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Calibrating Your Intuition: Capital Allocation for Market and Credit Risk

Paul Kupiec

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Prepared by Paul Kupiec

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Abstract

Value-at-risk (VaR) models often are used to estimate the equity investment that is required to limit the default rate on funding debt. Typical VaR "buffer stock" capital calculations produce biased estimates. To ensure accuracy, VaR must be modified by (1) measuring loss relative to initial market value, and (2) augmenting VaR to account for the interest income required by investors. While this issue has been identified in the market risk setting, it has yet to be recognized in the credit risk literature. Credit VaR techniques, as typically described, are not an appropriate basis for setting equity capital allocations.

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Author's E-Mail Address: pkupiec@imf.org

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I. Introduction

In the context of market risk, banking regulation, industry practice, and the risk management literature often equate market risk capital with a specific value-at-risk (VaR) measure or perhaps a multiple thereof.[1] In the context of credit risk, the Basel Committee on Banking Supervision reports[2] that banks typically set credit risk capital equal to a measure of credit risk called unexpected credit loss—the difference between the expected value and an extreme loss value of the probability distribution of a credit portfolio's potential future value.[3] Unexpected credit loss is the metric that is estimated in credit VaR models.

The widespread use of VaR techniques to estimate risk capital requirements owes in part to the intuitive appeal of VaR measures. Unfortunately, the simplistic intuition that underlies a VaR approach to capital allocation has serious shortcomings in both the market risk and credit risk settings. The flaw in the VaR capital allocation methodology identified in this paper is unrelated to the statistical accuracy of VaR measures. Perfectly accurate VaR models produce seriously biased estimates of risk capital requirements.

In the context of a rigorous equilibrium model of firm capital structure, this paper constructs accurate buffer stock capital allocations for both market and credit risk. These equity capital funding requirements differ from those recommended by the traditional VaR capital allocation process in two ways. One difference is in the construction of the VaR measure. For capital allocation purposes, it is demonstrated that VaR must be measured relative to a portfolio's initial market value. Many textbook discussions suggest that VaR should be measured relative to the mean of the end-of-period value (or return) distribution. Such measures are inappropriate for use in capital allocation calculations. The second source of bias in traditional VaR capital allocation estimates is that the VaR calculation ignores the interest payments that must be made on the funding debt. To accurately determine buffer stock equity capital requirements, a correctly constructed VaR estimate must be augmented by an estimate of the required interest payments on funding debt. This recipe holds for both market and credit risk capital allocations.

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[1] See for example, Wilson (1997a), Smithson (1997), or the discussion in Kupiec (2001).

[2] "Basel Committee on Banking Supervision, (1999), p. 13.

[3] In some cases, setting capital equal to unexpected credit loss is encouraged by supervisors. For example, the U.S. Comptroller of the Currency's *Handbook for Large Bank Supervision* (1995) states that, "capital is required as a cushion for a bank's overall risk of *unexpected* loss."

While the results show that VaR-like techniques can be used to set accurate buffer stock equity capital allocations, the appropriate VaR measure requires a significant recalibration of thinking, especially in the case of credit risk capital allocation. This issue has yet to be widely recognized in practice or in the literature. The credit VaR measure appropriate for capital allocation is not a measure of the credit risk of the purchased risky debt contract. Unlike a credit risk measure, the credit VaR measure appropriate for capital allocation purposes measures loss relative to an asset's initial value and not to the instrument's promised payment stream. This construction gives rise to VaR measures that are likely to be counter intuitive to many risk managers. For example, in many instances, credit risk capital VaR measures are negative.

An outline of the paper follows. Section II formally defines market risk and credit risk VaR measures. The intuition that links risk capital with a VaR risk exposure measure seemingly is transparent and consequently VaR-based capital allocation schemes have strong appeal. Section III identifies the flaw in the logic that underlies the common explanation that is used to support VaR-based approaches for capital allocation. Section IV discusses the accurate construction of buffer stock capital allocations in the context of the Black and Scholes (1973) and Merton (1974) (BSM) model. Section V provides explicit examples of alternative capital allocation calculations. Section VI concludes the paper.

II. DEFINING A VaR MEASURE

VaR is commonly defined to be the loss amount that could be exceeded by at most a maximum percentage of all potential future value realizations at the end of a given time horizon.[4] By this definition, VaR is determined by a specific left-hand critical value of a potential profit and loss distribution, and by convention, the loss it represents is reported as a positive value. The other determinant of VaR is the right boundary against which the loss is measured. While the importance of the right hand boundary of the VaR measure may seem puzzling when VaR is defined in terms of the profit and loss distribution, in practice VaR is often measured relative to the mean of the end-of-period value or return distribution and not relative to a portfolio or asset's initial market value. While this practice is sometimes obscured by the short horizons typically used in market risk calculations, it is clearly evident in many discussions describing the calculation of credit VaR measures.

[4] This definition can be found for example in Duffie and Pan, (1997), Hull and White (1998), Jorion (1995 and 1997), Beder (1995), Marshall and Siegel (1997), and Pritsker (1997).

A. Market Risk VaR

The "textbook" formulation for market risk VaR assumes that assets' returns are normally distributed over the interval of interest. In this setting, if the asset of interest has a present value of V_0, and a single-period normally distributed return, $\tilde{r}$, with a mean of μ and a variance of σ^2, then the 1 percent VaR measured relative to the asset's initial market value, V_0, over the single period horizon, $VaR(0.01)$, is given by,

$$VaR(0.01) = -V_0(\mu - 2.33\sigma) \tag{1}$$

This definition of VaR is consistent with the common description of VaR as a measure of potential loss exposure relative to the portfolio's initial value.

