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Riding the Yield Curve: Risk Taking Behavior in a Low Interest Rate Environment

by Ralph Chami, Thomas F. Cosimano, Céline Rochon, Julieta Yung

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Abstract

Investors seek to hedge against interest rate risk by taking long or short positions on bonds of different maturities. We study changes in risk taking behavior in a low interest rate environment by estimating a market stochastic discount factor that is non-linear and therefore consistent with the empirical properties of cashflow valuations identified in the literature. We provide evidence that non-linearities arise from hedging strategies of investors exposed to interest rate risk. Capital losses are amplified when interest rates increase and risk averse investors have taken positions on instruments with longer maturity, expecting instead interest rates to revert back to their historical average.

JEL Classification Numbers: E43; C58; G11; G12

Keywords: Interest rate risk; non-linear stochastic discount factor; investment portfolio; term structure model; risk aversion distribution; low interest rate environment

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I. Introduction

Financial market participants hedge against interest rate risk by taking long or short positions on bonds of different maturities, i.e. the "riding the yield curve" strategy. If investors anticipate an increase in interest rates (i_t), standard macroeconomic theory would suggest a decline in expected future cashflows, the "stochastic discount factor" or SDF,

$$\frac{1}{1+\uparrow i_t} = \downarrow \mathbb{E}_t\left[SDF_{t+1}\right]. \tag{1}$$

This theoretical relationship follows from an SDF that is positive and monotonically decreasing in economic activity, an argument that holds under the assumption of complete markets (Dybvig, 1988), and therefore provides limited opportunities to hedge against interest rate fluctuations.[1] There is substantial evidence, however, that the expected SDF varies over time and is non-monotonic.[2] Intuitively, this means that if investors anticipate a one percentage point increase in interest rates, their portfolio rebalancing strategy would depend on whether interest rates were lower than average (and therefore reverting back to the mean), or higher than average (and therefore moving even further away from the mean).

In this paper, we derive and empirically estimate a non-monotonic SDF and we identify investors' hedging strategy against interest rate risk as the source of non-linearities in their cashflow valuation process. This framework allows for investors to (i) respond to movements along the entire yield curve – not just the short rate, (ii) evaluate movements in interest rates in relation to their historical averages, and (iii) develop hedging strategies given their relative risk aversion. To illustrate the importance of non-linearities, we consider investors in a low interest rate environment as they optimize their portfolios to account for interest rate fluctuations.

In order to study risk-taking behavior in a low interest rate environment, we derive the market's stochastic discount factor from a canonical term structure model under the assumption of no arbitrage, in which the U.S. yield curve is described by observable interest-rate risk factors that capture cross-sectional variation in interest rates as in Joslin et al. (2011) and Adrian et al. (2013). The model is parsimonious and econometrically tractable, yet successful

[1] In particular, popular models used by Central Banks to understand the effects of monetary policy on the economy such as FRB/US (Brayton et al., 2014) at https://www.federalreserve.gov/econresdata/frbus/us-models-about.htm and Euro (Smets and Wouters, 2003, 2007; Christoffel et al., 2008), have this limitation in terms of reflecting financial market participants' attitudes towards risk.

[2] See Campbell (2014) for a discussion of the contributions of Eugene Fama, Lars Peter Hansen, and Robert Shiller to this line of research, which is the basis for their 2013 Nobel Memorial Prize in Economic Sciences.

at explaining interest rate movements and inferring the market prices of risk that describe how interest-rate risk factors relate to the investor's SDF.

First, we show that the probability density function of the SDF is log-normal, thus introducing time-variation and non-linearity with respect to the risk factors, consistent with empirical evidence in the macro-finance literature.[3] Following Chami et al. (2017) and Cosimano and Ma (2018), we derive the conditional expected SDF in the bond market, relative to each one of the interest-rate risk factors moving, while holding all else constant. We use the movements in the bond market following the 2016 U.S. presidential elections to illustrate how as the interest-rate risk factors change, expected cashflow valuations increase when the yield curve movements occur to the left of the conditional mean of the expected SDF, yet decrease when they are to the right. This feature is an important distinction that arises from the Gaussian properties of the probability density function of the SDF.

Second, we provide evidence that non-linearity in the SDF arises naturally as investors choose their portfolio positions to hedge against interest rate risk, even under standard constant relative risk averse preferences. Following the optimization problem in Sangvinatsos and Wachter (2005), we find the optimal portfolio rule for an investor that takes as given the linear relation between the expected excess return on bonds and the interest-rate risk factors. These portfolio rules are also linear in the Sharpe ratio of U.S. Treasury securities, a leverage constraint, and the risk factors, and are derived for an investor with a constant relative risk aversion utility for terminal wealth.[4] Hedging strategies arise from investors' conditional expectations of future interest rate movements. Investors go long on long-term securities when yields are above their historical average and vice versa. Given these hedging positions, investors suffer a capital loss when the future yields move away from their long-term mean.

Our framework allows us to study portfolio choices for investors with different levels of risk aversion and examine both the direct and indirect effects of interest rate fluctuations. By multiplying investors' portfolio rules (which state the percentage of wealth invested in each security) by their respective wealth levels, we obtain the demand for all the maturities of U.S. Treasury securities. We equate total demand to the total supply of government securities made available to the market, following Wang (1994)'s model of stock market volume and Vayanos and Vila (2009)'s model of preferred habitat in the market for Treasury securities,

[3]For example, Cochrane (2011) found substantial time variation in the discount rate or expected SDF across many financial assets including the expected excess return on longer-term bonds.

[4]If the investors also have distinct investment horizon, then the hedging demand strategy will depend on different mean and variance-covariances according to the investor's horizon.

which derives the investor's preferred habitat based on the absolute risk aversion of the investor.

The expected SDF for the market is an envelope of Gaussian distributions, since the equilibrium excess return on Treasury securities is an affine function of the interest-rate risk factors, which are normally distributed. As a result, we find that in a low interest rate environment, the market equilibrium is such that the more risk averse investors hold portfolios with longer duration relative to less risk averse investors, and therefore experience larger capital losses when the estimated level of the yield curve increases. For more risk averse investors, this change leads to a decrease in their valuation of cashflows; whereas less risk averse investors experience an increase in the valuation of their cash flows, and they take the opposite position on longer-term securities. This behavior leads to a second indirect effect of interest rate movements through changes in the distribution of the absolute risk aversion of the investors. Investors with capital gains have higher wealth in the future and lower absolute risk aversion, since wealthier individuals are more willing to take on risk. On the other hand, investors who suffer a capital loss have higher absolute risk aversion, making them even more conservative in their portfolio positions. Thus, an increase in the estimated level of the yield curve in a low interest rate environment changes the distribution of investors' absolute risk aversion, which increases the investors aversion to risk relative to future demand for Treasury securities.

We contribute to the literature that studies the properties of the empirical SDF. The pricing kernel puzzle as described by Beare and Schmidt (2016) documented the inconsistency between the proposed monotonicity of the SDF by Dybvig (1988) and estimated pricing kernels for various financial instruments.[5] Hens and Reichlin (2012) offered three solutions to the pricing kernel puzzle: (i) incomplete markets; (ii) alternatives to the risk averse expected utility for investors; and (iii) incorrect beliefs. These alternatives were also used to explain the equity premium puzzle of Mehra and Prescott (1985), which demonstrated that an SDF dependent only on consumption growth is insufficient to explain the excess return on stocks and the risk-free interest rate. Intuitively, small variation of the risk-free interest rate implies small variation in the expected SDF; however, high fluctuations in expected stock returns implies that the SDF must be non-linearly related to equity markets. Cochrane (1991) showed that time variation of the equity premium, predictability of returns, and excess stock volatility are all derived from the same properties of the expected SDF, and Cochrane (2017)

[5] From Beare and Schmidt (2016), given a pricing kernel, π, (i) the SDF is a random variable at a future time $t+1$; (ii) the current price of a financial instrument Y_t is given by $\mathbb{E}_t\left[SDF\ Y_{t+1}\right]$; (iii) $M^* = \mathbb{E}_t\left[SDF\ S_{t+1}\right]$, where S_{t+1} is the price of the market portfolio at time $t+1$; and (iv) the pricing kernel is defined such that $M^* = \frac{1}{1+i}\pi_t\left(S_{t+1}\right)$.

surveyed this work and proposed that an extra variable that increases risk aversion during bad times is necessary to capture the time variation of the expected SDF and hence the equity premium.

