



WP/17/57

IMF Working Paper

Public Debt Sustainability Under Uncertainty: An
Invariant Set Approach

by Rossen Rozenov

I N T E R N A T I O N A L M O N E T A R Y F U N D

IMF Working Paper

Fiscal Affairs Department

Public Debt Sustainability Under Uncertainty: An Invariant Set Approach

Prepared by Rossen Rozenov¹

Authorized for distribution by Bernardin Akitoby

March 2017

IMF Working Papers describe research in progress by the author(s) and are published to elicit comments and to encourage debate. The views expressed in IMF Working Papers are those of the author(s) and do not necessarily represent the views of the IMF, its Executive Board, or IMF management.

Abstract

The paper offers an approach to assessing the sustainability of public debt taking into account the effect of fiscal policy on output, as well as uncertainty in the model parameters and system dynamics. Uncertainty is specified in general terms, and the analysis is based on the notion of invariant sets. Examples are provided to illustrate how the method can be applied in practice.

JEL Classification Numbers: C61, C63, D81, E62, H63

Keywords: public debt sustainability, uncertainty, fiscal reaction function, set invariance, linear matrix inequalities

Author's E-Mail Address: rrozenov@imf.org

¹I would like to thank Bernardin Akitoby, Suman Basu, Vitor Gaspar, Fuad Hasanov, Mikhail Krastanov, Francesco Luna, Santiago Acosta Ormaechea, Alex Pienkowsky, Thomas Sargent, Abdelhak Senhadji, Chris Towe, Irina Yakadina, and the participants in the IMF FAD Seminar for valuable comments and suggestions. All remaining errors are my own.

Contents

1	Introduction	4
2	Analytical framework	8
3	Application	13
3.1	Scenario 1: Growth rates higher than interest rates	14
3.2	Scenario 2: Interest rates higher than growth rates	15
3.3	Scenario 3: Intermediate case	17
4	Conclusion	18
Appendix A Invariant sets, Lyapunov functions and LMIs		19

List of Figures

1	Example of set invariance.	10
2	Scenario 1: Invariant set for $r_l < r_h < g_l < g_h$.	14
3	Scenario 2a: Invariant set for $g_l < g_h < r_l < r_h$.	15
4	Scenario 2b: Invariant set for $g_l < g_h < r_l < r_h$, constrained control.	16
5	Scenario 2c: Invariant set for $g_l < g_h < r_l < r_h$, constrained output.	17
6	Scenario 3a: Invariant set for $g_l < r_l < g_h < r_h$.	17
7	Scenario 3b: Invariant set for $g_l < r_l < g_h < r_h$, constrained control.	18

1 Introduction

The question of what level of public debt a country can afford is central to fiscal policy. Vast literature is devoted to debt sustainability and an array of tools has been developed to guide assessment. A comprehensive review of previous work on the subject is beyond the scope of this paper and is not attempted here. We only mention a few studies to provide a context for the subsequent discussion and illustrate some of the difficulties that arise in the analysis.

Most of the earlier contributions have focused on ensuring solvency, interpreted as compliance with the government's intertemporal budget constraint. For example, Blanchard et al. (1990) derived a simple sustainability criterion according to which the current level of debt should not exceed the present value of all future primary surpluses. This condition follows directly from the dynamic equation describing the evolution of debt and is relatively mild in the sense that it only requires the discounted value of future debt to tend to zero. Thus, debt trajectories that increase indefinitely as time goes to infinity are not ruled out as long as the rate of increase is lower than the interest-growth differential. Variants of this solvency condition have been widely used in the literature, e.g., EU Commission (2006) and Escolano (2010), among others.

The solvency approach to debt sustainability has been often criticized for not being particularly stringent as it allows for large primary deficits to be run initially, provided that future surpluses are high enough to offset them. However, there is no guarantee that the government would follow such a path; moreover, large corrections are costly both politically and economically. There is usually a limit up to which the government can increase taxes before collections start to decline due to the distortions introduced (Laffer curve effect). Similarly, expenditures cannot be reduced below a certain threshold consistent with the minimum level necessary for the smooth functioning of government. Therefore, considerations regarding the feasibility of adjustment are important and they have been reflected, for example, in IMF's definition of debt sustainability (IMF, 2002).

For practical purposes, debt sustainability analysis (DSA) is often carried out by comparing a country's debt level—typically measured as a ratio to GDP—to some indicative benchmark value (the so-called threshold approach). Benchmarks can be derived in various ways. The solvency condition itself defines such a limit if the future flow of revenues and non-interest expenditures, hence the primary balance, is constrained. Historical episodes of debt crises may also provide some guidance. Empirical work has been carried out, mostly based on signal-to-noise ratios, to identify the level of debt above which the likelihood of a crisis rises significantly

(Baldacci et al., 2011).¹ The IMF DSA framework is also underpinned by debt limits derived based on the signal approach.

Notwithstanding the difficulties associated with the identification of thresholds, the approach has gained a lot of traction and many countries have adopted fiscal rules containing explicit debt benchmarks. According to IMF (2016a), about 70 countries currently have fiscal frameworks that include caps on gross public debt. In a number of cases, these reflect rules established at the supranational level such as the Maastricht convergence criteria for the EU member states, for example. Since debt cannot be directly controlled by the government, often debt ceilings are complemented with rules about deficits or expenditures.

An alternative to the threshold approach is the sustainability definition proposed by Arrow et al. (2004), which, applied to indebtedness, can be interpreted as a requirement for the net worth of the entity of interest to be non-decreasing. Cuerpo et al. (2015) and IMF (2016b) apply this sustainability framework to households and non-financial corporations. Wyplosz (2011) discusses it in relation to government debt, pointing out that the concept can be made operational by requiring the debt to GDP ratio to be stationary, and since stationarity is difficult to ascertain, the ratio should follow a declining trend, allowing only for temporary increases.

Another challenge to the traditional DSA is uncertainty. The solvency condition depends on the future dynamics of primary balances, interest rates and growth rate, none of which are known for sure. Celasun et al. (2007), building upon Garcia and Rigobon (2005), address this issue by developing a methodology which accounts for the risks surrounding the debt dynamics. Their approach is probabilistic and uses fan charts to illustrate the uncertainty around the central projection of debt. The proposed algorithm comprises an estimated fiscal reaction function, a vector autoregression (VAR) which models the non-fiscal determinants of debt dynamics (e.g., GDP growth rate, interest and exchange rates), and a debt equation. In stochastic simulations of the debt path, shocks are drawn from a distribution which has the same covariance matrix as the covariance matrix of the VAR residuals and forecasts of the non-fiscal variables are consistent with these shocks. In a similar vein, Cherif and Hasanov (2012) examine how macroeconomic shocks affect the debt dynamics in a VAR framework with debt feedback. A different approach is taken by Barnhill and Kopits (2003) who use Value at Risk techniques to estimate the risk-adjusted net worth of the government.

IMF's revised DSA methodology (IMF, 2011 and 2013) addresses some of the shortcomings of earlier methods. In particular, the definition of sustainable public

¹The method is based on minimizing the sum of type I and type II errors as a way to capture the trade-off in setting the threshold between missing crisis episodes and sending wrong signals.

debt is broadened to include a number of conditions, notably that “the primary balance needed to at least stabilize debt under both the baseline and realistic shock scenarios is economically and politically feasible, such that the level of debt is consistent with an acceptably low rollover risk and with preserving potential growth at a satisfactory level.”² The IMF DSA provides a set of tools to assess the realism of macroeconomic projections, vulnerabilities arising from the structure of debt, sensitivity to macro-fiscal shocks, and impact from realization of contingent liabilities. Stochastic simulations and fan charts showing ranges for the possible evolution of the debt ratio are also part of the toolkit.