Under a common alternative definition, VaR is a measure of the distance between a selected left-hand critical value and the mean of the end-of-period value (or return) distribution. When losses are measured relative to the asset's expected value, the 1 percent VaR for an asset with normally distributed returns with a mean of μ and a variance of σ^2 is given by,

$$VaR^{\mu}(0.01) = V_0(2.33\sigma), \tag{2}$$

where the notation VaR^{μ} is used to indicate that potential losses are being measured relative to the expected end-of-period asset value, i.e., relative to $V_0(1+\mu)$. VaR^{μ} is, for example, the measurement basis for the Basel Internal Models Approach for setting market risk capital requirements.

In typical short-horizon applications that assume normally distributed returns, VaR is intended to measure one-day exposures and μ is either approximately 0 or intentionally set to 0 to minimize the effects of errors associated with the estimation of short-horizon expected returns and there is no difference in the alternative measures. Capital allocation decisions, however, require VaR calculations for holding periods substantially longer than a day, and the differences in these alternative VaR measures can become substantial.[5]

B. Credit Risk VaR

In contrast to the market risk setting in which VaR methods were initially developed for monitoring trading book exposures over short horizons, credit VaR techniques were developed to measure risks over relatively long horizons primarily for use in capital

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[5] Kupiec (1999) provides additional discussion.

allocation and Risk-Adjusted Return On Capital (RAROC) decisions. It has been widely presumed that an appropriate approach for setting the equity share of funding for a credit portfolio is to set equity capital equal to an estimate of a portfolio's so-called unexpected credit loss. The credit risk modeling techniques used to estimate unexpected credit losses are generically called credit VaR models.[6]

In anticipation of the discussion that follows, it is worth reviewing the "current best practices" recommendations for credit risk capital allocation. The *CreditMetrics Technical Document* recommends using unexpected credit losses (the *CreditMetrics* so-called "percentile level" measure) to measure credit risk and set credit risk capital.[7] Wilson (1997b) and Saunders (1999) argue that unexpected credit losses are a measure of credit risk and an appropriate capital benchmark. While the Basel Bank Supervisors voice concerns regarding the empirical implementation of credit VaR models, they too subscribe to the view that the unexpected credit loss metric is, in principle at least, a sound measure of credit risk and an appropriate gauge of credit risk capital.[8] Beyond the credit risk measurement and capital literature, the view that credit VaR measures of credit risk are an appropriate benchmark for capital is also common in the literature on risk-adjusted performance or so-called RAROC analysis.[9]

A stylized credit VaR unexpected credit loss measure is illustrated in Figure 1. The probability distribution pictured in Figure 1 represents the true probabilities associated with all potential end-of-period values that may be realized on an asset (portfolio) with credit risk. The potential profit and loss distribution of interest is generated by potential changes in the value of credit risk sensitive exposures to individual counterparties over the horizon that has been selected to measure credit risk and set capital.[10] Credit VaR models attempt to estimate unexpected credit losses in either a mark-to-market (MTM) or a held to maturity (HTM) setting. If the asset has yet to mature in the horizon of interest, the end-of-period value distribution represents the asset's potential MTM values or its range of potential values in an early default. If the end of the period in question corresponds to the maturity of the asset, the variation in end-of-period values owes entirely to variation in default severity.

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[6] Basel Committee on Banking Supervision, (1999), p. 14.

[7] J.P. Morgan, April, (1997), p. 139.

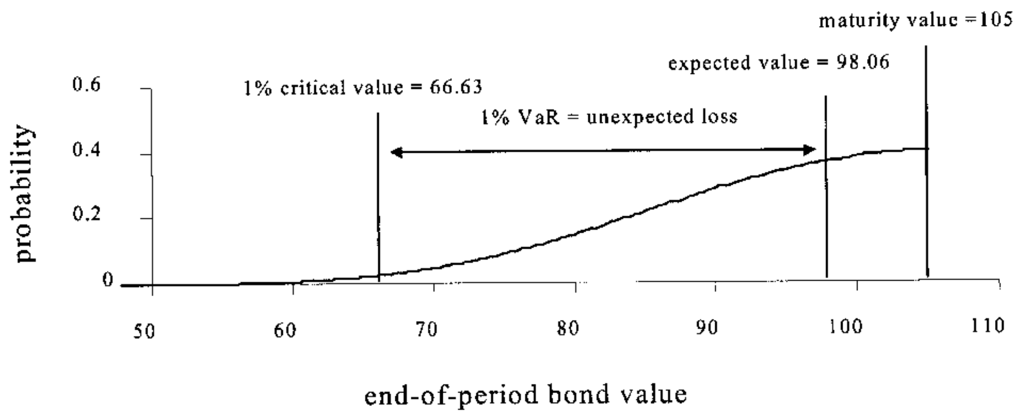
[8] Basel Committee on Banking Supervision, (1999), p. 13.

[9] See for example, Shimko (1997), Smithson (1997), or Matten (1997), Kupiec (2001) provides a critical assessment of these claims.

[10] See Basel Committee on Banking Supervision (1999) or Saunders (1999) for a discussion of alternative approaches for estimating the end-of-period value distribution in alternative credit VaR models.