Consistent with the literature, our results provide additional evidence of a non-monotonic SDF or pricing kernel. Non-monotonicity in risk premia has been identified in studies of combinations of forward rates (Fama and Bliss, 1987; Cochrane and Piazzesi, 2005), Treasury spreads (Campbell and Shiller, 1991), and equity returns (Parker and Julliard, 2005; Lustig and Van Nieuwerburgh, 2005; Yogo, 2006; Sousa, 2010); and explained by slow-moving habit driven by shocks to aggregate consumption (Campbell and Cochrane, 1999; Wachter, 2006), shocks to inflation (Brandt and Wang, 2003), countercyclicality (Ludvigson and Ng, 2009), transitory deviations from the common trend among consumption, aggregate wealth and labor income (Lettau and Ludvigson, 2001), and long-run risk (Bansal and Yaron, 2004), among other explanations. We show that investors' strategies to hedge against maturity risk – which depend on the shape of the yield curve and the mean-reverting properties of interest rates – also yield a non-linear expected SDF.

II. Non-linearities in Bond Market Valuations

A. Prices of Risk in a Term Structure Model

We begin with a canonical term structure model with observable factors as in Joslin et al. (2011). The zero-coupon nominal yield to maturity $r_{\tau,t}$ is driven by an affine process mapping the yield of each maturity to a parsimonious number of underlying factors, Y_t, such that

$$r_{\tau,t}\left(Y_t\right) = A_\tau + B_\tau Y_t, \tag{2}$$

where the time subscript t corresponds to today's date and τ is the maturity date. The parameters A_τ and B_τ for each maturity are set so that there is no arbitrage opportunity for investors in the bond markets.

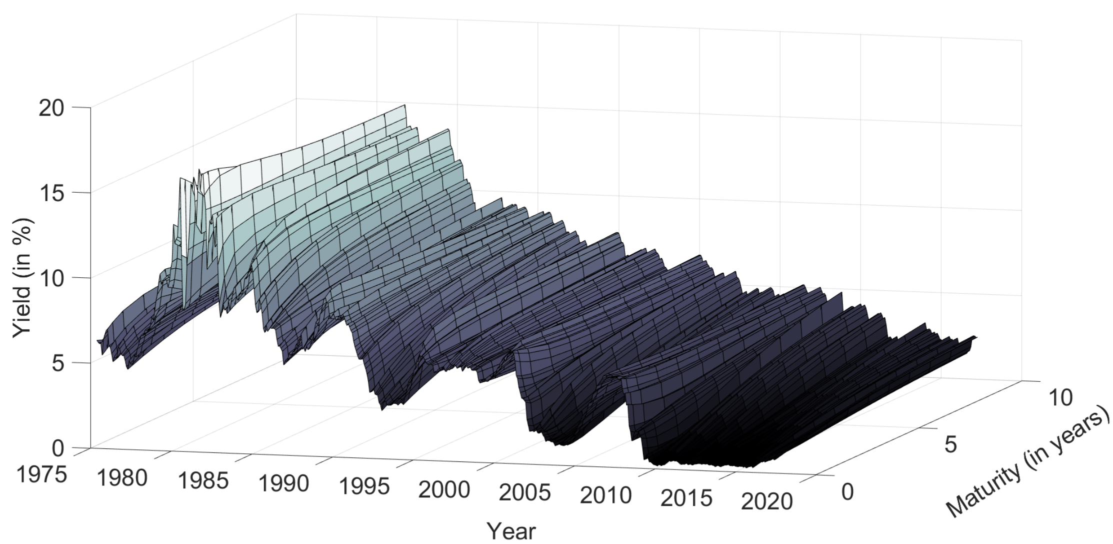
The state vector $Y_t = \left[\begin{array}{ccc} Y_{1t} & Y_{2t} & Y_{3t} \end{array}\right]'$ contains the set of observable "interest-rate risk" factors constructed by principal component analysis as weighted averages of the yields for all the maturities, y_t^{obs}, with weight vector W:

$$Y_t = W y_t^{obs}.$$

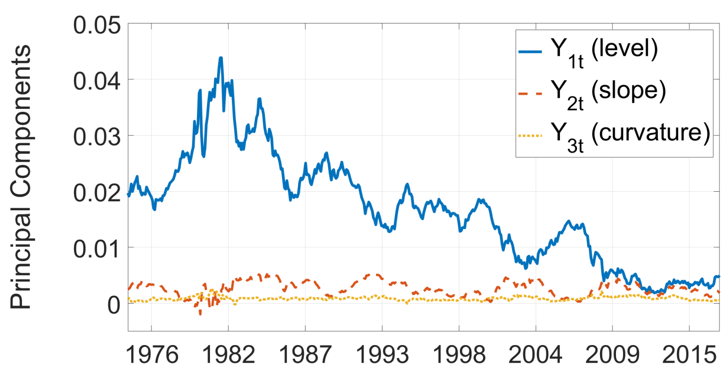
Fig. 1 shows the U.S. yield curve during the 1975–2017 period in panel (a) and the three factors extracted by principal component analysis in panel (b). Consistent with the literature identifying the first three principal components to account for over 99% of the cross-sectional variation in the yield curve, we refer to these factors as "level" (88.5%), "slope" (8.5%) and "curvature" (2.1%) as in Litterman and Scheinkman (1991). The level factor corresponds to the average of all yields, the slope factor is 90% correlated with the 10-year–3-month spread, and the third factor is 70% correlated with the Diebold and Li (2006) definition of curvature, $y^{obs}_{10y,t} + y^{obs}_{3m,t} - 2 * y^{obs}_{2y,t}$. The estimated factors in our term structure model track the empirical level, slope, and curvature of the yield curve and hence inherit their name.

Figure 1: U.S. Yield Curve and Interest-Rate Risk Factors

(a) Yield Curve



(b) Interest-Rate Risk Factors



Notes: Annualized yield curve data in percentage from 3 months to 10 years from January 1975 to March 2017 come from Yung (2017). Factors are extracted by principal components analysis as the eigenvalue-eigenvector decomposition of the variance-covariance matrix of yields.

These factors are assumed to follow an autoregressive process of order one under the actual or physical distribution $\mathbb{P}$,

$$Y_{t+1} = K_0^{\mathbb{P}} + \left(K_1^{\mathbb{P}} + I\right) Y_t + \Sigma \varepsilon_{t+1}^{\mathbb{P}}, \tag{3}$$

where $K_0^{\mathbb{P}}$ and $K_1^{\mathbb{P}}$ are constants, I is the identity matrix, and Σ is a lower triangular matrix obtained from the variance covariance of the innovations to the factors $\left(\varepsilon_{t+1}^{\mathbb{P}}\right)$, which are i.i.d. This stochastic process is mean reverting to $-\left(K_1^{\mathbb{P}}\right)^{-1} K_0^{\mathbb{P}}$, as long as all the eigenvalues of $K_1^{\mathbb{P}}$ are strictly negative. If all the eigenvalues are zero, then Eq. (3) is a random walk (with drift if $K_0^{\mathbb{P}} \neq 0$). In this case, there is no change in the conditional expected future yields and hence no benefit to hedging against interest-rate risk. On the other hand, under a mean reverting stochastic process with $K_1^{\mathbb{P}} < 0$, investors expect the factors (and hence yields) to move back to their long-term unconditional means and therefore adopt hedging positions to take advantage of these expectations.