This paper contributes to the literature by offering an alternative framework for assessing debt sustainability which features the following characteristics.

First, the proposed approach explicitly takes into account the effect of fiscal policy on the output gap and makes it endogenous in the analysis. Since fiscal policy can affect demand, large fiscal contractions, especially over extended periods of time, may not be feasible or desirable, including due to possible hysteresis effects, whereby long and deep recessions destroy the economy’s productive potential. The IMF DSA also recommends that the potential impact of fiscal adjustment on growth and interest rates is factored in, but this impact is not fully endogenized.

Second, the methodology provides an estimate of a debt threshold which is entirely forward-looking and depends on assumptions about fiscal multipliers, interest and growth rates, as well as uncertainties around the future paths of these variables. Thus, the threshold is system-specific and does not rely on estimation and/or averaging based on past data as in other works. In addition, the method delivers a fiscal reaction function in the form of a linear feedback rule which prescribes how the primary balance should respond to debt and the output gap. Unlike most of the literature where fiscal reaction functions are estimated empirically based on past data, our rule is normative and is designed to stabilize the economy given the parameters of the model and system disturbances.

Third, uncertainty is inherent in the analysis– it affects the system not only through additive shocks to the dynamics, but also through key model parameters.

Uncertainty is described in rather general terms, reflecting the view that a probabilistic representation may not be available (especially since the focus is on the future) or an outcome “on average” may not be desirable. Instead, the decision-maker would seek policies that perform well under a range of possibilities, for example, when fiscal multipliers fall in the interval $[0.5, 1.5]$, or when future interest rates could be anywhere between 3 and 7 percent. A similar type of bounds can be assumed for additive shocks, either based on historical experience or other relevant information. In this respect, the analysis is closely

²IMF (2013), p.4

related to the literature on robustness (Hansen and Sargent, 2008). A robust approach can be justified based on model misspecification as decision makers seldom have complete knowledge of the underlying dynamics.

Fourth, the framework assumes an infinite time horizon. This is another important difference from the prevailing approaches to debt sustainability under uncertainty. Typically, DSAs fix a specific time frame and examine the debt path during this period. For example, IMF uses a 6-year horizon, consistent with the availability of World Economic Outlook projections. As argued by Wyplosz (2011), finite horizons are a major constraint since in theory the debt sustainability concept requires infinite time. In fact, the stochastic approach to debt sustainability works well only over relatively short intervals. For instance, in a fan chart presentation, uncertainty around the central debt path typically becomes very large after 5-10 periods, rendering the tool of little practical value for longer horizons.

The analysis in this paper is based on Lyapunov's stability theory for dynamical systems and the related notion of set invariance. In a nutshell, to capture the effect of fiscal policy on growth, the standard debt equation is complemented with an equation describing the evolution of the output gap. Formally, in each period the two variables representing the state of the system – output gap and the debt ratio – are viewed as a point in the two-dimensional space. We define debt as sustainable if it does not increase indefinitely, i.e., if it remains bounded at all times. This definition can be operationalized as follows: find a set of points with the property that whenever the initial state falls into this set, it stays there forever. In other words, we are interested in all possible combinations of output gaps and debt levels for which it is guaranteed that debt will not embark on an explosive path when subjected to shocks of a given size, also taking into account uncertainty in the system parameters. Such sets, if they exist, will be called invariant. We note that while invariant set membership is a sufficient condition for debt to remain bounded, it is not necessary. Even if the initial point is outside the invariant set, or if the state is pushed out of it later by a large one-off shock, it would still be possible to stabilize the system. This is so because, in the case of linear dynamics the invariant sets are also attracting, so all trajectories that start outside the set tend to approach it as time tends to infinity (Khlebnikov et al., 2011). It cannot be guaranteed, however, that in the process of convergence some underlying constraints, e.g., on the size of the fiscal adjustment in any given period, will be satisfied.

Finally, it is important to stress that assessing the sustainability of public debt is a complex problem and the simple model presented here is not meant to substitute a full-fledged DSA. In particular, it cannot capture many of the qualitative elements involved in such assessments. Still, the invariant set method can be useful as a supplementary tool to more elaborate frameworks.

The rest of the paper is organized as follows. Section 2 presents the model and describes the methodology for debt sustainability assessment; section 3 illustrates how the methodology is applied to specific examples, and section 4 concludes. Appendix A contains all the technical material, including proofs of some of the results used in the main text.

2 Analytical framework

To study debt sustainability when the effect of fiscal policy on growth is explicitly taken into account, we consider the following system of equations:

$$y_{t+1} = \rho y_t + \alpha_t u_t + \delta_1 w_{1,t} \quad (1)$$

$$d_{t+1} = \frac{(1 + r_{t+1})}{(1 + g_{t+1})} d_t + u_t + \delta_2 w_{2,t} \quad (2)$$

$$y_0 = \bar{y}_0, \quad d_0 = \bar{d}_0,$$

where y_t denotes the output gap at time t , defined as $y_t = Y_t/Y_t^p - 1$, with Y_t and Y_t^p denoting actual GDP and potential GDP, respectively. The output gap is assumed to follow a first order autoregressive process. Without a fiscal action, and assuming that $0 < \rho < 1$, the output gap would eventually close, with degree of persistence determined by the value of ρ . Although in practice the output gap is not directly observable, the level of current economic activity relative to its natural level is an important consideration in economic decision making, and various techniques have been developed to measure potential output and deviations from it.

A key assumption of the model is that the pace at which the output gap closes can be influenced by a change in the primary deficit u_t . The effect of the fiscal action on the output gap depends on the size of the fiscal multiplier α_t . An increase in the primary deficit relative to the baseline (here assumed to be zero), however, increases the debt level d_t , defined as the ratio of public debt to potential GDP.³ Besides the primary deficit, the evolution of the debt ratio depends on the growth rate of potential output g_t and the interest rate r_t . Both output gap and public debt are subject to exogenous shocks captured by the additive terms $w_{i,t}$, $i = 1, 2$ (scaled by the parameters δ_i , $i = 1, 2$). No probabilistic assumptions are made about the nature of the shocks; they are only assumed to belong to some compact set. It is

³The focus on debt relative to potential GDP is motivated by the long-term nature of the analysis; it is the capacity of the economy to generate income that is more relevant for its ability to repay the debt, rather than the output at any particular point in time. In addition, if actual GDP is used in the denominator, the system would become nonlinear and likely intractable.

convenient to work with ellipsoidal sets of the type $\mathcal{W}_t = \{w_t : w_t' W_t^{-1} w_t \leq 1\}$ given their analytical advantages and potential links to statistical inference. In particular, the shape matrix of the disturbance ellipsoid can be calibrated to an estimated covariance matrix of shocks.⁴ This is a generalization of the standard assumption in the engineering literature that $\|w_t' w_t\| \leq 1$.

Further to the additive uncertainty, there is a strong case for introducing uncertainty in the model parameters. Indeed, since debt sustainability assessments are forward-looking and the evolution of the debt ratio depends critically on future interest and growth rates that are not known precisely, it appears natural to allow these parameters to vary within a certain range, e.g., $r_t \in [r_l, r_h]$, $g_t \in [g_l, g_h]$, where subscripts “l” and “h” stand for “low” and “high”. Similarly, the fiscal multiplier is uncertain and most likely not constant, with higher values typically reported during economic downturns and when interest rates are low. While ideally these variables would be modeled as functions of the state of the system (e.g., higher debt triggers higher interest rates), such an approach is technically very challenging due to the nonlinearities involved.