The unexpected credit loss measure is defined as the difference between the mean of the end-of-period value distribution and the loss associated with a user-selected critical value in the loss tail of the distribution.[11] In Figure 1, for example, the unexpected credit loss measure is 31.43 when measured using the distribution's 1 percent critical value.

Figure 1. Stylized Credit VaR



Source: Stylized 1 percent unexpected credit loss calculation over a one-year horizon for a discount bond with a maturity of one-year and a par value of 105.

III. Calibrating Intuition Linking VaR and Capital Allocation

It is useful to review the conventional intuition that underlies the use of VaR approaches for setting buffer stock capital allocations. A buffer stock capital allocation is the equity portion of a funding mix that can be used to finance an asset (portfolio) in a way that maximizes the use of debt finance subject to a maximum acceptable probability of default on the funding debt.[12] The analysis in this study is limited to portfolios composed of traditional financial assets such as bonds or equities for which the maximum value that can be lost is the current market value of the portfolio.

[11] This definition appears, for example, in the *CreditMetrics Technical Document* (1996), Wilson (1997a), Saunders (1999), and Basel Committee on Banking Supervision (1999).

[12] This study makes no claim that this objective function formally defines a firm's optimal capital structure—indeed it almost certainly does not. It is, however, the objective function that is consistent with VaR-based capital allocation schemes and an approach commonly taken by banks according to the Basel Committee on Banking Supervision's (1999) survey results.

Consider the use of a 1 percent, one-year VaR measure of an asset's risk to determine the necessary amount of equity funding for the position under a buffer stock approach for setting debt funding objectives. By definition, there is less than a 1 percent probability that the asset's value will ever post a loss that exceeds its 1 percent VaR risk exposure measure. That is, if we choose an amount of equity finance equal to its 1 percent VaR, the implication is that there is less than a 1 percent chance that any loss in its underlying assets' values will ever exceed the value of the firm's equity. A common interpretation is that this equity financing share will ensure that there is at most a 1 percent chance that the firm will default on its debt. This intuition is, however, flawed.

Assume that VaR will be measured from the asset's initial market value and that VaR measures are completely accurate in this sense that there is no statistical error in measuring the asset's end-of-period market value distribution. In this case VaR can never exceed V_0. If the firm were to set the share of equity funding equal to the asset's 1 percent VaR measure, $VaR(0.01)$, the amount of debt finance required to fund the asset would be $V_0 - VaR(0.01)$. The flaw in the aforementioned VaR capital allocation logic is that if the firm borrows $V_0 - VaR(0.01)$, *it must pay back more than* $V_0 - VaR(0.01)$ *if it is to avoid default.* The simple intuition that underlies the VaR approach for capital allocation ignores the interest payment that must be made on funding debt. An unbiased buffer stock capital allocation rule is to set equity capital equal to 1 percent VaR (calculated appropriately) *plus the accrued interest on funding debt.*

The upshot is, if one uses the correct VaR measure—one in which the VaR's right-side boundary is set by the asset's initial market value—and the VaR estimate is augmented by the interest payments that will be required by investors that purchase the funding debt, the VaR methodology can provide perfectly accurate measures of buffer stock capital for bond or equity type investments. This is true in both the market risk and the credit risk setting. The required VaR calculation, while modified compared to many discussions of VaR measures, does not present any technical issues. The complication is introduced by the necessity of obtaining estimates of the required interest payments on funding debt. The following section describes the capital allocation process in the context of a specific equilibrium asset pricing model that will allow for the determination of the required interest payments on funding debt.

IV. Using VaR to Set Accurate Risk Capital Calculations

If there are no taxes, transactions are costless, short sales are possible, trading takes place continuously, if borrowers and savers have access to the debt market on identical risk-adjusted terms, and investors in asset markets act as perfect competitors, Merton (1974) established that the Modigliani-Miller capital structure irrelevance theorem holds in the presence of risky debt. That is, the market value of the firm is completely independent of capital structure and the probability of default can be chosen freely by management.

If the risk-free term structure is flat and a firm issues only pure discount debt and asset values follow geometric Brownian motion, Black and Scholes (1973), and Merton (1974) have demonstrated that the market value of a firm's debt is equal to the market value the issue would have if it were default free, less the market value of a Black-Scholes put option written on the value of the firm's assets. The put option has a maturity equal to the maturity of the debt issue and strike price equal to the par value of the discount debt. If B_0 represents the bond's initial equilibrium market value, and Par represents its promised payment at the maturity date M, the BSM model requires,

$$B_0 = Par\ e^{-r_f M} - Put(A_0, Par, M, \sigma\ ), \tag{3}$$

where r_f represents the risk free rate and $Put(A_0, Par, M, \sigma\ )$ represents the value of a Black-Scholes put option on an asset with an initial value of A_0, a strike price of Par, a maturity of M, and an instantaneous return volatility of σ. The default (put) option is a measure of the credit risk of the bond. While Merton (1974) shows that the model will generalize (as to term structure assumptions, coupon debt issuance, and generalized volatility assumptions), the capital allocation discussion that follows will be based upon the simplest formulation of the BSM model.[13]

A. Market Risk Capital

In the BSM model, the firm's underlying assets evolve in value according to geometric Brownian motion and have future values that exhibit "market risk" in the vernacular of risk managers. In this setting, the selection of the firm's debt-equity funding mix under an objective of achieving a target default rate on its funding debt is a market risk capital allocation problem. In the market risk setting, the VaR calculation is applied to the physical probability distribution for the firm's asset value at a horizon equal to the desired maturity of the firm's funding debt.