The risk neutral distribution $\mathbb{Q}$ of the factors satisfies

$$Y_{t+1} = K_0^{\mathbb{Q}} + \left(K_1^{\mathbb{Q}} + I\right) Y_t + \Sigma \varepsilon_{t+1}^{\mathbb{Q}}. \tag{4}$$

Eq. (4) adjusts the mean of the physical distribution for the price of risk per unit of volatility.[6] Thus, bonds of any maturity can be priced as if investors in the bond markets were risk neutral.

The risk-free interest rate, r_t, is also a linear function of the factors:

$$r_t\left(Y_t\right) \equiv r_t = \rho_0 + \rho_1 Y_t, \tag{5}$$

such that the constant ρ_0 and the vector ρ_1 are independent of time.

To determine the price of zero-coupon bonds, the SDF, M_{t+1}, is assumed to have the following exponential quadratic form

$$M_{t+1} = \exp\left\{-r_t - \frac{1}{2}\Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1}^{\mathbb{P}}\right\}, \tag{6}$$

so the price of risk that characterizes investors' attitude toward risk, Λ_t, is affine in the

[6] Following Beare and Schmidt (2016) as in footnote 5, suppose $Y_{t+1} = f_t(S_{t+1})$ is the price at time $t+1$ of a market portfolio for some payoff function f_t on a contingent security. Thus, $Y_t = \mathbb{E}_t\left[SDF_t\ f_t(S_{t+1})\right] = \frac{1}{1+r_t}\int_0^\infty f_t(x) q_t(x) dx$, where $q(x)$ is the risk neutral distribution. $Y_t = \frac{1}{1+r_t}\mathbb{E}_t\left[\pi_t(S_{t+1})\ f_t(S_{t+1})\right] = \frac{1}{1+r_t}\int_0^\infty f_t(x)\pi_t(x) p_t(x) dx$, where $p_t(x)$ is the physical distribution. This means $\pi_t(x) = \frac{q_t(x)}{p_t(x)} = M^*(x)(1+r_t)$.

factors,

$$\Sigma\Lambda_t = K_0^{\mathbb{P}} - K_0^{\mathbb{Q}} + \left(K_1^{\mathbb{P}} - K_1^{\mathbb{Q}}\right) Y_t. \tag{7}$$

The adjustment for risk in the SDF, $-\frac{1}{2}\Lambda_t'\Lambda_t$, follows from the shocks to the interest-rate risk factors being a log-normal probability distribution. This risk adjustment is given by

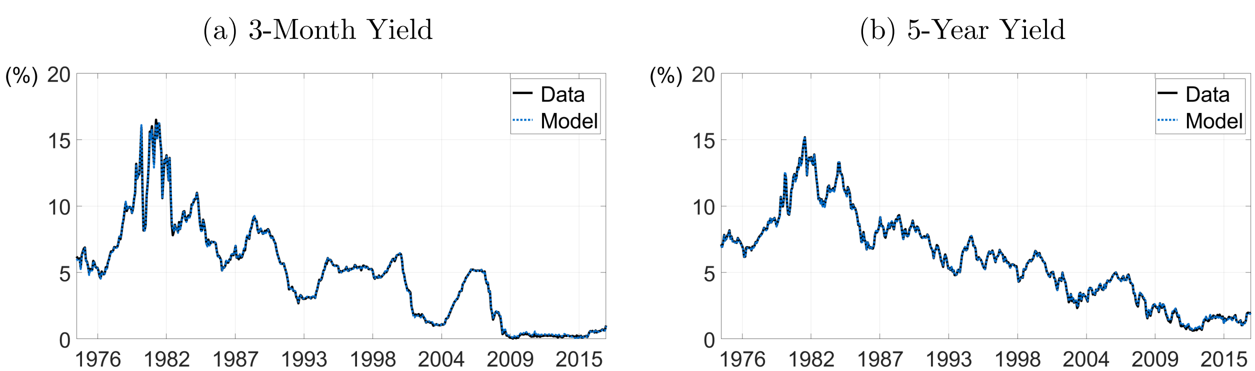
$$-\frac{1}{2}\Lambda_t'\Lambda_t = -\frac{1}{2}\left[K_0^{\mathbb{P}\prime} - K_0^{\mathbb{Q}\prime} + Y_t'\left(K_1^{\mathbb{P}} - K_1^{\mathbb{Q}}\right)\right](\Sigma'\Sigma)^{-1}\left[K_0^{\mathbb{P}} - K_0^{\mathbb{Q}} + \left(K_1^{\mathbb{P}} - K_1^{\mathbb{Q}}\right)Y_t\right]. \tag{8}$$

Importantly, the logarithm of the SDF exhibits a quadratic shape. This property is a consequence of time-varying interest-rate risk, implying a Gaussian bond risk premium – the adjustment for maturity risk in bond markets.

We estimate the model parameters using monthly U.S. Treasury yields data from January 1975 to March 2017. In the estimation, we use 12 maturities: 3 and 6 months, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 years. An advantage of the two-stage procedure of Joslin et al. (2011) is that Eq. (3) can be separately estimated by OLS and all other parameters can be rotated such that the maximum likelihood algorithm immediately converges to the global optimum.

In Fig. 2, we plot the actual yield to maturity relative to the estimated values from the model for the three-month and five-year bonds. As is standard in the term structure literature, the model captures the movement in yields over time really well, both in the short and the long end.

Figure 2: Model Interest-Rate Fit



Notes: The three-month and five-year yields in annualized percentage from the data (in black) are compared to the model-implied yields (in blue) from Eq. (2).

B. The Expected Stochastic Discount Factor

Given the properties of the log normal probability distribution, the expected SDF for investments paying off in one month and conditional on information at time t, i.e., $Y_t = Y$, is given by

$$\mathbb{E}_t[M_{t+1}|Y_t] \equiv \mathcal{M}(Y) = \mathcal{M}_1 \exp\left\{ -\frac{1}{2}\Big(Y - \mu_{\mathcal{M}}\Big)' (\sigma_{\mathcal{M}}\sigma_{\mathcal{M}}')^{-1} \Big(Y - \mu_{\mathcal{M}}\Big)\right\}. \tag{9}$$

$\mu_{\mathcal{M}}$ is the mean of the expected SDF, and $\sigma_{\mathcal{M}}$ is a lower triangular matrix obtained from the variance covariance of the innovations to the factors with the corresponding standard deviations on the diagonal. Taking the conditional expectation converts the shock into a time horizon-dependent term only, which we include in the constant $\mathcal{M}_1$ to simplify the notation.[7]

Fig. 3 shows the model-implied one-month expected cashflow valuation function for an investor, characterized by an SDF as in Eq. (9), conditional on the level $\mathcal{M}(Y_1)$, slope $\mathcal{M}(Y_2)$, or curvature $\mathcal{M}(Y_3)$ factor moving, while all other factors are held constant at their stationary values. In each case, the expected SDF exhibits a Gaussian shape centered around a conditional mean (indicated with the vertical dashed line), along three times its conditional standard deviation on either side. In the case of the level, the one-month ahead expected SDF is at its maximum of 0.9978 whenever the level factor is at $\mu_{\mathcal{M}}^{level} = 0.0061$. From Eq. (1), this value corresponds to a one-month discount rate of 0.22%.