In matrix notation, system (1)- (2) can be written as:

$$x_{t+1} = A_t x_t + B_t u_t + D w_t, \quad (3)$$

x_0 given

$$w_t' W_t^{-1} w_t \leq 1$$

For the matrices A_t and B_t we assume that $A_t \in co\{A_1, A_2\}$ and $B_t \in co\{B_1, B_2\}$ (here co denotes convex hull), with

$$A_1 = \begin{pmatrix} \rho & 0 \\ 0 & (1+r_h)/(1+g_l) \end{pmatrix}, A_2 = \begin{pmatrix} \rho & 0 \\ 0 & (1+r_l)/(1+g_h) \end{pmatrix}$$

$$B_1 = \begin{pmatrix} \alpha_l \\ 1 \end{pmatrix}, B_2 = \begin{pmatrix} \alpha_h \\ 1 \end{pmatrix}.$$

The autoregressive parameter ρ is kept constant for simplicity. With these assumptions, it is enough to consider the convex hull of only two A matrices, rather than all four possible combinations of growth and interest rates; the intermediate cases $(1+r_l)/(1+g_l)$ and $(1+r_h)/(1+g_h)$ can be obtained as convex

⁴Ellipsoidal sets can be thought of as confidence regions of normally distributed random variables. Even if a non-normal distribution is assumed for the error terms in a statistically estimated model, as long as the underlying set from which the shocks are drawn is compact, the ellipsoidal set can be viewed as an approximation of the original set.

combinations of A_1 and A_2 . The proposed framework remains applicable when an interval for ρ is specified as well but this would entail adding more constraints to the ensuing optimization problem (see below). Note that formulation (3) is more general than (1)- (2) in that it allows for the possibility of shocks to the output gap equation to affect also debt and vice versa, if matrix D is non-diagonal.

The main question of interest is under what combinations of growth, interest rates, fiscal multipliers, and shocks debt will remain sustainable. This, according to our definition, is equivalent to finding the largest set of points E (if it exists) that is invariant with respect to the dynamics described by (3). Invariance is understood in the following sense: for any initial $x_0 \in E$, the state x_t is guaranteed to stay in E for all $t > 0$. In the context of debt sustainability, the existence of such a set implies that debt will not grow indefinitely even in the presence of uncertainties as specified above. As in the case of additive disturbances, we restrict the class of invariant sets to ellipsoids. This choice is motivated by two main reasons: first, any non-empty compact convex set can be reasonably well approximated by an ellipsoid, and second, ellipsoidal sets arise naturally from the application of Lyapunov's stability theory to linear systems. The link can perhaps be best understood through a simple example in continuous time.

Consider the following system of ordinary differential equations defined on a domain $\Omega \subset \mathbb{R}^2$:

$$\frac{dx(t)}{dt} = f(x(t)).$$

and suppose that the system has an equilibrium at zero, i.e., $f(0)=0$. We want to know under what conditions the origin is a stable equilibrium in the sense that every trajectory that starts close to zero at t_0 remains so for all $t > t_0$. A famous result by A. Lyapunov states that if there exists a continuously differentiable function $V : \Omega \rightarrow \mathbb{R}$, such that $V(0) = 0$, $V(x) > 0, \forall x \in \Omega \setminus \{0\}$ and $\dot{V}(x) \leq 0, \forall x \in \Omega$, then $x = 0$ is stable. The last condition is equivalent to requiring that $\nabla V(x) \cdot f(x) \leq 0$, i.e., the gradient of $V(x)$ and the velocity vector $f(x)$ do not form an acute angle, implying that on the boundary of the sublevel set $V(x) \leq a$, all velocity vectors point to the interior of the set when the angle is obtuse (Figure 1). Thus, within the invariant set a point can wander around, but once it gets close to the border it will move back.

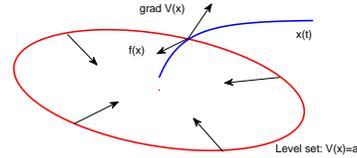


Figure 1: Example of set invariance.

In physical systems, the Lyapunov function typically has the meaning of total energy and the stability condition requires that total energy decreases along every trajectory. In an economics context, one could interpret $V(x)$ as the value function of an infinite-horizon linear-quadratic control problem whereby the decision-maker minimizes a loss function that penalizes deviations of the state from equilibrium. With this interpretation, the definition of sustainability adopted here is closely related to the criterion of Arrow et al. (2004). This criterion is derived from an optimization problem where a representative agent maximizes intertemporal utility. Sustainability is achieved when the value function associated with this problem is non-decreasing with time. Since in our setup we can think of the Lyapunov function as the value function of a minimization problem, it is natural to require that the loss function be non-increasing.

In a discrete-time framework, the condition $\dot{V}(x) \leq 0$ is replaced with $V(x_{t+1}) - V(x_t) \leq 0$. Further, it is well known that for linear systems the Lyapunov function has the form:

$$V(x) = x_t' Q x_t$$

where Q is some positive definite matrix. Note that the sublevel sets of $V(x)$ are ellipsoids, so there is a close link between Lyapunov functions and the invariant ellipsoids introduced earlier. Specifically, since Q is positive definite, the set $E = \{x_t : x_t' P^{-1} x_t \leq 1\}$ where $P^{-1} = Q$ defines an ellipsoid. In order for this ellipsoid to be invariant, for any $x_t \in E$ we need $x_{t+1} \in E$ as well.

If the matrix Q (or equivalently P) were known, we would only need to verify whether the current state falls inside the set E ; if it does, then we can ascertain that the debt trajectories will remain bounded even in case of realization of the worst combination of parameters and shocks, as long as they are within the specified limits. All that it takes to stabilize the system is to adjust the primary deficit in accordance with a linear rule which reacts to the output gap and the debt ratio, i.e., to follow the rule $u_t = K x_t$, where K in this case is a (1×2) matrix. The problem, however, is that the matrix Q is not known in advance and there can be many such matrices or none. In the latter case, the problem is not feasible, implying that debt may not be sustainable, and in the former case, we need a criterion to choose among the possible options. It appears reasonable to aim for the largest invariant set since typically policy-makers and markets are interested in the highest debt ratio that can be stabilized and eventually reduced. Thus, a natural criterion is to seek the invariant ellipsoid with the largest volume. Since the volume of an ellipsoid is proportional to the determinant of its shape matrix, this leads to a constrained maximization problem of the following kind:

$$\log \det P \rightarrow \max \tag{4}$$

subject to

$$\begin{pmatrix} P - (1 - \tau)^{-1} D W D' & (A_i P + B_j Y) \\ (A_i P + B_j Y)' & \tau P \end{pmatrix} \geq 0, \quad (5)$$

$$\begin{pmatrix} P & P C' \\ C P & z_{max}^2 I \end{pmatrix} \geq 0, \quad (6)$$

$$P > 0. \quad (7)$$

Problem (4)-(7) is not standard in the sense that the variable of interest is not a scalar or vector but a matrix, and this matrix must possess certain properties, notably to be symmetric and positive definite, and to satisfy additional constraints arising from the nature of the problem. Such are constraints (5)-(6) which represent linear matrix inequalities or LMIs (see Boyd et al., 1994).

The first LMI ensures that the ellipsoidal set defined by the shape matrix P^{-1} is invariant (see Appendix A). The second LMI is necessary because of economic considerations; it arises from the definition of the output gap. Clearly, neither actual, nor potential GDP can be negative, so as a minimum we have to impose the requirement $y \geq -1$. More generally, it may be desirable to constrain the size of the output gap to some value y_{max} , i.e., to have $\|y_t\| \leq y_{max}$ for all t . To incorporate this condition (which has to be symmetric in order to be able to use LMI techniques), it is convenient to introduce an auxiliary equation of the form

$$z_t = C x_t,$$

where z_t is a vector that depends linearly on the state x_t , and to establish a more general result based on the constraint $\|z_t\| \leq z_{max}$ (see Proposition 2 in Appendix A). The constraint related to the output gap definition is a special case of inequality (6) with $z_{max} = 1$ and C being a (2×2) matrix with an entry of 1 in the (1,1) position and zeroes elsewhere.