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[13] That is, it assumed that the term structure is flat, asset volatility is constant, the underlying asset pays no dividend or convenience yield, and all debt securities are pure discount issues.

Under the assumptions of the BSM model, the value of the firm's assets evolve following,

$$dA = \mu A\, dt + \sigma A\, dz\,, \tag{4}$$

where dz is a standard Weiner process. If A_0 represents the initial value of the firm's assets, and A_T the value of the firm's assets at time T, Ito's lemma implies,

$$ln\, A_T - ln\, A_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \quad \sigma\sqrt{T}\right], \tag{5}$$

where $\phi[a,b]$ represents the normal density function with a mean of "a" and a standard deviation of "b". Equation (5) defines the physical probability distribution for the end-of-period value of the firm's assets,

$$\tilde{A}_T \sim A_0 \quad e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\ \tilde{\varepsilon}}\,, \tag{6}$$

where $\tilde{\varepsilon} \sim \phi[0,1]$.

Let $\Phi(x)$ represent the cumulative density function for a standard normal random variable evaluated at x, and $\Phi^{-1}(\alpha)$, the inverse of this function evaluated at $0 \le \alpha \le 1$. The market risk VaR measure that is appropriate for calculating an equity capital allocation consistent with a target default rate of α for a funding debt maturity of T is given by,

$$VaR(\alpha) = A_0 - A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\ \Phi^{-1}(\alpha)} = A_0\left[1 - e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\ \Phi^{-1}(\alpha)}\right] \tag{7}$$

$A_0 - VaR(\alpha) = A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\ \Phi^{-1}(\alpha)}$ is the maximum par value of discount debt that can be issued without violating the firm's target default rate. The BSM debt pricing condition (expression (3)) can then be used to determine the initial market value of this debt issue. The difference between the initial market value of the debt and its par value is the equilibrium interest compensation that must be offered to the firm's debt holders. In the BSM model setting, the interest payments are,

$$A_0\left[e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)} - e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha) - r_f T}\right] + Put\left(A_0\,,\, A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\ \Phi^{-1}(\alpha)}, T, \sigma\right) \tag{8}$$

This interest amount must be added to $VaR(\alpha)$ to calculate the true equity capital needed to achieve the target default rate on funding debt. The true amount of equity required to achieve a target default rate of α on funding debt of maturity T is given by,

$$A_0\left[1 - e^{\left(\mu-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha) - r_f T}\right] + Put\left(A_0\ , A_0\ e^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\,\Phi^{-1}(\alpha)}, T,\ \sigma\right) \qquad (9)$$

The components of the equity capital allocation are instructive. The first component of expression (9) increases equity over $VaR(\alpha)$ to allow funding debt holders to receive a risk free return on their investment. The second term in expression (9) further increases equity capital above $VaR(\alpha)$ to ensure that the funding debt holders receive the proper credit risk interest spread on their investment.

B. Credit Risk Capital

In order to illustrate the buffer stock capital allocation technique that is appropriate for assets with credit risk, it necessary to introduce a modified version of the BSM model in order to value the funding debt of a firm that purchases credit risky assets. Consider the case in which a firm's only asset is a risky BSM discount debt issue. Assume that the firm will fund this bond with its own discount debt and equity issues. In this setting, the firm's funding debt issue is a compound option.

Let $\tilde{A}_T$ represent the time T value of the assets that support the purchased discount debt. Let Par_P represent the par value of the purchased discount bond and Par_F represent the par value of the discount bond that is used to fund the asset purchase. If the maturity of the firm's funding debt matches the maturity of the firm's asset (both equal to M), then the end-of-period cash flows that accrue to the firm's debt holders are given by,

$$Min\left[Min\left(\tilde{A}_M, Par_p\right), Par_F\right] \qquad (10)$$

If the funding debt is of a shorter maturity (T) than the purchased discount bond (M), then the end-of-period cash flows that accrue to the firm's funding debt holders are given by,

$$Min\left[\left(Par_p e^{-r_f(M-T)} - Put\left(\tilde{A}_T, Par_p, M-T, \sigma\right)\right), Par_F\right] \qquad (11)$$

Equilibrium absence of arbitrage conditions impose restrictions on the underlying asset's Brownian motion's drift term, $\mu = r_f + \lambda\sigma$, where λ is the market price of risk associated with the firm's assets. Define $dA^{\eta} = (\mu - \lambda\sigma)A^{\eta}dt + A^{\eta}\sigma\, dz$ to be the "risk neutralized" process that is used to value derivative claims after an equivalent martingale change of measure. The probability distribution of the underlying end-of-period asset values after the equivalent martingale change of measure, $\widetilde{A}_M^{\eta}$, is,

$$\widetilde{A}_T^{\eta} \sim A_0\, e^{\left(r_f - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\widetilde{\varepsilon}} \qquad (12)$$

When the maturity of the firm's funding debt matches the maturity of the firm's asset (both equal to M), the equivalent martingale probability distribution of the end-of-period asset's value, $\widetilde{A}_M^{\eta}$, is used to calculate the initial market value of the funding discount bond by discounting (at the risk free rate) the expected value of (10) taken with respect to probability density of $\widetilde{A}_M^{\eta}$.[14]

In the alternative case in which the funding debt is of a shorter maturity (T) than the purchased discount bond (M), because the value of the put option in the purchased discount bond depends only on the risk free rate, the time to maturity on this bond (both deterministic), and the underlying value of the supporting assets, is straight forward to calculate the initial equilibrium value of the funding debt as the discounted value (at the risk free rate) of the expected value of expression (11), where the expectation is taken with respect to the equivalent martingale probability density $\widetilde{A}_T^{\eta}$.

