from the authors upon request.

[33] Benhabib et al. (2001a) also present results of cyclical perfect foresight equilibria (see their Proposition 7). However, for these cyclical equilibria they do not show the isomorphism between MIUF and MIF set-ups, since they consider that money and consumption are separable in utility.

economy (SOE) with a fixed endowment of traded goods, $y_t^T = y$, and with non-traded goods produced by the following technology $y_t^N = y^N(m_t) = k^\theta m_t^{1-\theta}$. Labor is inelastically supplied and utility takes the following form $U_t = \frac{c_t^{1-\sigma}}{1-\sigma}$, where $c_t = \left(c_t^T\right)^\alpha \left(c_t^N\right)^{1-\alpha}$. By solving the model for a PFE, we can show the followings, under the MIPF set-up:

- We still have a modified Fisher equation as equation (22), i.e.,

$$\left(\frac{e_{t+1}}{e_t}\right)^{1-\alpha} = \beta \frac{R_t}{\pi_{t+1}}$$

- The global dynamics are described by a non-linear difference equation with the same form of equation (23), but

$$\chi \equiv \frac{(1-\alpha)\,\sigma\theta}{\alpha\sigma + (1-\theta)(1-\alpha)} > 0$$

By the same arguments and methodology used in Proposition 3, we can state and prove the existence of equilibrium cycles.

**Proposition 7** ***Equilibrium cycles under the MIPF set-up:*** *Suppose the government follows an active forward-looking rule like (16) in the paper. Then, in a MIPF-SOE model described above, there exist threshold values $\sigma^c > 0$ and $\alpha^c \in (0,1)$ such that period-2 equilibrium cycles occur around the active steady state for any $\alpha \geq 0$ when $\sigma \in (0, \sigma^c)$, and for $\alpha > \alpha^c$ when $\sigma \geq \sigma^c$.*

**Proof.** The global non-linear PFE dynamics are still described by the mapping $f$ defined in (27)-(28) but with $\chi \equiv \frac{(1-\alpha)\sigma\theta}{\alpha\sigma+(1-\theta)(1-\alpha)} > 0$. By the same arguments used in the proof of Proposition 3 (point 2), one can show that equilibrium period-2 cycles exist around the active steady state if $\chi < \Upsilon^d$, where $\Upsilon^d \equiv \frac{1}{2}(R^* - 1)\left(1 - \frac{R^*}{A}\right) > 0$. That is, by the new definition of $\chi$ and simple algebra, it is straightforward to show that there exists a threshold value $\sigma^* \equiv \frac{1-\theta}{\theta}\Upsilon^d > 0$ such that:

a) $\chi < \Upsilon^d$ and hence equilibrium cycles exist for any $\alpha \in (0,1)$ if $\sigma \in (0, \sigma^*)$.

b) $\chi < \Upsilon^d$ and hence equilibrium cycles exist for $\alpha > \alpha^c$ if $\sigma \in (0, \sigma^*)$, where $\alpha^c \equiv \frac{\sigma\theta - (1-\theta)\Upsilon^d}{\sigma\theta - (1-\theta)\Upsilon^d + \sigma\Upsilon^d} \in (0,1)$. ■

A quick comparison between this Proposition and point 2 of Proposition 3 of the main text reveals that it is then possible to construct a SOE-MIPF model which is isomorphic to the MIUF (with Edgeworth substitutability), in terms of the global properties (period-2 cycles) of active interest rate rules.

# References


Alstadheim, R. and D. Henderson, 2006, "Price-Level Determinacy, Lower Bounds on the Nominal Interest Rate, and Liquidity Traps," *The B.E. Journal of Macroeconomics*, Vol. 6, pp. 1437-1437.

Airaudo, M., and L.F. Zanna, 2012, "Inflation Measures, Interest Rate Rules, and Cycles," Manuscript.

Bask, M., 2002, "A Positive Lyapunov Exponent in Swedish Exchange Rates?," *Chaos, Solitons and Fractals*, Vol. 14, pp. 1295-1304.

Benhabib, J., S. Schmitt-Grohé, and M. Uribe, 2001a, "Monetary Policy Rules and Multiple Equilibria," *American Economic Review*, Vol. 91, pp. 167-184.

———, 2001b, "The Perils of the Taylor Rules," *Journal of Economic Theory*, Vol. 96, pp. 40-69.

———, 2002a, "Chaotic Interest Rate Rules" *American Economic Review*, Vol. 92, pp. 72-78.

———, 2002b, "Avoiding Liquidity Traps," *Journal of Political Economy*, Vol. 110, pp. 535-563.

Benigno, G. and P. Benigno, 2008, "Exchange Rate Determination under Interest Rate Rules," *Journal of International Money and Finance*, Vol. 27, pp. 971-993.

Bentolilla, S. and G. Saint-Paul, 2003, "Explaining Movements in the Labor Share," *The B.E. Journal of Macroeconomics*, Vol. 3, pp. 1103-1103.

Boivin, J., 2006, "Has U.S. Monetary Policy Changed? Evidence from Drifting Coefficients and Real-Time Data," *Journal of Money, Credit and Banking,* Vol. 38, pp. 1149-1174.

Bullard, J. and A. Singh, 2008, "Worldwide macroeconomic stability and monetary policy rules," *Journal of Monetary Economics*, Vol. 55, pp. S34-S47.

Bullard, J., 2010, "Seven Faces of 'The Peril'," *Federal Reserve Bank of St. Louis Review*, Vol. 92(5), pp. 339-52 (St. Louis: Federal Reserve Bank).