In the above problem, Y is an auxiliary matrix which is related to matrix K appearing in the feedback law and τ is a free parameter taking values in $(0, 1)$.⁵

In order for problem (4)-(7) to have a solution, the LMI constraints must be feasible. Even for simple problems, feasibility is often difficult to establish explicitly, so numerical methods are used. While a complete analytical characterization of feasibility in our case is challenging as well, it can be shown

that if $\frac{\delta_1^2 w_{11}}{1 - \tau} > y_{max}^2$, i.e., when the shock on the output gap equation is large

⁵In fact the parameter τ is related to the spectral radius of the matrix $A + BK$, where $K = Y P^{-1}$ so that $\lambda_{max}^2 < \tau < 1$, where λ_{max} is the largest eigenvalue of $A + BK$ (see Nazin et al., 2007).

relative to the maximum admissible output gap, LMI (6) is not feasible and the problem does not admit a solution (Appendix A).

In summary, the invariant set approach to debt sustainability reduces the problem of stabilizing debt and output in the presence of uncertainty to that of finding a matrix P with certain properties, as described above. The existence of such a matrix ensures that the debt ratio will eventually converge, and moreover, if at any point in time the state of the system (represented by the combination of debt level and output gap) falls within the ellipsoidal set determined by P , all future states will remain within this set, as long as the shocks do not exceed the specified bounds. If in a given period there is a large one-off shock that takes the state outside the invariant set, the system would still be stabilizable; however, the required adjustment will likely be larger, and it will take longer to move to the equilibrium. On the other hand, if no such matrix exists, then the system cannot be stabilized and the debt ratio would likely grow indefinitely. An important feature of the approach is that whenever debt is sustainable it provides a fiscal rule – a linear function of the state variables which guarantees stabilization even if the worst admissible combination of shocks occurs.

Below we illustrate how this analytical framework can be used to determine the stability regions for specific values of the parameters involved. The numerical solutions have been obtained using CVX, a package for specifying and solving convex programs (Grant and Boyd, 2008 and 2013) and the figures have been generated with the aid of the Ellipsoidal Toolbox by A. Kurzhanski and P. Varaiya (2007).

3 Application

As an application of invariant set methods to debt sustainability, we consider three distinct scenarios depending on the interest-growth differential. Scenario (1) assumes that future growth rates are consistently higher than interest rates, that is $g_l > r_l$; Scenario (2) makes the opposite assumption, namely that $r_l > g_h$ and finally, Scenario (3) considers an intermediate case where $g_l < r_l < g_h < r_h$. In all simulations, we assume that the fiscal multiplier α takes values in the interval $[0.3, 1]$ and the autoregressive parameter ρ is equal to 0.7, values consistent with the typical findings in the literature.⁶ The shape matrix of the disturbance ellipsoid W^{-1} is diagonal with entries 0.01^2 and 0.02^2 which correspond to persistent

⁶The fiscal multiplier is assumed positive, implying that an increase in the primary deficit relative to the baseline has a positive effect on the output gap. We do not consider the case of “expansory fiscal consolidations” which is of lesser interest as there is no tension between reducing debt and supporting economic activity.

shocks to the output gap of up to 1 percent of potential GDP per year and to debt of up to 2 percent of potential GDP per year. These values are purely illustrative and do not draw on any particular empirical work. Further, the matrix D is the identity matrix and for simplicity we fix the free parameter $\tau = 0.95$. In principle, we cannot optimize with respect to τ but we can do a grid search to find the value that maximizes the objective. The results presented below, however, are not very sensitive to the choice of this parameter. In all simulations, the initial condition is set at $(-0.02; 0.6)$, that is, the economy starts with a 2 percent negative output gap and 60 percent debt to potential GDP ratio. For the graphical presentation of the state trajectory, we take $A = 0.5A_1 + 0.5A_2$, $B = 0.5B_1 + 0.5B_2$ and the disturbances are generated randomly from uniform distributions on $[-0.01, 0.01]$ for w_1 and $[-0.02, 0.02]$ for w_2 , respectively.

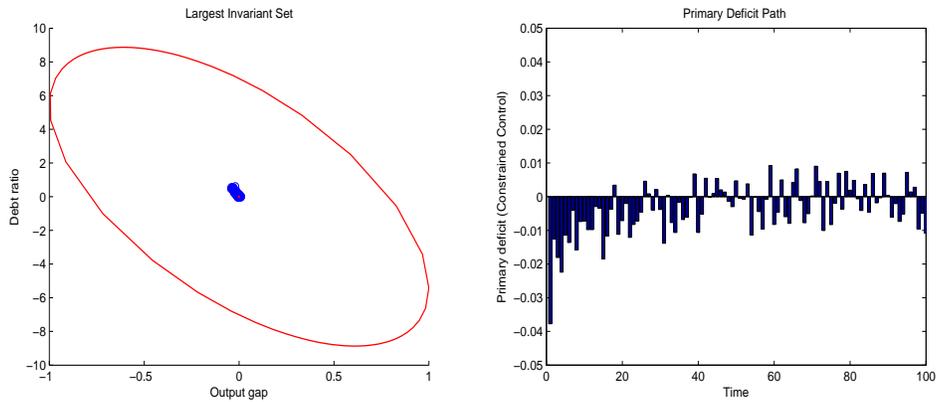


Figure 2: Scenario 1: Invariant set for $r_l < r_h < g_l < g_h$.

3.1 Scenario 1: Growth rates higher than interest rates

In this scenario, we assume that future interest rates are consistently lower than growth rates: $r_l < r_h < g_l < g_h$. Specifically, $r \in [0.03, 0.045]$ and $g \in [0.05, 0.06]$. The maximal invariant set and the simulated primary deficit path are shown in Figure 2. As one would expect, under these favorable interest and growth assumptions, debt sustainability is not an issue; initial debt can be 8 times larger than potential GDP, and the state would still remain in the invariant set. Although Scenario 1 may correspond to the economic realities that many countries currently face in an environment of historically low interest rates, it is unlikely that this situation will be sustained in the long run. Therefore, from a policy point of view, the case where the growth rate is always higher than the interest rate is perhaps the least interesting one.

3.2 Scenario 2: Interest rates higher than growth rates

Scenario 2 assumes that $g_l < g_h < r_l < r_h$. In this scenario, $r \in [0.05, 0.07]$, $g \in [0.03, 0.04]$ and it corresponds to a dynamically efficient economy. As one would expect, maintaining debt sustainability is more challenging in this case. Figure 3 confirms this; the invariance bounds for debt are more than twice as narrow as in the previous scenario. Moreover, the implied adjustment of the primary balance in the first period of 12 percent of potential GDP is unrealistically large. It is underpinned by the high (in absolute terms) coefficient on debt in the fiscal reaction function, which for this example has the form $F = -0.64y - 0.22d$. For comparison, the coefficient on debt in fiscal reaction functions typically found in the empirical literature ranges between 0.02 and 0.10 (Berti et al., 2016).