Given the equilibrium valuation relationships that must be satisfied by the firm's funding debt, we now consider the buffer stock capital allocation process for assets with credit risk. Assume that the firm's objective is to maximize the use of debt funding subject to limiting the default rate on its funding debt to a maximum acceptable rate. Recall that the firm's asset is a BSM risky discount bond of maturity M. We consider initially capital allocation when the maturity of the funding debt is equal to the maturity of the purchased bond.

C. Held-to-Maturity Credit VaR

At maturity, the payoff on the firm's purchased bond is given by, $Min\left[Par_P, \widetilde{A}_M\right]$. The credit risk VaR measure appropriate for credit risk capital allocation is given by,

$$VaR_{Credit}(\alpha) = B_0 - Min\left[Par_P,\ A_0 e^{\left[\mu - \frac{\sigma^2}{2}\right]M + \sigma\sqrt{M}\,\Phi^{-1}(\alpha)}\right] \qquad (13)$$

[14] Alternatively, Geske (1977 and 1979) provides a closed-form expression for the value of the compound option.

where B_0 is the initial market value of the purchased discount debt given by expression (3), and α is the target default rate on the funding debt. If α is sufficiently small (which will be assumed), the expression $Min\left[Par_P,\ A_0 e^{\left[\mu-\frac{\sigma^2}{2}\right]M+\sigma\sqrt{M}\,\Phi^{-1}(\alpha)}\right]$ simplifies to

$A_0\, e^{\left[\mu-\frac{\sigma^2}{2}\right]M+\sigma\sqrt{M}\,\Phi^{-1}(\alpha)}$, and consequently, the expression for credit VaR is,

$$VaR_{Credit}(\alpha, M, M) = B_0 - A_0 e^{\left[\mu-\frac{\sigma^2}{2}\right]M+\sigma\sqrt{M}\,\Phi^{-1}(\alpha)} \qquad (14)$$

The notation for credit VaR includes three arguments, the target default rate α, the maturity of the funding debt issue, M (the second argument), and the maturity of the credit risky asset, M (the third argument). The utility of this unusual notation will become clear in the subsequent section.

$B_0 - VaR_{Credit}(\alpha, M, M) = A_0\, e^{\left[\mu-\frac{\sigma^2}{2}\right]M+\sigma\sqrt{M}\,\Phi^{-1}(\alpha)}$, is the maximum par value of the funding debt that is consistent with the target default rate. The initial market value of this funding debt issue is given by,

$$E^{\eta}\left[Min\left[Min\left(\widetilde{A}_M, Par_P\right), A_0\, e^{\left[\mu-\frac{\sigma^2}{2}\right]M+\sigma\sqrt{M}\,\Phi^{-1}(\alpha)}\right]\right] e^{-r_f M} \qquad (15)$$

where the notation $E^{\eta}[\ ]$ denotes the expected value operator with respect to the probability density for $\widetilde{A}_M^{\eta}$. Using these relationships, it should be clear that the equilibrium required interest payment on the funding debt is given by,

$$A_0\, e^{\left[\mu-\frac{\sigma^2}{2}\right]M+\sigma\sqrt{M}\,\Phi^{-1}(\alpha)} - E^{\eta}\left[Min\left[Min\left(\widetilde{A}_M, Par_P\right), A_0\, e^{\left[\mu-\frac{\sigma^2}{2}\right]M+\sigma\sqrt{M}\,\Phi^{-1}(\alpha)}\right]\right] e^{-r_f M} \qquad (16)$$

Expressions (14) and (16) imply that the initial equity allocation consistent with the target default rate α is given by,

$$B_0 - E^{\eta}\left[Min\left[Min\left(\widetilde{A}_M, Par_P\right), A_0\, e^{\left[\mu-\frac{\sigma^2}{2}\right]M+\sigma\sqrt{M}\,\Phi^{-1}(\alpha)}\right]\right] e^{-r_f M} \qquad (17)$$

D. Mark-to-Market Credit VaR

The credit VaR profit-and-loss distribution differs according to whether the VaR horizon corresponds to the maturity of the credit risky asset or a shorter period of time. When calculating credit VaR for buffer stock capital purposes, the credit VaR horizon must be equal to the maturity of the funding debt issue. Any other credit VaR horizon will produce capital allocations with default rates that differ from the intended target.[15]

In some instances in which the end-of-period value of the purchased bond is less than the par value of its funding debt, it is possible for the firm to refinance its debt and avoid default without an equity injection. In such instances, however, the implied default rate on the new debt issue is necessarily much higher than the firm's original target default rate because, in order to refinance, the firm must dilute its equity value by promising a greater share of the end-of-period cash flows to the new bond investors.[16] Regardless of whether the firm is actually forced to default when the value of the purchased debt falls below the funding debt's par value at maturity, the firm's capital allocation objective has been violated and the firm cannot avoid default and continue in business while funding at its optimal target default rate unless the shareholders inject new equity capital.