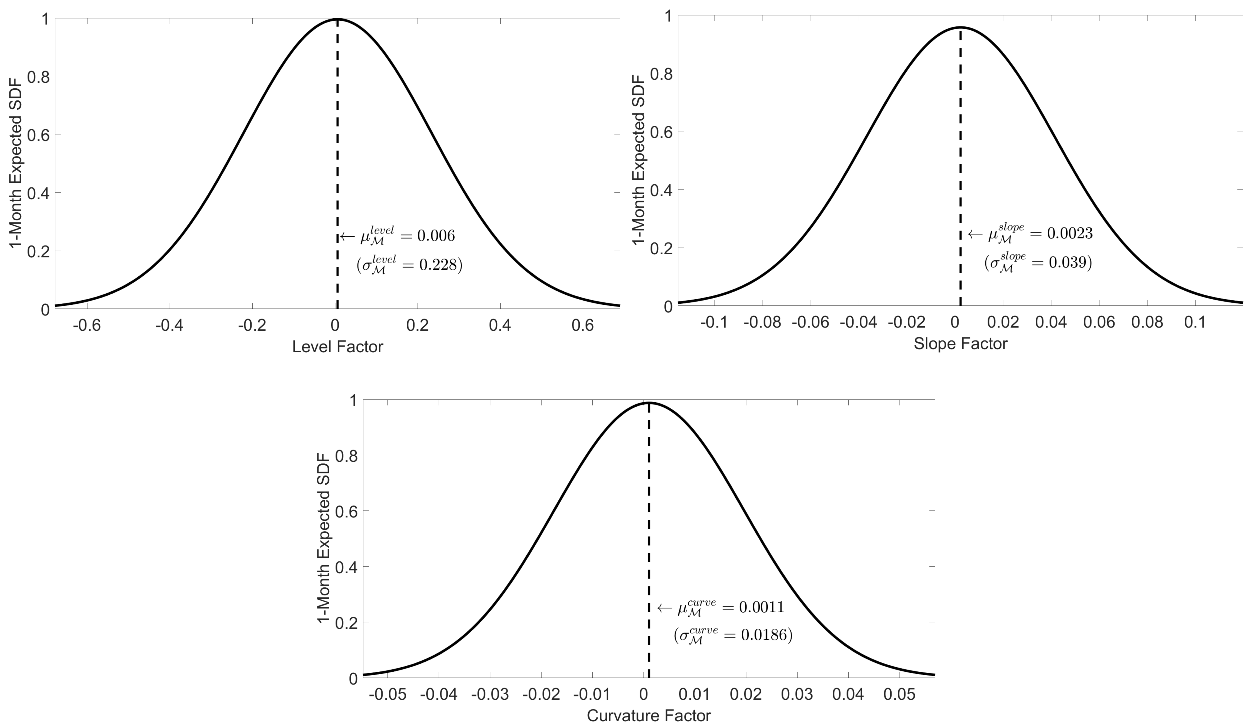
An increase in the level of the yield curve is associated with higher cashflow valuations only when the level change occurs from the left of the conditional mean, i.e. $\Delta Y_{t\to t+1}^{level} < 0.0061$. If the level increases from the right of the conditional mean, $\Delta Y_{t\to t+1}^{level} > 0.0061$, then investors' expected SDF actually decreases. Similarly, the expected SDF is at its maximum of 0.9597 whenever the slope factor is at $\mu_{\mathcal{M}}^{slope} = 0.0023$ and 0.9902 whenever the curvature factor is at $\mu_{\mathcal{M}}^{curve} = 0.0011$.

In practical terms, this framework can be used to quantify how changes in the level, slope, and curvature impact investors' expected SDF. Two takeaways emerge. First, the effect on expected cashflows depends on the current state of the yield curve and hence the economy. In other words, investors discounting function depends on how interest-rate risk factors

[7] $\mathcal{M}_1 = \exp\{-(\rho_0 + \frac{1}{2}(K_0^{\mathbb{P}} - K_0^{\mathbb{Q}})'\Sigma^{-1'}\Sigma^{-1}(K_0^{\mathbb{P}} - K_0^{\mathbb{Q}})) + \frac{1}{2}(\rho_1' + (K_0^{\mathbb{P}} - K_0^{\mathbb{Q}})'\Sigma^{-1'}\Sigma^{-1}(K_1^{\mathbb{P}} - K_1^{\mathbb{Q}}))(\frac{1}{2}(K_1^{\mathbb{P}} - K_1^{\mathbb{Q}})'\Sigma^{-1'}\Sigma^{-1}(K_1^{\mathbb{P}} - K_1^{\mathbb{Q}}))^{-1}(\rho_1' + (K_0^{\mathbb{P}} - K_0^{\mathbb{Q}})'\Sigma^{-1'}\Sigma^{-1}(K_1^{\mathbb{P}} - K_1^{\mathbb{Q}}))\}$. See Cosimano and Yung (2019) for the discrete-time derivation of the expected SDF for any horizon $k = \{1, ..., K\}$ in months and Cosimano and Ma (2018) for the continuous-time counterpart.

change, relative to the conditional mean of the expected SDF. Second, investors' expected SDF depends on movements across the entire maturity spectrum; hence decomposing the impact of interest rates by one factor at a time helps understand the net effect of different yield curve shapes on bond valuation through the non-linear risk adjustment in investors' expected SDF.

Figure 3: Expected SDF versus Level, Slope and Curvature



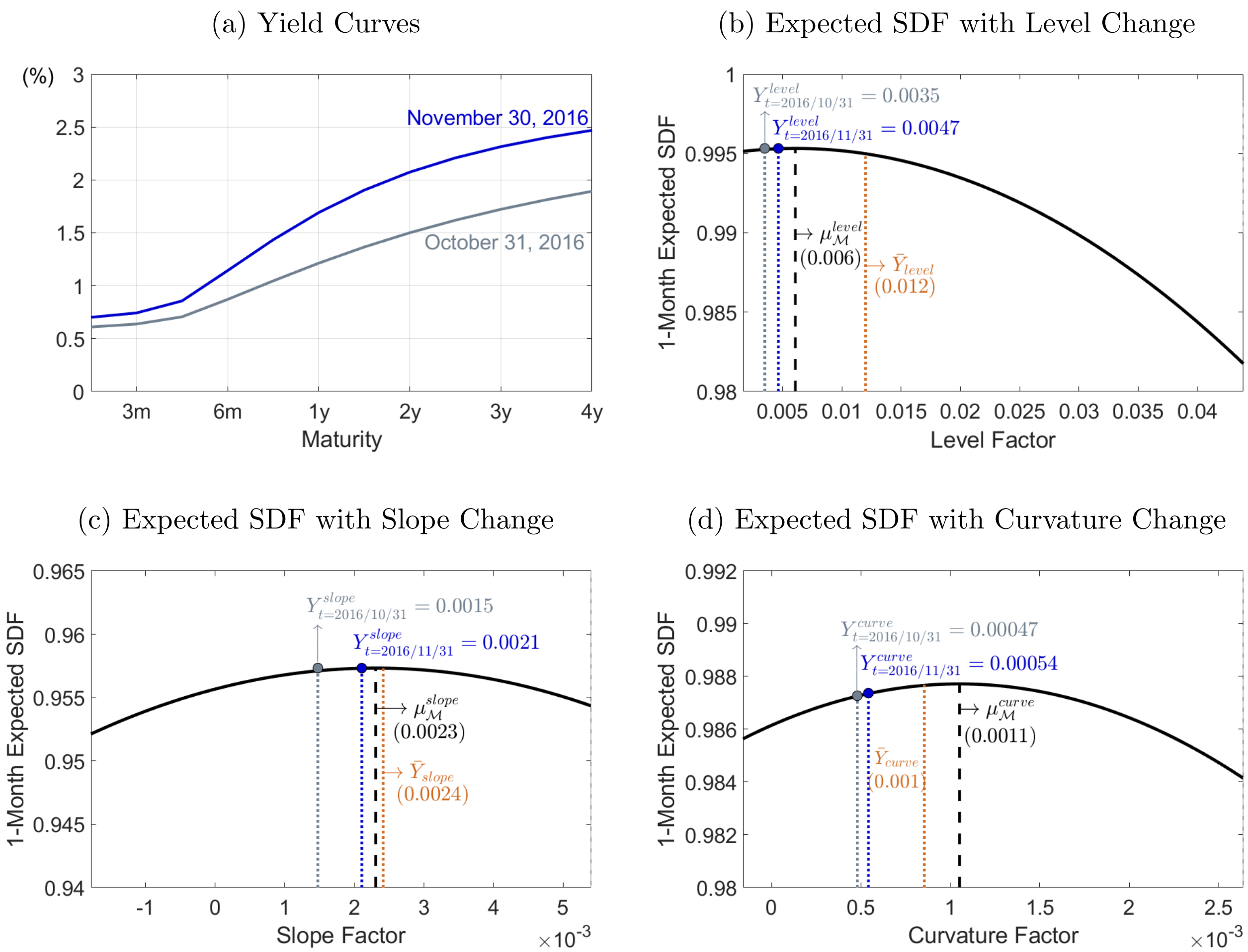
Notes: Each plot shows the one-month expected SDF conditional on each factor moving, while the others are held constant, as given by Eq. (9), where the dashed line represents the value at which the expected SDF is at its conditional mean $\mu_{\mathcal{M}}$ and the conditional standard deviation is indicated in parentheses.