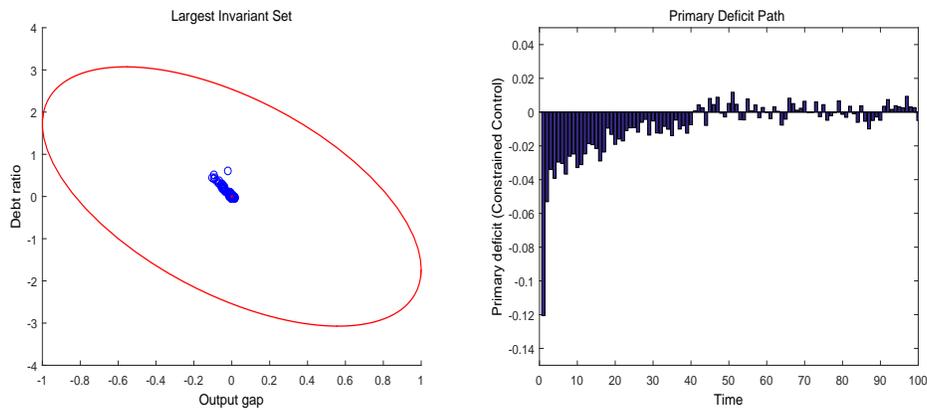


Figure 3: Scenario 2a: Invariant set for $g_l < g_h < r_l < r_h$.

Given that adjustment of the magnitude suggested by the model is not feasible in most cases, introducing an additional constraint to limit the size of the primary balance change in each period seems warranted: $\|u_t\| \leq u_{max}$.⁷ This gives rise to another LMI:

$$\begin{pmatrix} P & Y' \\ Y & u_{max}^2 I \end{pmatrix} \geq 0. \quad (8)$$

The proof that this LMI is equivalent to the constraint on the control variable follows the same steps as Proposition 2 in Annex A and can be found in Nazin et al. (2007), for example. In line with the literature (see IMF, 2016a), we choose the maximum adjustment to be 4 percent of potential GDP per year in the subsequent simulations.

⁷In this case the control variable is one-dimensional, so the constraint is simply an interval centered at zero. Symmetry is important for the applicability of the method.

As Figure 4 suggests, when the constraint on the primary deficit is taken into account, the invariant set shrinks further, so that the initial condition is no longer inside it. Still, eventually the state moves toward equilibrium (reflecting the attracting property of the invariant ellipsoid), but outside of the set the constraint on control is violated (right panel). Once the point gets into the invariant set all constraints are satisfied. The calculated feedback rule prescribes zero response to the output gap and deficit reduction of 0.08 percent of potential GDP for each percentage point of the debt ratio. If we relax slightly the constraint on the primary balance from 4 percent to 5 percent, the initial point falls into the stability region and all constraints are satisfied. A similar result is obtained if we reduce the shock to the debt equation from 2 percent to below 1 percent of potential GDP. If, on the other hand, the size of the admissible permanent shocks is doubled (i.e., matrix W^{-1} has diagonal elements 0.02^2 and 0.04^2), then an invariant set no longer exists and debt cannot be stabilized.

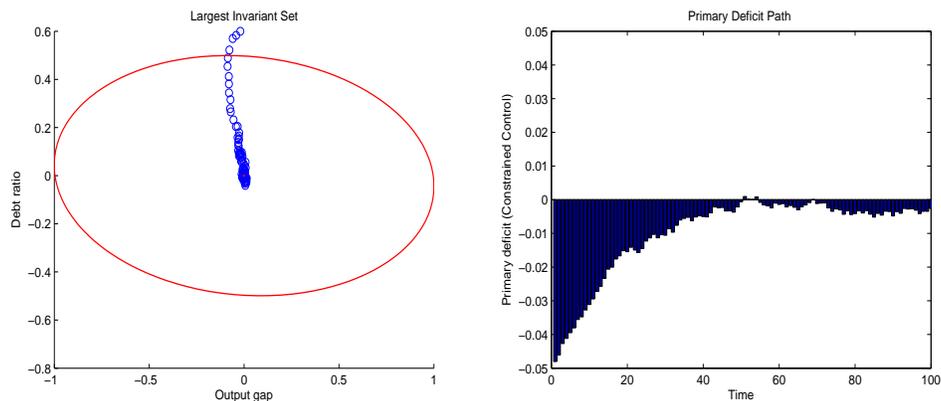


Figure 4: Scenario 2b: Invariant set for $g_l < g_h < r_l < r_h$, constrained control.

As an alternative to imposing an explicit limit on the improvement of the primary balance, one can opt for a tighter constraint on the output gap, such that at no point in time actual GDP is allowed to deviate from potential by more than a specified amount. Figure 5 presents the case when the maximum output gap is set at 15 percent. Under this assumption, the invariant set becomes smaller and the required adjustment to move the state to the stability region is again very large in the initial period (the fiscal reaction function is $F = -0.61x_1 - 0.30x_2$), so constraining the output gap does not appear to be a viable alternative to limiting the size of adjustment.

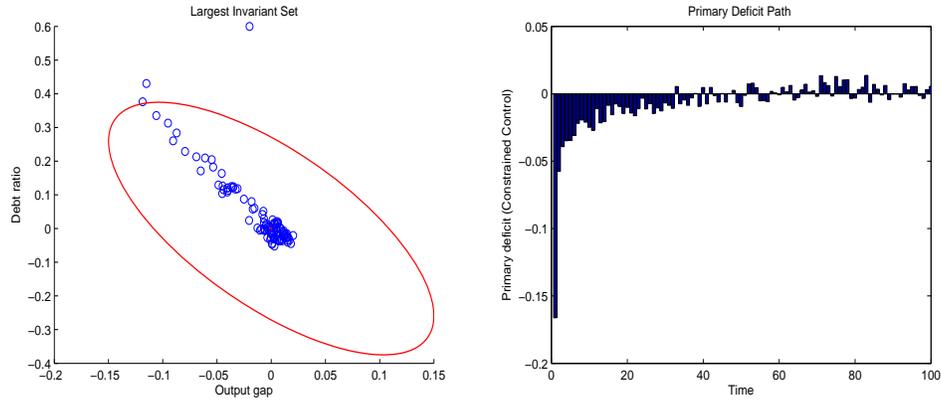


Figure 5: Scenario 2c: Invariant set for $g_l < g_h < r_l < r_h$, constrained output.

3.3 Scenario 3: Intermediate case

Finally, Scenario 3 considers the intermediate, and perhaps most realistic case where in some periods interest rates are higher than potential GDP growth rates and in others they are lower ($g_l < r_l < g_h < r_h$). Nominal growth under this scenario is assumed to be between 4 and 6 percent and interest rates between 5 and 7 percent. The results are shown on Figure 6. Again, the adjustment in the initial period is quite substantial, which calls for imposing constraints on the change in the primary balance. The maximal invariant set under the 4 percent constraint is presented on Figure 7.

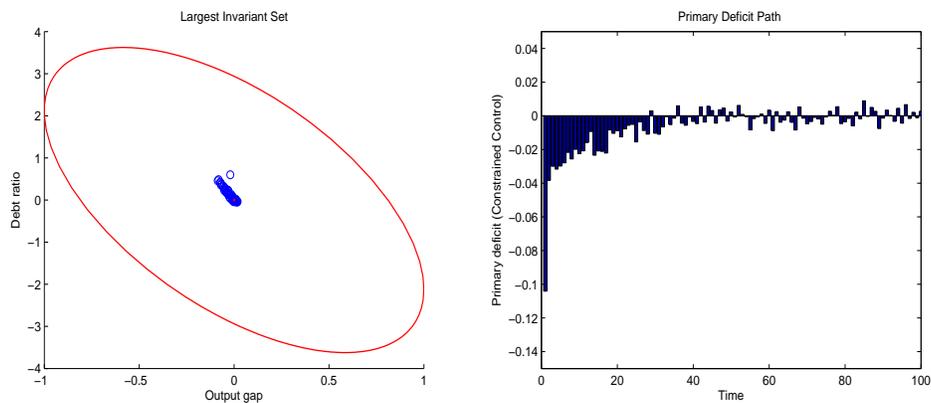


Figure 6: Scenario 3a: Invariant set for $g_l < r_l < g_h < r_h$.