When the firm's funding debt matures at date T before the purchased risky discount bond's maturity, M, $T < M$, the α level credit VaR is given by,

$$VaR_{Credit}(\alpha,T,M) = B_0 - \left(Par_P\, e^{-r_f(M-T)} - Put\left(A_0\, e^{\left[\mu - \frac{\sigma^2}{2}\right]T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)}, Par_P, M-T, \sigma \right)\right) \qquad (18)$$

Again, $B_0 - VaR_{Credit}(\alpha,T,M)$ determines the maximum par value of the funding debt that satisfies the target default rate constraint; this value is given by,

$$Par_F(\alpha,T,M) = Par_P\, e^{-r_f(M-T)} - Put\left(A_0\, e^{\left[\mu - \frac{\sigma^2}{2}\right]T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)}, Par_P, M-T, \sigma \right) \qquad (19)$$

where the arguments in the notation for $Par_F(\alpha,T,M)$ conform with those in $VaR_{Credit}(\alpha,T,M)$.

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[15] For further discussion, see Kupiec (2002).

[16] See Kupiec (2002).

Using the expression for $Par_F(\alpha,T,M)$, the initial market value of the funding debt issue is,

$$E^{\eta}\left[Min\left[\left(Par_P \;\; e^{-r_f(M-T)} \;\; -Put\left(\tilde{A}_T, Par_p, M-T, \sigma\right)\right),\; Par_F\left(\alpha,T,M\right)\right]\right] \; e^{-r_f T} \qquad (20)$$

and the equilibrium required interest payment on the funding debt is,

$$Par_F(\alpha,T,M)-E^{\eta}\left[Min\left[\left(Par_P \;\; e^{-r_f(M-T)} \;\; -Put\left(\tilde{A}_T, Par_p, M-T, \sigma\right)\right) ,\; Par_F(\alpha,T,M)\right]\right] e^{-r_f T} \qquad (21)$$

Expressions (18), (19), and (21) imply that the equity allocation consistent with a target default rate of α is given by,

$$B_0 - E^{\eta}\left[Min\left[\left(Par_P \;\; e^{-r_f(M-T)} \;\; -Put\left(\tilde{A}_T, Par_p, M-T, \sigma\right)\right) ,\; Par_F\left(\alpha,T,M\right)\right]\right] \; e^{-r_f T} \qquad (22)$$

E. Remarks

Equations (17) and (22) are respectively the equity capital allocations necessary to achieve the target default rates of the firm's funding debt in the HTM and the MTM credit risk cases. In both of these expressions, the equity capital requirement is determined by the α critical value of the risky discount debt's supporting asset distribution. Notice that the underlying capital allocation credit VaR measures (expressions (14) and (18)) are not measures of the credit risk of the risky asset. Credit risk measures are defined relative to the promised maturity payment on a fixed income asset, not its initial value. In contrast to credit VaR measures of unexpected loss, accurate equity capital allocation requires that VaR be measured relative to the fixed income security's (or portfolio's) initial market value, not its promised, expected maturity, or expected future MTM value. These results challenge the long-standing tradition of linking the processes of credit risk measurement and capital allocation. These results also highlight the importance of establishing an accurate estimate of the initial MTM value for a credit portfolio. Accurate buffer stock capital estimates require that initial market values can be accurately measured. Such a task is often thought to be particularly complicated in the case of bank loans.

Another implication of the analysis is that capital allocation credit VaR measures (expressions (14) and (18)) can be, and indeed frequently are negative. The negative credit VaR measures that are appropriate for credit risk capital allocation are unlikely to conform with the intuition of many risk mangers who typically expect to find a positive relationship between VaR measures, risk, and capital allocations.

V. Some Examples

This section illustrates the capital allocation process for both market and credit risk in the context of the BSM model of firm capital structure and valuation. For each of these examples, consider a firm with an initial asset value of 100, a instantaneous return volatility of $\sigma = 0.20$, and a market price of risk $\lambda = 0.15$, in an equilibrium where the risk free rate is 5 percent. Under these assumptions, the asset's physical and risk-neutral value distribution functions are given by,

$$\widetilde{A}_T \sim 100\, e^{\left(.08-\frac{.2^2}{2}\right)T+.2\sqrt{T}\,\tilde{\varepsilon}} \tag{23}$$

$$\widetilde{A}_T^{\eta} \sim 100\, e^{\left(.05-\frac{.2^2}{2}\right)T+.2\sqrt{T}\,\tilde{\varepsilon}} \tag{24}$$