Burnstein, A., J. Neves, and S. Rebelo, 2003, "Distribution Costs and Real Exchange Rate Dynamics," *Journal of Monetary Economics*, Vol. 50, pp. 1189-1214.

Campa, J. M., and L. Goldberg, 2006, "Pass Through of Exchange Rates to Consumption Prices: What has Changed and Why?," NBER Working Papers No. 12547, (Cambridge, Massachusetts: National Bureau of Economic Research).

Carlstrom, C., and T. Fuerst, 2001, "Timing and Real Indeterminacy in Monetary Models," *Journal of Monetary Economics*, Vol. 47, pp. 285-298.

Carlstrom, C.T., T.S. Fuerst, and F. Ghironi, 2006, "Does It Matter (For Equilibrium Determinacy) What Price Index the Central Bank Targets?," *Journal of Economic Theory*, Vol. 128, pp. 214-231.

Clarida, J., J. Galí and M. Gertler, 1998, "Monetary Policy Rules in Practice: Some International Evidence," *European Economic Review*, Vol. 42, pp. 1033-1067.

Cogley, T. and T. Sargent, 2005, "Drifts and Volatilities: Monetary Policy and Outcomes in the Post WWII US," Review of Economic Dynamics, Vol. 8, pp. 262-302.

De Fiore, F. and L. Zheng, 2005, "Does Trade Openness Matters for Aggregate Instability?" *Journal of Economic Dynamics and Control*, Vol. 29, pp. 1165-1192.

Dib, A., 2003, "An Estimated Canadian DSGE Model with Nominal and Real Rigidities," *Canadian Journal of Economics*, Vol. 36(4), pp. 949-972.

Diks, C., C. Hommes, V. Panchenko, and R. Van der Weide, 2008, "E&F Chaos: A User Friendly Software Package for Non-Linear Economic Dynamics," *Computational Economics*, Vol. 32, pp. 221-244.

Eusepi, S., 2005, "Comparing Forecast-Based and Backward-Looking Taylor Rules: a Global Analysis," *Federal Reserve Bank of New York Staff Report*, No. 198 (New York City, Federal Reserve Bank).

Eusepi, S., 2007, "Learnability and Monetary Policy: A Global Perspective," *Journal of Monetary Economics*, Vol. 54, pp. 1115-1131.

Evans, G., E. Guse, and S.Honkapohja, 2008, "Liquidity Traps, Learning and Stagnation," *European Economic Review*, Vol. 52, pp. 1438-1463.

Galí, J., and T. Monacelli, 2005, "Monetary Policy and Exchange Rate Volatility in a Small Open Economy," *Review of Economic Studies*, Vol. 72, pp. 707-734.

Gogas, P., and A. Serletis, 2000, "Purchasing Power Parity, Non-linearity and Chaos," *Applied Financial Economics*, Vol. 10, pp. 615-622.

Holman, J., 1998, "GMM Estimation of a Money-in-the-Utility-Function Model: The Implications of Functional Forms," *Journal of Money, Credit and Banking*, Vol. 30, pp. 679-698.

Horvarth, M., 2000, "Sectoral Shocks and Aggregate Fluctuations," *Journal of Monetary Economics*, Vol. 45, pp. 69-106.

Leith, C. and S. Wren-Lewis, 2009, "Taylor Rules in the Open Economy," *European Economic Review*, Vol. 53, pp. 971-995.

Llosa, G., and V. Tuesta, 2008, "Determinacy and Learnability of Monetary Policy Rules in Small Open Economies," *Journal of Money, Credit and Banking*, Vol. 40, pp. 1033-1063.

Lorenz, H., 1993, *Nonlinear Dynamical Economics and Chaotic Motion* (New York: Springer-Verlag).

Lubik, T. and F. Schorfheide, 2007, "Do Central Banks Target Exchange Rates? A Structural Investigation," *Journal of Monetary Economics*, Vol. 54, pp. 1069-1087.

Mendoza, E., 1995, "The Terms of Trade, The Real Exchange rate and Economic Fluctuations," *International Economic Review*, Vol. 36, pp. 101-137.

Meng, Q., and A. Velasco, 2003, "Indeterminacy in a Small Open Economy with Endogenous Labor Supply," *Economic Theory*, Vol. 22, pp. 661-670.

Monacelli, T., 2005, "Monetary Policy in a Low Pass-Through Environment," *Journal of Money Credit and Banking*, Vol. 37, pp. 1047-1066.

Ogaki, M., J.D. Ostry, and C.M. Reinhart, 1996, "Saving Behavior in Low and Middle-Income Developing Countries: A Comparison," *IMF Staff Papers*, Vol. 43, pp. 38-71.

Petursson, T., 2004, "The Effects of Inflation Targeting on Macroeconomic Performance" Central Bank of Iceland Working Papers No. 23(June).

Schmitt-Grohé, S. and M, Uribe2003, "Closing Small Open Economy Models," *Journal of International Economics*, Vol. 61, pp. 137-139.

Weder, M., 2001, "Indeterminacy in a Small Open Economy Ramsey Growth Model," *Journal of Economic Theory*, Vol. 98, pp. 339-356.

Woodford, M., 2003, *Interest and Prices: Foundations of a Theory of Monetary Policy* (Princeton: Princeton University Press).

Zanna, L. F., 2003, "Interest Rate Rules and Multiple Equilibria in the Small Open Economy," International Finance Discussion Papers No. 7895.