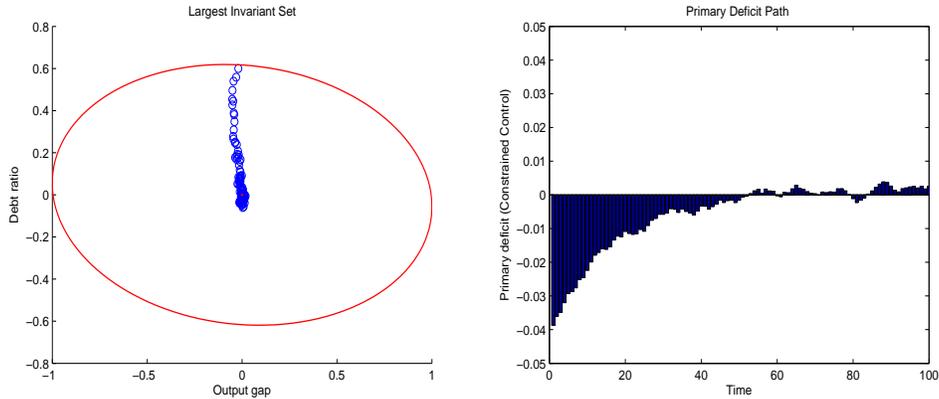


Figure 7: Scenario 3b: Invariant set for $g_l < r_l < g_h < r_h$, constrained control.

4 Conclusion

This paper discusses how invariant set techniques can be used to aid debt sustainability assessments. The method provides an estimate of a debt threshold and a fiscal reaction function which is guaranteed to stabilize debt taking into account bounded parameter uncertainty and exogenous shocks. In the case of linear systems, there is a close link between invariant ellipsoids and Lyapunov functions, which in turn can be related to the debt sustainability criterion proposed by Arrow et al. (2004).

Application of the invariant set approach leads to an optimization problem which involves a system of linear matrix inequalities. This problem can be efficiently solved using numerical methods and the paper provides several examples which differ in terms of key model parameters. Simulations suggest that debt sustainability issues arise in scenarios where future interest rates are consistently higher than potential growth rates and the size of the primary balance adjustment is constrained. In those scenarios, the resulting invariant sets are relatively tight; debt can still be stabilized (if shocks are not large) but at the cost of violating the constraints on adjustment.

Although the analysis in the paper is restricted to a simple system of two equations, the method has a rather general applicability. It can be used for an arbitrary linear system, e.g., an estimated VAR. Besides debt sustainability, invariant sets could be useful in addressing problems of stabilization in other areas, such as monetary policy. Deriving analytical criteria for the existence of invariant sets, however, remains a challenge.

Appendix A Invariant sets, Lyapunov functions and LMIs

Proposition 1 below demonstrates that the existence of an invariant ellipsoid for system (3) is equivalent to establishing the feasibility of a set of LMIs. Specifically, we show how LMIs (5)- (7) can be derived based on purely geometric considerations. The proofs are standard and closely follow the existing literature, e.g., Nazin et al. (2007), Luca et al. (2009), Khlebnikov et al. (2011), but we provide all the details since our model is slightly different from the models considered in the control literature. As noted earlier, the difference pertains to the bounds for the additive disturbances; the standard assumption is that $\|w_t\| \leq 1$, whereas we assume that $w_t \in \mathcal{W}_t$ as defined in the main text. All results are formulated for constant matrices A and B not to overload notation but the respective LMIs should be satisfied for all (A_i, B_j) , $j = 1, 2$. The following lemmas will be used in the proofs:

Lemma 1 (*Schur complement*) *The linear matrix inequality (LMI)*

$$\begin{pmatrix} Q & S \\ S' & R \end{pmatrix} \geq 0$$

where $Q = Q', R = R'$ is equivalent to

$$(i) R \geq 0, Q - SR^{-1}S' \geq 0,$$

$$(ii) Q \geq 0, R - S'Q^{-1}S \geq 0,$$

where the sign \geq used for matrices means positive semidefiniteness.

Lemma 2 (*S-procedure; Polyak, 1998*) *Given quadratic forms $x'A_0x$, $x'A_1x$, $x'A_2x$ in R^n and numbers $\alpha_0, \alpha_1, \alpha_2$, suppose for $n \geq 3$, there exist numbers μ_1, μ_2 and vector x_0 such that $\mu_1A_1 + \mu_2A_2 > 0$ and $x_0'A_1x_0 < \alpha_1$, $x_0'A_2x_0 < \alpha_2$. Then,*

$$x'A_0x \leq \alpha_0 \text{ for each } x \text{ such that } x'A_1x \leq \alpha_1, x'A_2x \leq \alpha_2$$

if and only if there exist numbers $\tau_1 \geq 0$, $\tau_2 \geq 0$ such that

$$A_0 \leq \tau_1A_1 + \tau_2A_2 \text{ and } \alpha_0 \geq \tau_1\alpha_1 + \tau_2\alpha_2.$$

Lemma 3 (*S-procedure; Boyd et al., 1994*) Let F_0, \dots, F_p be quadratic functions in $\xi \in \mathbb{R}^n$:

$$F_i(\xi) = \xi' T_i \xi + 2u_i' \xi + v_i$$

where $T = T'$. Consider the following condition on F_0, \dots, F_p :

$$F_0(\xi) \geq 0 \text{ for all } \xi : F_i(\xi) \geq 0, i = 1, \dots, p. \quad (9)$$

If there exist numbers $\tau_1 \geq 0, \dots, \tau_p \geq 0$ such that for all ξ

$$F_0(\xi) - \sum_{i=1}^p \tau_i F_i(\xi) \geq 0 \quad (10)$$

then (9) holds. Inequality (10) can be written also as

$$\begin{pmatrix} T_0 & u_0 \\ u_0' & v_0 \end{pmatrix} - \sum_{i=1}^p \tau_i \begin{pmatrix} T_i & u_i \\ u_i' & v_i \end{pmatrix}$$

Proposition 1 Consider the discrete-time system (3). Assume that the control law takes the state feedback form $u_t = Kx_t$, where K is a constant matrix of appropriate dimension. Then, the existence of an invariant ellipsoid $E = \{x : x' P^{-1} x \leq 1\}$ for (3) is equivalent to the feasibility of the following LMI:

$$\begin{pmatrix} P - (1 - \tau)^{-1} D W D' & (A P + B Y) \\ (A P + B Y)' & \tau P \end{pmatrix} \geq 0, \quad (11)$$

where $Y = K P$ and $0 < \tau < 1$.

Proof. Invariance implies that starting from any point in the set E at time t , i.e., any x_t such that $x_t' P^{-1} x_t \leq 1$, all future states will remain in E ; in particular, x_{t+1} will lie in the set. This leads to the following inequalities (setting $P^{-1} = Q$):

$$x_t' Q x_t \leq 1$$

$$x_{t+1}' Q x_{t+1} \leq 1$$

$$w_t' W^{-1} w_t \leq 1$$

By replacing x_{t+1} with the right-hand side of (3), the second inequality is equivalent to

$$[(A + BK)x_t + Dw_t]' Q [(A + BK)x_t + Dw_t] \leq 1$$

$$x_t'[(A+BK)'Q(A+BK)]x_t + x_t'(A+BK)'QDw_t \\ + w_t'D'Q(A+BK)x_t + w_t'D'QDw_t \leq 0$$

or

$$\begin{pmatrix} x_t \\ w_t \end{pmatrix}' \begin{pmatrix} (A+BK)'Q(A+BK) & (A+BK)'QD \\ D'Q(A+BK) & D'QD \end{pmatrix} \begin{pmatrix} x_t \\ w_t \end{pmatrix} \leq 1.$$