A. Market Risk Capital Allocation

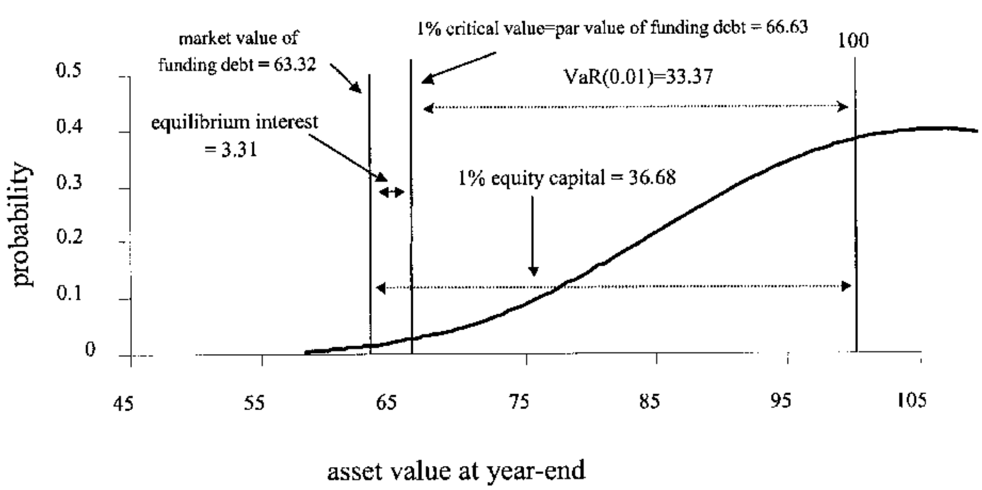
Assume that a firm wishes to fund a risky asset with characteristics given by expression (23) using the maximum amount of one-year discount debt possible subject to the constraint that the probability of default on the funding debt cannot exceed 1 percent. The probability distribution for the asset's value after one-year is pictured in Figure 2. The asset's future value is distributed log-normally, with a left-hand 1 percent critical value of 66.63. The 1 percent, one-year market risk VaR measure appropriate for capital allocation is 33.37, or the difference between the asset's initial market value (100) and the 1 percent critical value of its future value distribution. 66.63 is the maximum par value of the funding debt that can be issued while constraining the funding debt's default rate to 1 percent. Using the BSM valuation model, this discount debt issue will sell for 63.32. The difference between 66.63 and 63.32 is the equilibrium interest compensation required by the funding debt holders. The 1 percent equity capital requirement is equal to 36.68, the sum of $VaR(0.01) = 33.37$, and the equilibrium interest payments to funding debt holders, 3.31. In contrast to these estimates, the "traditional" VaR capital allocation recommendation—setting equity equal to the difference between the mean of the future value distribution (108.33) and its 1 percent critical value (66.63)—would imply equity capital of 41.70.

B. Held-to-Maturity Credit Risk Capital Allocation

Figure 3 illustrates the HTM credit risk capital allocation for a risky BSM discount bond that has a par value of 66.63, an initial market value of 63.32, and is supported by assets with an initial market value of 100 and future values that evolve according to expression (23). In this example, a firm purchases the risky discount bond that is sold by the firm that funded the asset acquisition illustrated in the prior MTM example. Since the purchased discount bond is the firm's only asset, if this bond is completely funded with debt, the funding debt will have a probability of default identical to that of the purchased bond (1 percent). If this bond is in part funded with equity, the funding debt's probability of default must be less than 1 percent.

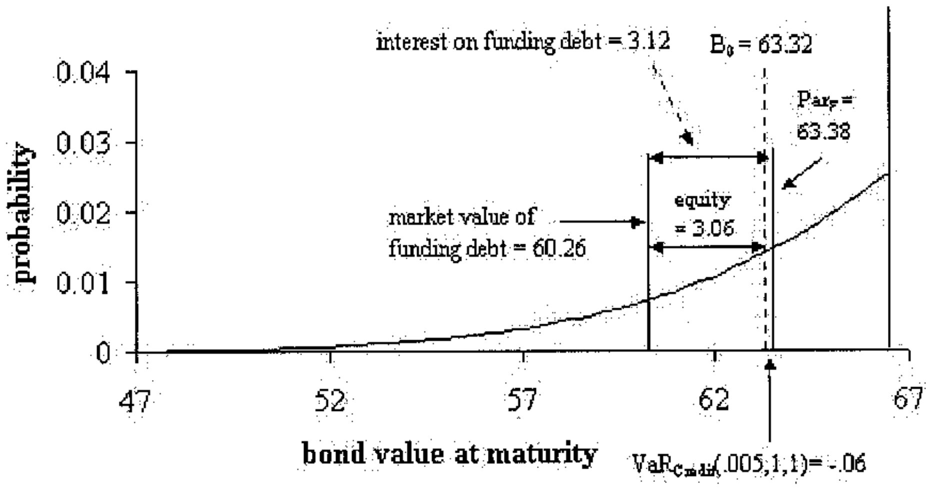
Assume that the objective of the firm that purchases the risky BSM discount bond is to maximize the use of debt finance, subject to limiting the probability of default on its funding debt to $^1/_2$ of 1 percent. The $^1/_2$ of 1 percent critical value of the underlying asset's future value distribution is 63.38. Under this target default rate objective, the capital allocation credit VaR measure is given by $63.32-63.38=-0.06$, or *negative* 6 cents. The par value of the funding debt consistent with the $^1/_2$ percent target default rate is 63.38, and using expressions (15) and (24), the initial value of the funding debt is calculated to be 60.26. The required interest payment on the funding debt is 3.12, and so the required equity funding—the credit VaR estimate plus funding debt interest—is 3.06. Notice that in this example the capital allocation credit VaR measure is negative. For comparison purposes, we note that if credit risk capital were set equal to unexpected credit losses—the difference between the mean of the end-of-period value distribution (66.59) and its $^1/_2$ percent critical value (63.38)—equity capital would be set equal to 3.21.

Figure 2. Market Risk Capital Allocation



Source: Market risk capital allocation example for an asset with an initial value of 100 and a future value that evolves according to geometric Brownian motion with an instantaneous drift rate of 8 percent, and an instantaneous return standard deviation of 20 percent. The risk free rate is assumed to be 5 percent.