C. Application: The November 2016 Change in Yields

We now consider the movement in interest rates following the 2016 U.S. presidential elections. Fig. 4(a) shows the 3-month to 10-year Treasury yield curve on October 31, 2016 (gray) and November 30, 2016 (blue). The pronounced change in the shape of the yield curve reflects investors' reassessment of the future course of the economy, in this case marked by higher interest rates and a salient steepening of the yield curve. In terms of interest-rate risk factors, the level increased by 35%, the slope steepened by 40%, and the curvature went up by 15%.

In order to study the implications of these movements on investors' discounting of future expected cashflows, panels 4(b), (c), and (d) show the impact of the level, slope, and curvature by zooming into the specific range through which the factors have historically moved since 1975. In all three cases, the factors moved towards their unconditional means ($\bar{Y}_{level} = 0.012, \bar{Y}_{slope} = 0.0024, \bar{Y}_{curve} = 0.0009$).

Figure 4: Investors' Discounting as Interest-Rate Risk Factors Change



Notes: Panel (a) shows the U.S. Treasury yield curve for October 31, 2016 and November 30, 2016 from 3 months to 10 years. Panels (b), (c), and (d) show the expected SDF conditional on either the level, slope, or curvature factor moving, while all other interest-rate risk factors are held constant, as a zoomed-in version of Fig. 3. The corresponding yield curve changes are shown along with the factors' steady state values ($\bar{Y}$) and the expected SDF conditional means ($\mu_{\mathcal{M}}$).

Typically, an increase in the level of the yield curve implies a decline in investors' discounting function, as it is likely to occur from the right of the conditional mean of the expected SDF. However, given that interest rates have been at historic lows, the increase in the level happened from the left of the conditional mean, implying an increase in investors' valuation of cashflows. For this particular event, the steepening of the yield curve and the increase in the curvature also increased the expected SDF, as the factors moved from the left of the conditional mean. Thus, this specific movement in the yield curve following the results of the U.S. presidential elections led to an overall increase in the expected value of future cashflows received by investors in the bond market.

III. Investors' Risk Aversion and Portfolio Positions

We obtain the equilibrium expected return on Treasury securities as a linear function of the interest-rate risk factors, the supply of Treasury securities, and the average leverage of investors in the market for Treasuries. The dependence on the interest-rate risk factors reflects Merton's hedging demand.[8] In equilibrium, the coefficients on these factors are an average of the responses by all the investors in the economy, which we can think of as the marginal investor's response to changes in interest rate risk.

A. Portfolio Rules

Chami et al. (2017) and Cosimano and Ma (2018) show that the discrete-time model for the factors can be transformed into a continuous-time stochastic process allowing the factors to coincide at each date of observation, i.e. $X(s) = Y_t$. This mapping allows us to derive the probability distribution for unobservable interest-rate risk factors estimated by Kalman filter over any given time horizon.

The continuous time stochastic process for the factors is

$$dX(s) = \left[\gamma^{\mathcal{P}} - A^{\mathcal{P}} X(s)\right] ds + \Sigma_X d\epsilon_s,$$

where ϵ_s is a Brownian motion and the mapping from the discrete-time to the continuous-time model is given by

$$A^{\mathcal{P}} = -\ln\left(K_1^{\mathbb{P}} + I\right) \quad \text{and} \quad \gamma^{\mathcal{P}} = A^{\mathcal{P}}\left(I - e^{-A^{\mathcal{P}}}\right)^{-1} K_0^{\mathbb{P}}.$$

In addition, the continuous-time equivalent price of risk is now given by

$$\Lambda\left(X(s)\right) = \left(\Sigma_X\right)^{-1}\left(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}\right) - \left(\Sigma_X\right)^{-1}\left(A^{\mathcal{P}} - A^{\mathcal{Q}}\right) X(s), \tag{10}$$

[8]This demand is based on a comparison of the current factors with the expected utility desired by each investor; which is scaled by the variance-covariance matrix for the investor's expected utility and multiplied by the beta from regressing the return on Treasury securities on the risk factors.

where $\gamma^{\mathcal{Q}}$ and $A^{\mathcal{Q}}$ are risk-adjusted parameters such that the variance-covariance matrix of the residuals, $\Sigma_X\Sigma_X^{'}$, is invariant across both distributions, following the diffusion invariance principle (Girsanov, 1958). The excess holding period return on a zero coupon security with maturity τ, where $P_{\tau,s}$ is the price, is given by

$$\begin{aligned} \frac{dP_{\tau,s}}{P_{\tau,s}} - r(s) =& b_\tau \left[(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}) - (A^{\mathcal{P}} - A^{\mathcal{Q}})X(s)\right] ds + b_\tau \Sigma_X d\epsilon_s \\ =& \left[\mu_\tau(s) - r(s)\right] ds + b_\tau \Sigma_X d\epsilon_s. \end{aligned} \tag{11}$$

b_τ is a constant proportional to the bond pricing coefficient, B_τ, $r(s)$ is the continuous-time risk-free rate and $\mu_\tau(s)$ is the expected return on a maturity-τ bond.

Let investors who take Eq. (11) as given be grouped into J investment buckets. Without loss of generality, each bucket of investors $j = 1, ..., J$ chooses how to optimize a portfolio with U.S. Treasury securities of different maturities $\tau = \{1, 2, ..., n\}$, subject to a liquidity constraint that limits the percentage invested in these securities to ξ^j. Investors have a constant relative risk aversion coefficient, γ^j, and seek to maximize expected lifetime utility over a fixed terminal wealth at time horizon h^j. W^j is the sum of total investment by all investors in bucket j.[9]

The total demand for Treasury securities, $D(s)$, is

$$D(s) = \sum_{j=1}^{J} \omega^j(s) W^j, \tag{12}$$

where $\omega^j(s)$ is the optimal percentage of wealth invested by the j^{th} investor:

$$\omega^j(s) = \frac{1}{\gamma^j}\Bigg[\underbrace{\omega_1\left(\mu(s) - \mu_\tau(s)\iota\right)}_{\text{Sharpe Ratio}} + \underbrace{\omega_1\omega_2\xi^j}_{\text{Leverage}} + \underbrace{\omega_1\omega_3\gamma^j\left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1}\left[X(s) - \mu_j(h^j)\right]}_{\text{Hedging Demand Against Interest-Rate Risk}}\Bigg], \tag{13}$$

for $\omega_1^j(s) = \xi^j - \iota'\omega^j(s)$ representing the wealth invested on the 1^{st} bond, where $\iota' = (1 \quad ... \quad 1)$.[10]

The terms $\omega_1, \omega_2, \omega_3$ are constants defined as follows:

$$\omega_1 \equiv \left[b\Sigma_X\Sigma_X'b' + \iota\iota' b_\tau\Sigma_X\Sigma_X'b_\tau' - 2b\Sigma_X\Sigma_X'b_\tau'\iota'\right]^{-1},$$

[9] By limiting the utility function to a constant relative risk aversion coefficient we know that the non-linearity of the expected SDF arises from the hedging behavior of investors.