In addition, the inequalities

$$x_t'Qx_t \leq 1 \\ w_t'W^{-1}w_t \leq 1$$

can be represented as

$$\begin{pmatrix} x_t \\ w_t \end{pmatrix}' \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ w_t \end{pmatrix} \leq 1, \\ \begin{pmatrix} x_t \\ w_t \end{pmatrix}' \begin{pmatrix} 0 & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} x_t \\ w_t \end{pmatrix} \leq 1.$$

Using the S-procedure (Lemma 2), with $\alpha_0 = \alpha_1 = \alpha_2 = 1$ we obtain

$$\begin{pmatrix} (A+BK)'Q(A+BK) & (A+BK)'QD \\ D'Q(A+BK) & D'QD \end{pmatrix} \leq \begin{pmatrix} \tau_1 Q & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \tau_2 W^{-1} \end{pmatrix}.$$

which is equivalent to

$$\begin{pmatrix} (A+BK)'Q(A+BK) - \tau_1 Q & (A+BK)'QD \\ D'Q(A+BK) & D'QD - \tau_2 W^{-1} \end{pmatrix} \leq 0.$$

together with

$$\tau_1 + \tau_2 \leq 1.$$

Taking $\tau_2 = 1 - \tau_1$ and setting $\tau_1 = \tau$, yields

$$\begin{pmatrix} (A+BK)'Q(A+BK) - \tau Q & (A+BK)'QD \\ D'Q(A+BK) & D'QD - (1-\tau)W^{-1} \end{pmatrix} \leq 0. \quad (12)$$

The above inequality is not linear in K , so we need a few additional steps to arrive at an LMI. One approach would be to follow Nazin et al. (2007). From the Schur complement formula, (12) is equivalent to

$$\tau Q - (A+BK)'Q(A+BK) - (A+BK)'QD((1-\tau)W^{-1} - D'QD)^{-1}D'Q(A+BK) \geq 0$$

$$\tau Q \geq (A+BK)'[Q + QD((1-\tau)W^{-1} - D'QD)^{-1}D'Q](A+BK)$$

Using a variant of the Woodbury matrix identity

$(L - MNM')^{-1} = L^{-1} - L^{-1}M(M'L^{-1}M - N^{-1})^{-1}M'L^{-1}$ with $L^{-1} = Q$, $M = D$ and $N^{-1} = \tau W^{-1}$, we obtain

$$\tau Q \geq (A+BK)'(Q^{-1} - (1-\tau)^{-1}DWD')^{-1}(A+BK)$$

Applying the Schur formula again yields:

$$\begin{aligned} & \begin{pmatrix} \tau Q & (A+BK)' \\ (A+BK) & Q^{-1} - (1-\tau)^{-1}DWD' \end{pmatrix} = \\ & = \begin{pmatrix} \tau P^{-1} & (A+BK)' \\ (A+BK) & P - (1-\tau)^{-1}DWD' \end{pmatrix} \geq 0 \end{aligned}$$

which by Schur's formula (version (ii)) is equivalent to

$$P - \frac{1}{1-\tau}DWD' - \frac{1}{\tau}(A+BK)P(A+BK)' \geq 0$$

Set $Y := KP$. Then,

$$\begin{aligned} & P - \frac{1}{1-\tau}DWD' - \frac{1}{\tau}(A+BK)PP^{-1}P(A+BK)' = \\ & = P - \frac{1}{1-\tau}DWD' - \frac{1}{\tau}[(AP+BY)P^{-1}(AP+BY)'] \geq 0 \end{aligned}$$

and finally

$$\begin{pmatrix} P - (1-\tau)^{-1}DWD' & (AP+BY) \\ (AP+BY)' & \tau P \end{pmatrix} \geq 0.$$

■

Remark 1 It is possible to obtain LMI (11) based on the Lyapunov function $V(x) = x'Qx$ and requiring that the stability condition

$$V(x_{t+1}) - V(x_t) = x'_{t+1}Qx_{t+1} - x'_tQx_t \leq 0$$

holds for all x_t outside the boundaries of the invariant ellipsoid, i.e., for all x_t , such that $V(x_t) \geq 1$.

Indeed, the stability condition can be rewritten as

$$\begin{pmatrix} x_t \\ w_t \end{pmatrix}' \begin{pmatrix} (A+BK)'Q(A+BK) - Q & (A+BK)'QD \\ D'Q(A+BK) & D'QD \end{pmatrix} \begin{pmatrix} x_t \\ w_t \end{pmatrix} \leq 0.$$

Similar to Proposition 1, the constraints $x_t'Qx_t \geq 1$ and $w_t'W^{-1}w_t \leq 1$ can be stated as

$$\begin{pmatrix} x_t \\ w_t \end{pmatrix}' \begin{pmatrix} -Q & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ w_t \end{pmatrix} \leq -1,$$

$$\begin{pmatrix} x_t \\ w_t \end{pmatrix}' \begin{pmatrix} 0 & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} x_t \\ w_t \end{pmatrix} \leq 1.$$

Apply Lemma 2 with $\alpha_0 = 0$, $\alpha_1 = -1$ and $\alpha_2 = 1$. Then, the existence of $\tau_1 \geq 0$, $\tau_2 \geq 0$ such that

$$\begin{pmatrix} (A+BK)'Q(A+BK) - Q & (A+BK)'QD \\ D'Q(A+BK) & D'QD \end{pmatrix} \leq \begin{pmatrix} -\tau_1 Q & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \tau_2 W^{-1} \end{pmatrix},$$

$$\tau_2 - \tau_1 \leq 0$$

implies the first inequality. Take $\tau_2 = \tau_1 = 1 - \tau$ and substitute above. Inequality (12) obtains. The rest follows as in Proposition 1.

The next proposition establishes that LMI (6) corresponds to the constraint on the system state $\|z_t\| \leq z_{max}$.

Proposition 2 *In addition to (11), the LMI*

$$\begin{pmatrix} P & PC' \\ CP & z_{max}^2 I \end{pmatrix} \geq 0. \quad (13)$$

ensures that the constraint $\|z_t\| \leq z_{max}$ is satisfied.

Proof. The constraint

$$\|z_t\| \leq z_{max}$$

is equivalent to

$$\|z_t\|^2 = \|Cx_t\|^2 = x_t' C' C x_t \leq z_{max}^2$$

The above should hold true for any x_t in the invariant ellipsoid, i.e., any x_t such that $x_t' P^{-1} x_t \leq 1$. Applying the S-procedure (Lemma 3), if there exists $\tau \geq 0$ such that

$$\begin{pmatrix} C' C & 0 \\ 0 & z_{max}^2 \end{pmatrix} - \begin{pmatrix} \tau P^{-1} & 0 \\ 0 & \tau \end{pmatrix} \leq 0$$

or

$$\begin{pmatrix} C' C - \tau P^{-1} & 0 \\ 0 & z_{max}^2 - \tau \end{pmatrix} \leq 0 \quad (14)$$

then the inequality $x_t' C' C x_t \leq z_{max}^2$ will hold. For the above matrix to be negative semi-definite we need $z_{max}^2 - \tau \leq 0$, so we can assume $z_{max}^2 = \tau$ and then (14) becomes equivalent to

$$z_{max}^2 P^{-1} - C' C \geq 0.$$

Pre- and post-multiplying by P and dividing by $z_{max}^2 > 0$ yields:

$$P - \frac{1}{z_{max}^2} P C' C P \geq 0$$

which is equivalent (by Lemma 1) to

$$\begin{pmatrix} P & P C' \\ C P & z_{max}^2 I \end{pmatrix} \geq 0.$$

■

Although powerful numerical methods have been developed to determine the feasibility of LMIs, it is of interest whether analytical criteria can be found for relatively simple systems such as the one considered here. For the debt sustainability problem the relevant LMIs are (5) and (6) (and also (8) if a constraint on the control is imposed).