Figure 3. Held-to-Maturity Credit Risk Capital Allocation



Source: Held-to-maturity credit risk capital allocation for a one-year BSM risky discount bond with a par value of 66.63 that is supported by assets that have an initiation market value of 100 and future values that evolve according to geometric Brownian motion with an instantaneous drift rate of 8 percent, and an instantaneous standard deviation of 20 percent. The initial market value of the bond is 63.32 and the risk free rate is assumed to be 5 percent.

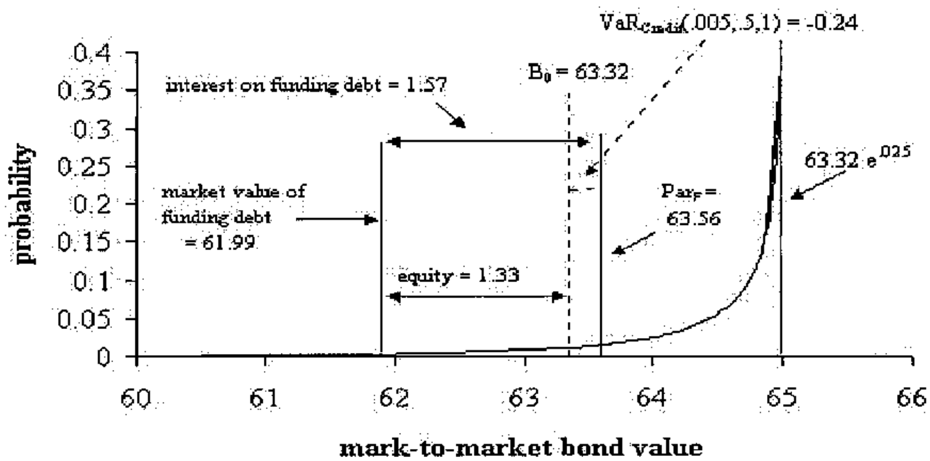
C. Mark-to-Market Credit Risk Capital Allocation

Suppose that the bond issued in the prior market risk example was purchased by another firm and funded for only six-months. Assume that the purchasing firm wanted to fund the issue with as much six-month debt as possible while limiting the default rate on its debt to $^1/_2$ of 1 percent. Figure 4 illustrates the capital allocation calculations in this example.

The end-of-period bond valuation distribution is generated using the BSM discount bond valuation condition (expression(3)), and the distribution for the bond's supporting assets (expression (23)) setting $T = \frac{1}{2}$. Using this future asset value distribution, the $^1/_2$ of 1 percent critical value of the BSM bond's end of period value distribution is 63.56, and its corresponding VaR measure, $VaR\left(0.005, \frac{1}{2}, 1\right)$ is -0.24, or *negative* 24 cents.

The maximum par value of the funding debt that can be issued without violating the target default rate constraint is 63.56, and expressions (20) and (24) are used to calculate the initial equilibrium value of this debt which is 61.99. The equilibrium interest payment required by funding debt investors is 1.57, and so the required amount of equity funding needed to achieve the firm's funding objective is 1.33, the VaR amount, -0.24, plus the required interest on funding debt, 1.57. If alternatively, capital were set equal to unexpected losses has been traditionally recommended, equity capital would be set equal to 3.06.

Figure 4: Mark-to-Market Credit Risk Capital Allocation



Source: Six-month mark-to-market credit risk capital allocation for a one-year BSM risky discount bond with a par value of 66.63 that is supported by assets that have an initial market value of 100 and future values that evolve according to geometric Brownian motion with an instantaneous drift rate of 8 percent, and an instantaneous standard deviation of 20 percent. The initial market value of the bond is 63.32 and the risk free rate is assumed to be 5 percent.

VI. Conclusions

Buffer stock capital allocations cannot be accurately estimated using the VaR measures that are often described in the literature. In both the market and credit risk setting, accurate buffer stock equity capital allocation requires that VaR estimates be calculated relative to the initial market value of the assets or portfolio in question. The future portfolio value distribution used in the VaR calculation must accurately account for the expected return (or expected drift rate) that determines the future value of the assets (or portfolio). Using an appropriate VaR measure, the amount of equity funding consistent with the target funding debt default rate is given by the VaR estimate plus an estimate of the equilibrium rate of return that will be required by the investors who purchase the funding debt. The interest compensation calculation and, in the case of non-traded or thinly traded debt instruments, the estimate of the initial market value of the portfolio, will generally require the use of an asset pricing model.

While this paper discusses capital allocation using VaR techniques, the analysis clearly demonstrates that there is really no need to calculate a VaR measure in order to calculate equity capital requirements. "All" that is actually required for this calculation is the critical value of the asset's end-of-period value distribution to set the par value of the funding debt, and an asset pricing model to estimate the equilibrium initial market value of the funding debt. The initial market value of the asset and the proceeds from the funding debt issue determine the required amount of equity needed to fund the asset's purchase. This is true for both market risk and credit risk. In the case of credit risk, the portfolio's so-called "unexpected loss" is irrelevant for constructing a buffer stock capital allocation. This paper's findings regarding credit risk capital allocation have not been widely recognized. They have important implications both for practitioners who calculate internal capital allocations as well as for regulators who are responsible for calibrating bank capital requirements for credit risks.

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