[10] For derivations see Chami et al. (2017) and Cosimano and Ma (2018).

$$\omega_2 \equiv 2\left(b\Sigma_X\Sigma_X' b_\tau' - \iota b_\tau\Sigma_X\Sigma_X' b_\tau'\right),$$

$$\omega_3 \equiv \left(b - \iota b_\tau\right)\Sigma_X\Sigma_X'.$$

$b' = [b_{2\tau} \quad ... \quad b_{n\tau}]$ is a vector of bond price elasticities with respect to the interest-rate risk factors, such that $b' - \iota b_\tau = \left[\begin{array}{ccc} b_{2\tau} - b_\tau & ... & b_{n\tau} - b_\tau \end{array}\right]$ captures the elasticity of the 2^{nd} to n^{th} bond relative to the elasticity of the 1^{st} bond, b_τ.

The first term in portfolio rule Eq. (13) is the traditional Sharpe ratio adjusted for investors' coefficient of relative risk aversion γ^j. The expected return on the 2^{nd} to n^{th} bond, $\mu(s)' = [\mu_{2\tau}(s) \quad ... \quad \mu_{n\tau}(s)]$ relative to the 1^{st} bond, $\mu_\tau(s)$, is given by

$$\mu(s) - \mu_\tau(s)\iota \equiv \left(b - \iota b_\tau\right)\left[(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}) - (A^{\mathcal{P}} - A^{\mathcal{Q}})X(s)\right]. \tag{14}$$

Notice that the excess return on bonds is measured relative to the 1^{st} Treasury security in Eq. (14), rather than the risk-free rate as in Eq. (11). Consequently, the price of risk $(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}) - (A^{\mathcal{P}} - A^{\mathcal{Q}})X(s)$ from Eq. (10) is multiplied by the relative bond elasticity parameters, $b - \iota b_\tau$. The ω_1 term in the Sharpe ratio adjusts the variance-covariance of the last $n-1$ bonds, $b\Sigma_X\Sigma_X' b'$, by the variance of the first bond, $b_\tau\Sigma_X\Sigma_X' b_\tau'$ and the covariance of the last $n-1$ bonds with the first, $b\Sigma_X\Sigma_X' b_\tau'$.

The second term in Eq. (13) is an adjustment to ensure that the portfolio weights add up to ξ^j, so that leverage is limited. The term $\omega_1\omega_2$ represents the coefficient from regressing the excess return for the last $n-1$ bonds against the return for the first bond.

The last term in portfolio rule Eq. (13) is the hedging demand for Treasury securities from Merton (1969, 1971), which arises under mean reverting factors, given that $K_1^{\mathbb{P}} < 0$. The term $\omega_1\omega_3$ is the ratio of the covariance between excess returns and the estimated interest-rate risk factors relative to the variance-covariance of excess returns. As a result, it can be interpreted as the beta coefficient from regressing expected excess returns on bonds on the lifetime utility of the investor. $\mu_j(h^j)$ represents the mean of the investors' expected utility. $\sigma_j(h^j)\sigma_j(h^j)'$ is the variance-covariance matrix for the investor's expected utility, where $\sigma_j(h^j)$ is a lower triangular matrix such that the diagonal elements are the standard deviations of the investor's expected utility for each factor. With these definitions, $\gamma^j\left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1}\left[X(s) - \mu_j(h^j)\right]$ captures the sensitivity of the expected lifetime utility for investor j with respect to the factors. This latter term can be interpreted as the risk adjusted duration of investors' portfolio for horizon h^j. If the factor $X(s)$ is equal to the expected mean of the investor's utility, $\mu_j(h^j)$, then there is no benefit to hedging interest-rate risk, whereas the hedging demand is positive for $X(s) > \mu_j(h^j)$ and negative for $X(s) < \mu_j(h^j)$.

To sum up, the investment strategy's expected return is a function of the estimated interest-rate risk factors, such that interest-rate risk shocks impact the decisions of the investor as she re-balances her portfolio of Treasury securities.

To evaluate the impact of yield curve moves on the investor's capital gains, we consider the case for two particular investor buckets $j = 1, 2$ with different levels of risk aversion. For simplicity, let all investors choose the optimal allocation of their wealth between a 3-month and a 5-year bond to maximize expected lifetime utility for an investment horizon of $h^j = 1$ year. We assume, for each investor, a subjective discount rate of 5%, leverage ratio $\xi^j = 1$, and estimate all parameters for the January 1999–December 2007 period as in Chami et al. (2017). While the first bucket of investors has a higher aversion to risk, i.e., $\gamma^{j=1} = 10$, the second investment group is less risk averse, i.e., $\gamma^{j=2} = 5$. For this shorter time period, the unconditional means are $(\bar{X}_{level} = -0.0177, \bar{X}_{slope} = -0.0159, \bar{X}_{curve} = 0.0045)$.

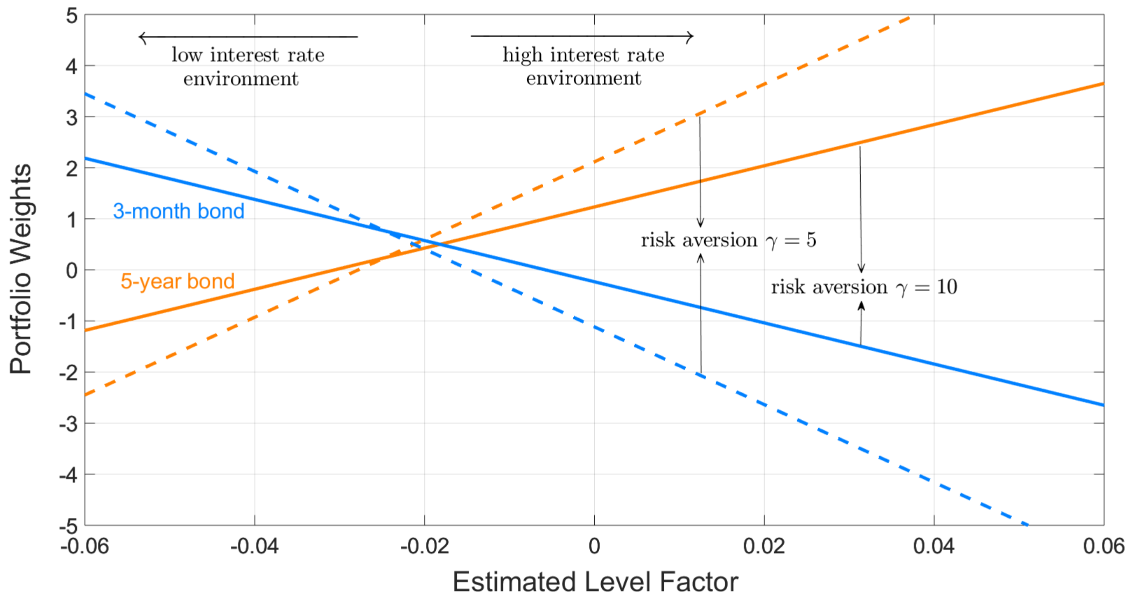
The more risk averse investor's expected lifetime utility is a Gaussian function with conditional mean $\mu_{j=1}(h^{j=1} = 1) = -0.0593$ and standard deviation $\sigma_{j=1}(h^{j=1} = 1) = 0.1512$ for changes in the estimated level of the yield curve, while holding all other factors constant. The conditional mean is $\mu_{j=2}(h^{j=2} = 1) = -0.0567$ and the standard deviation is $\sigma_{j=2}(h^{j=2} = 1) = 0.1197$ for the less risk averse group. A similar pattern arises for the gross rate of growth for the investor's wealth.[11]

Fig. 5 shows investors' portfolio allocation strategies given their risk aversion, conditional on the estimated level of the yield curve changing, while all other factors are held constant at their steady state values. When the estimated level of the yield curve is at its steady state $X^{level} = -0.0177$, the more risk averse investor allocates 46% of her wealth to the 3-month bond (solid blue) and 54% to the 5-year bond (solid orange).[12] The demand due to hedging risk is 46%, so that only 8% of the investor's wealth is allocated to the 5-year bond based on the Sharpe ratio and leverage constraint. The less risk averse investor places 86% of her wealth into the 5-year bond (dashed orange) with 14% in the 3-month bond (dashed blue). Of the 86% of the investor's wealth in the 5-year bond, 69% is due to the hedging demand.