One possible approach to establishing infeasibility is based on a generalization of Sylvester's criterion: for a real symmetric matrix to be positive semi-definite, it is necessary and sufficient that all its principal minors are non-negative. Applied to (6) with C as defined in the main text, the criterion requires checking the principal minors of the matrix

$$\begin{pmatrix} P & P C' \\ C P & z_{max}^2 I \end{pmatrix} =$$

$$\begin{pmatrix} p_{11} & p_{12} & p_{11} & 0 \\ p_{12} & p_{22} & p_{12} & 0 \\ p_{11} & p_{12} & y_{max}^2 & 0 \\ 0 & 0 & 0 & y_{max}^2 \end{pmatrix}.$$

This is straightforward, so we skip the calculations of the various determinants and only state the resulting inequalities:

$$\begin{aligned} y_{max}^2 &\geq p_{11} \\ p_{11}p_{22} &\geq p_{12}^2 \\ p_{22} &\geq p_{11} \end{aligned}$$

Also, from the condition that the elements of the main diagonals of (5) should be non-negative, it follows (with D being a diagonal matrix with elements δ_i) that:

$$\begin{aligned} p_{11} &\geq \frac{\delta_1^2 w_{11}}{1 - \tau} \\ p_{22} &\geq \frac{\delta_2^2 w_{22}}{1 - \tau} \end{aligned}$$

Therefore, if $\frac{\delta_1^2 w_{11}}{1 - \tau} \geq y_{max}^2$, the LMI is not feasible.

Calculating all principal minors of (5), however, is a daunting task. Instead, one can resort to results from control theory pertaining to controllability and stabilization. If a system is controllable, then it is also stabilizable, which implies the existence of a matrix P as above. This in turn requires that the matrix $A + BK$ is stable. Therefore, we need to see under what conditions this matrix is stable. In our example,

$$A + BK = \begin{pmatrix} a_{11} + b_1 k_1 & b_1 k_2 \\ b_2 k_1 & a_{22} + b_2 k_2 \end{pmatrix}.$$

We want the eigenvalues of this matrix to be between -1 and 1. The characteristic polynomial of the above matrix is the quadratic equation:

$$\lambda^2 - \lambda(a_{11} + a_{22} + b_1 k_1 + b_2 k_2) + a_{11}a_{22} + a_{22}b_1 k_1 + a_{11}b_2 k_2 = 0,$$

so we have an explicit formula for the eigenvalues. It remains to establish for which values of k_1 and k_2 the roots of the above equation will fall into the desired range.

References

- Arrow, K., P. Dasgupta, L. Goulder, G. Daily, P. Ehrlich, G. Heal, S. Levin, K.-G. Maler, S. Schneider, D. Starrett, and Walker, B. (2004), "Are We Consuming Too Much?" *Journal of Economic Perspectives* No. 18 (3), pp. 14772.
- Baldacci, E., Petrova, I., Belhocine, N., Dobrescu, G. and Mazraani, S. (2011), "Assessing Fiscal Stress", *IMF Working Paper* WP/11/100.
- Barnhill, T. and Kopits, G. (2003), "Assessing Fiscal Sustainability Under Uncertainty", *IMF Working Paper* WP/03/79.
- Blanchard, O., Chouraqui, J.C., Hagemann, R. and Sartor, N. (1990), "The Sustainability of Fiscal Policy: New Answers to an Old Question", *OECD Economic Studies*, No.15.
- Boyd, S., El Gahoui, L., Feron, E. and Balakrishnan, V. (1994) , *Linear Matrix Inequalities in System and Control Theory*, SIAM Philadelphia.
- Berti, K., Colesnic, E., Despons, C., Pamies, S. and Sail, E. (2016), "Fiscal Reaction Functions for European Union Countries", *EU Commission Discussion Paper No. 028*, April 2016.
- Celasun, O., Bebrun, X. and Ostry, J. (2007), "Primary Surplus Behavior and Risks to Fiscal Sustainability in Emerging Market Countries: A Fan-Chart Approach" , *IMF Staff Papers* Vol. 53, No. 3, pp. 401-425.
- Cherif, R. and Hasanov, F. (2012), "Public Debt Dynamics: The Effects of Austerity, Inflation, and Growth Shocks " , *IMF Working Paper* WP/12/230.
- Cuerpo, C., I. Drumond, J. Lendvai, P. Pontuch, and Raciborski, R. (2015), "Private Sector Deleverage in Europe", *Economic Modelling* No 44, pp. 37283.
- Escolano, J. (2010), "A Practical Guide to Public Debt Dynamics, Fiscal Sustainability, and Cyclical Adjustment of Budgetary Aggregates", *Technical Notes and Manuals*, Fiscal Affairs Department, International Monetary Fund.
- European Commission (2006), "The Long-term Sustainability of Public Finances in the European Union", *European Economy*, No. 4.
- Garcia, M. and Rigobon, R. (2005), "A Risk Management Approach to Emerging Markets Sovereign Debt Sustainability with an Application to Brazilian Data," in *Inflation Targeting, Debt, and the Brazilian Experience, 1999 to 2003*, MIT Press.

Grant, M. and Boyd, S. (2013), "CVX: Matlab software for disciplined convex programming, version 2.0 beta" <http://cvxr.com/cvx>

Grant, M. and Boyd, S. (2008), "Graph implementations for nonsmooth convex programs, Recent Advances in Learning and Control (a tribute to M. Vidyasagar)" in V. Blondel, S. Boyd, and H. Kimura, editors, *Lecture Notes in Control and Information Sciences*, Springer, pp. 95-110

Hansen, L. and Sargent, T.(2008), *Robustness*, Princeton University Press.

International Monetary Fund (2002), "Assessing Sustainability" *IMF Policy Paper*, Washington DC, May 2002.

International Monetary Fund (2011), "Modernizing the Framework for Fiscal Policy and Public Debt Sustainability Analysis", *IMF Policy Paper*, Washington DC, August 2011.

International Monetary Fund (2013), "Staff Guidance Note for Public Debt Sustainability Analysis in Market-access Countries", Washington DC, May 2013.

International Monetary Fund (2016), "Analyzing and Managing Fiscal RisksBest Practices", *IMF Policy Paper*, Washington DC, June 2016.

International Monetary Fund (2016), "Debt Use it Wisely", *IMF Fiscal Monitor*, Washington DC, October 2016.

Khlebnikov, M., Polyak, B. and Kuntsevich, V. (2011), "Optimization of Linear Systems Subject to Bounded Exogenous Disturbances: The Invariant Ellipsoid Technique", *Automation and Remote Control*, Vol. 72, No. 11, pp. 2227- 2275.

Kurzhanski, A. and Varaiya, P. (2007), *Ellipsoidal Toolbox*, University of California, Berkeley.

Luca, A., Rodriguez-Ayerbe, P. and Dumur, D. (2009), "Invariant Sets Method for State-feedback Control Design", *17th Telecommunications forum TELFOR 2009*, Belgrade, pp. 681-684.

Nazin, S., Polyak, B. and Topunov, M. (2007), "Rejection of Bounded Exogenous Disturbances by the Method of Invariant Ellipsoids", *Automation and Remote Control*, Vol. 68, No. 3, pp. 467- 486.

Polyak, B. (1998), "Convexity of Quadratic Transformations and Its Use in Control and Optimization", *Journal of Optimization Theory and Applications*, Vol. 99, No. 3, pp. 553-583.

Wyplosz, C. (2011), "Debt Sustainability Assessment: Mission Impossible", *Review of Economics and Institutions*, Vol. 2, No.3, Article 1, pp. 1-37.