[11]The gross rate of growth for investors' wealth has conditional mean −0.0640 and standard deviation 0.1065 for the more risk averse group; and conditional mean −0.0366 and standard deviation 0.0798 for the less risk averse group.

[12]Changing the discount rate has a small impact on the portfolio decision of the investor. For example, a discount rate of 10% instead of 5%, leads to a 48-52% allocation, compared to 46-54%.

Figure 5: Portfolio Positions Conditional on the Estimated Level of the Yield Curve



Notes: The solid lines indicate the portfolio weights of a risk-averse investor ($\gamma = 10$) that chooses wealth allocation between a 3-month bond (blue) and a 5-year bond (orange), for different values of the estimated level, holding all other factors constant. The dashed lines represent the portfolio weights of a less risk averse investor ($\gamma = 5$). In both cases, leverage is limited ($\xi^j = 1$) and the investment horizon is one year ($h^j = 1$).

If the estimated level is higher than its steady state value, $X^{level} > -0.0177$, anticipated mean reversion implies that the investor expects the level of the yield curve to fall, and hence longer-duration bonds would lead to a larger capital gain. As a result, the portfolio is longer on 5-year bonds and shorter on 3-month bonds during high interest-rate periods, as in the example, with the less risk averse investor allocating 86% to the 5-year bond and 14% to the short term bond. If the random change in the level is, however, positive, then the investor would suffer a larger capital loss. The portfolio position is reversed in a low interest rate environment, $X^{level} < -0.0177$, since mean reversion implies that the investor expects the level of the yield curve to move back to its steady state value and the portfolio is therefore shorter on longer term bonds. If the random change in the level is negative and hence declines even further, then the investor suffers a capital loss. In the case of $\mu_{j=1}(h^{j=1} = 1) = -0.0593 < -0.0177$, the more risk averse investor places -113% of her wealth in 5-year bonds and 213% in the 3-month bond. This allocation is accomplished with no hedging demand, since the investor is at the maximum expected utility conditional on the estimated level of the yield curve. Thus, these allocations are based on the Sharpe ratio and the leverage constraint from Eq. (13).

We also consider the portfolio positions of a second bucket of investors with a lower aversion to risk (dashed lines). These two types of investors could be representative, for example, of the investment positions of small and large banks, respectively, since the smaller banks would have less wealth and a higher risk aversion (i.e. $\gamma^{j=1} = 10.$). The investor with a lower aversion to risk increases the magnitude of the bet on the level of the yield curve reverting to its steady state value. This implies that the less risk averse investor will choose a portfolio with higher duration in a high interest rate environment, relative to the investor that is more risk averse.

The portfolio position is reversed in the lower interest rate environment, $X^{level} < -0.0177$, since mean reversion implies that the investor expects the level of the yield curve to move back to its steady state value. This expectation implies that the investor anticipates the bonds to have a capital loss (as the price of the bond decreases) and this capital loss is higher for the longer term bond. If the random change in the level is negative and hence declines even further, then the bond would receive a capital gain (as the price of the bond goes up). As a result, the investor goes short the five-year bond and long the three-month bond. Consequently, the investor expects to have a capital gain when the level of the yield curve moves back to its mean. See the blue lines in Fig. 5 under the low interest rate environment. This behavior is exaggerated by the less risk averse investors, the blue dashed line, since they are more willing to take on risk for a higher rate of return.

During an even lower interest rate environment –from June 2008 to December 2016– the level of the yield curve for which the expected utility of wealth is maximized is smaller, since $\mu_j(h^j)$ would be smaller. In this case, the portfolio positions in Fig. 5 would all shift to the left, but would not change their relative positions, because these portfolio positions are linear in $\mu_j(h^j)$. The investors behave this way, since they have lowered their estimate of the level for the yield curve in the lower interest rate environment.

B. Hedging Strategies in a Low interest Rate Environment

We next illustrate how the Treasury market for each maturity clears given the optimal portfolio behavior discussed in the previous section. Let $S(s)$ be a vector for the supply of Treasury securities in the bond markets at each maturity provided by the decisions of the U.S. Treasury and the Federal Reserve Board. Consequently, the equilibrium condition in the market for Treasury securities is given by

$$D(s) = S(s).$$

For simplicity, suppose the inverse of ω_1 exists, so that the number of independent securities is the same as the number of interest-rate risk factors (otherwise, we would have to use its pseudo inverse). Then from Eqs. (12) and (13), we find that the expected excess return on bonds is dependent on the behavior of all the investors in the Treasury market in equilibrium:

$$\mu(s) - \mu_\tau(s)\iota = \frac{1}{\sum_{j=1}^{J} \frac{W^j}{\gamma^j}} \Bigg\{ (\omega_1)^{-1} S(s) + \omega_3 \sum_{j=1}^{J} \gamma^j \left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1} \mu_j(h^j) \frac{W^j}{\gamma^j} - \omega_2 \sum_{j=1}^{J} \xi^j \frac{W^j}{\gamma^j} - \omega_3 \sum_{j=1}^{J} \gamma^j \left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1} \frac{W^j}{\gamma^j} X \Bigg\}.$$

$\sum_{j=1}^{J} \frac{W^j}{\gamma^j}$ is the sum of the inverse of the absolute risk aversion coefficients across all investors, such that $\frac{1}{\sum_{j=1}^{J} \frac{W^j}{\gamma^j}} \sum_{j=1}^{J} \left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1} \mu_j(h^j) \frac{W^j}{\gamma^j}$ is the weighted average of the desired Sharpe ratio and $\frac{1}{\sum_{j=1}^{J} \frac{W^j}{\gamma^j}} \sum_{j=1}^{J} \left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1} \frac{W^j}{\gamma^j}$ is the weighted average of the market portfolio's variance.

Define $\theta \equiv \frac{1}{\sum_{j=1}^{J} \frac{W^j}{\gamma^j}}$ to be one divided by the sum of the inverse of the absolute risk aversion of all investors, while $\theta^j \equiv \frac{W^j}{\gamma^j}\theta$ is the individual investor's contribution to this value. The price of risk coefficients from Eq. (14) in equilibrium are as follows:

$$(b - \iota b_\tau)\left(\gamma^{\mathcal{P}} - \gamma^{\mathcal{Q}}\right) = \theta\,(\omega_1)^{-1} S(s) - \omega_2 \sum_{j=1}^{J} \theta^j \xi^j + \omega_3 \sum_{j=1}^{J} \theta^j \gamma^j \left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1} \mu_j(h^j), \tag{15}$$

$$(b - \iota b_\tau)\left(A^{\mathcal{P}} - A^{\mathcal{Q}}\right) = \omega_3 \sum_{j=1}^{J} \theta^j \gamma^j \left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1}. \tag{16}$$

Eq. (15) is the constant in risk pricing Eq. (14). This term is positively related to the quantity of Treasury securities made available to the markets, weighted by total absolute risk aversion. This constant is also negatively related to the leverage ratio of all investors, weighted by absolute risk aversion relative to its value for the marginal investor. Given that $\sum_{j=1}^{J} \theta^j = 1$, we can think of these weights as the probability distribution of the inverse of the absolute risk aversion for each group of investors in the Treasury markets. In this case $\sum_{j=1}^{J} \theta^j \xi^j$ is the mean value of the leverage constraint for the probability distribution of investors. Finally, the last term in Eq. (15) is related to the expected lifetime utility for each bucket of investors in the market, adjusted for the investor's coefficient of risk aversion. This effect is multiplied by the covariance between the expected excess return on bonds and the interest-rate risk factors, ω_3. Thus, this last term captures how much a change in interest-rate risk factors influences the risk adjusted return on all investors' expected lifetime utility; and can hence be interpreted as the amount of risk in the market when factors are at their steady state values.

Eq. (16) represents the slope of the price of risk in the Treasury market, which captures how changes in interest-rate risk factors impact expected excess returns through the marginal investor's variance-covariance of her expected utility. This term is amplified by ω_3 and the absolute risk aversion for investors in each bucket. If we combine the last term in Eq. (15) with Eq. (16) times the current yield curve factors, we then obtain the impact of the hedging demand on the equilibrium excess return on bonds

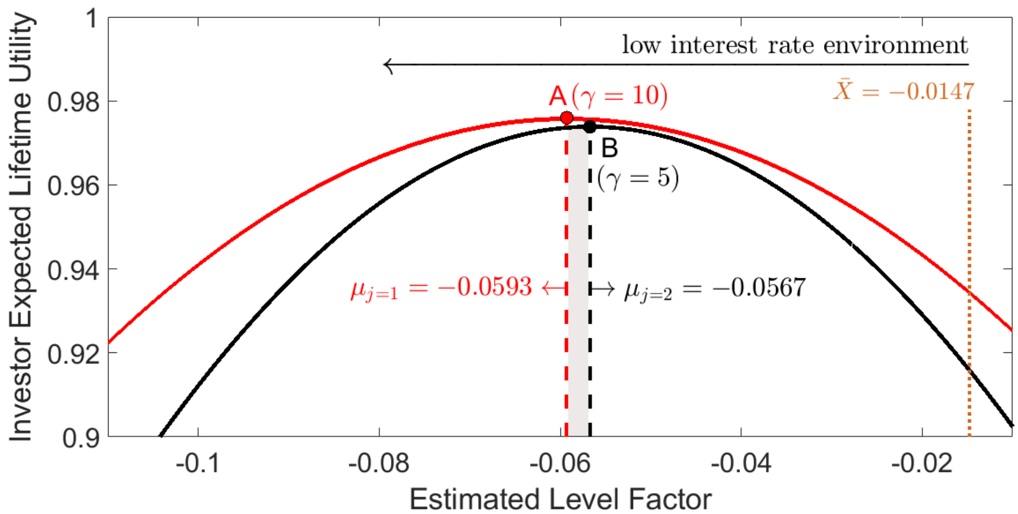
$$Hedging = \omega_3 \sum_{j=1}^{J} \gamma^j \theta^j \left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1} \left(X - \mu_j(h^j)\right). \tag{17}$$

Thus, the expected excess return is negatively related to the expected current factors $\left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1} X$ relative to the conditional mean $\left(\sigma_j(h^j)\sigma_j(h^j)'\right)^{-1} \mu_j(h^j)$ desired by the investors in bucket j.

Notice that while the portfolio position of investors is dependent on their constant relative risk aversion γ^j, the price of risk for the Treasury market is also a function of their wealth W^j and hence, the probability distribution for absolute risk aversion for investors (defined as γ^j/W^j). In addition, if the coefficient of relative risk aversion is fixed, then the less wealthy investor takes on a portfolio of Treasury securities that has less longer-term Treasury securities to reduce their risk.

For the market, we can find the equilibrium return as the fixed point of Eqs. (15) and (16) with the two types of investors ($\gamma = 10$ and $\gamma = 5$). In Fig. (6), we show each investor's expected lifetime utility conditional on the estimated level of the yield curve to emphasize the difference in the position of the two investors at a fixed point. Each investor has a maximum expected lifetime utility under their optimal portfolio strategy, which occurs when the estimated level of the yield curve is -0.0593 for the more risk averse investor (red) and -0.0567 for the less risk averse investor (black). These maxima are 0.9757 at point A and 0.9738 at point B for each corresponding investor. From Fig. 5, we know that the duration of the portfolio for the more risk averse investor is always higher relative to the less risk averse investor. In addition, the Gaussian functions of the expected lifetime utility have a standard deviation of 0.1512 for $\gamma = 10$ and 0.1197 for $\gamma = 5$, so that the dispersion is larger for the more risk averse group.

Figure 6: Finding the Equilibrium Return with Two Investors



Notes: Expected lifetime utility is depicted in red for an investor with high risk aversion ($\gamma = 10$) and in black for an investor with low risk aversion ($\gamma = 5$). Point A and B represent their mean, respectively, conditional on the estimated level of the yield curve.

We can now discuss properties of the equilibrium in the bond markets. Initially we consider a market in which there is zero net supply among the investors. For the excess returns in the Treasury market to be in equilibrium, it must be the case that the distribution across the two investors implies that one investor is shorter and the other is longer on longer-term Treasury securities. This means the market equilibrium cannot be to the left of point A or to the right of point B in Fig. (6). In particular, we see in Fig. (6) that if the equilibrium is between points A and B, the more risk averse investor, A, is long in the long-term Treasury securities, while the less risk averse investor is short the long-term Treasury securities. This result occurs since the more risk investor expects the level factor to fall which yields a bigger capital gain in long term government securities. If $S(s)$ is sufficiently positive, both investors will be long the long-term Treasury securities, but the more risk averse investor would be even longer on these securities.

Now consider an unanticipated increase in the estimated level of the yield curve during a low interest rate environment. In such an environment, investors with a higher level of absolute risk aversion (investors $\gamma = 10$) would be long on long-term Treasury securities relative to the less risk averse investors (investors $\gamma = 5$). As a result of the interest rate increase, $\gamma = 10$ investors would suffer a capital loss, while $\gamma = 5$ investors would experience a smaller capital loss $\frac{|\Delta W^{\gamma=5}|}{W^{\gamma=5}} < \frac{|\Delta W^{\gamma=10}|}{W^{\gamma=10}}$ for $W^{\gamma=5} > W^{\gamma=10}$. Hence, the absolute risk aversion increases more for the more risk averse investors and widens the spread between more and less absolute risk investors. In particular, $\frac{W}{\frac{\gamma}{W}} \frac{\partial \frac{\gamma}{W}}{\partial W} = -1$, so that a percentage change in wealth leads to the same percentage decrease in the level of absolute risk aversion. Thus, an increase in the estimated level of the yield curve in this case, implies a decrease in hedging behavior by the more risk averse investors relative to the less risk averse investors.

IV. Conclusion

Using a no-arbitrage asset pricing model for the U.S. Treasury market, we describe how the critical nature of the non-linearities in the SDF matters for the transmission of interest rate risk across investors' portfolio positions. Whenever interest-rate risk factors move away from the conditional mean of the expected SDF,this leads to a decline of all expected future cashflows. During the November 2016 U.S. presidential elections, we find that cashflow valuations increased as the factors reverted back towards their mean.

We illustrate the impact of the variation of discount rates on individual portfolio choices, by solving for the optimal portfolio of investors that base their decisions on the expected bond market SDF derived from the no-arbitrage model. When the interest-rate risk factors are above their long-term mean, investors go long on long-term bonds, since they anticipate a future fall in the factors leading to a bigger capital gain. These portfolio positions are reversed when the factors are below their means as investors anticipate increases in the factors. Finally, we show that the estimated no-arbitrage model of the term structure is an equilibrium in which all investors follow the optimal portfolio rules with different degrees of absolute risk aversion. In this equilibrium, the more risk averse investors have a longer position in long-term bonds relative to the less risk averse investors. Thus, the more risk averse investors suffer a larger capital loss from the average increase in interest rates, which leads to a larger increase in the absolute risk aversion.

Our findings suggest that interest rate risk exposure has direct and indirect effects on the bond markets. First, we identify changes in the entire yield curve to influence the market prices of risk non-linearly depending on the current value of interest-rate risk factors relative to their historical averages. Second, we find that changes in the yield curve also influence portfolio positions by changing the distribution of investors' absolute risk aversion in equilibrium, amplifying the risk exposure of risk-averse or less wealthy investors.

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