CONSUMER PRICE INDEX THEORY

Chapter Titles

CHAPTER 5: THE ECONOMIC APPROACH TO INDEX NUMBER THEORY [1]

Draft: April 16, 2021

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[1] Much of the material in sections 2-8 of this Chapter is drawn from Chapter 17 of *The Consumer Price Index Manual: Theory and Practice* and from Diewert (2009). The author thanks Paul Armknecht, Bert Balk, Peter Hill, Alice Nakamura, Mick Silver and Kim Zieschang for helpful comments on the material in this chapter that was taken from the previous Manual. The author thanks Charles Andrew Barclay, Kevin Fox, Ismail Hossain, Ronald Johnson, Daniel Melser, Chihiro Shimizu, Paul Schreyer, Rui Xiao and Clément Yélou for helpful comments on the current draft.

1. Introduction

Economics is the study of choice under constraints. Thus the economic approach to index number theory applied to households generally involves the assumption of cost minimizing or utility maximizing behavior on the part of consumers subject to one or more constraints. It is unlikely that actual consumer behavior is completely described by the optimization models that will be considered in this chapter but it seems that the economic approach to index number theory allows us to address some difficult measurement problems that other approaches to index number theory cannot address.

Some of the material in this chapter relies on advanced microeconomic theory. Some attempt is made to explain the various theories but if the explanations are not adequate, references to the underlying literature are given.

In section 2, the Konüs Cost of Living Index (COLI) for a single household is explained. This section is a fundamental one. It allows us to conceptualize the role of substitution as a response to changes in relative prices. In this section, the underlying utility or preference function is a general one. In section 3, the theory described in section 2 is specialized to the case of homothetic preferences. Preferences are homothetic if they can be represented by a linearly homogeneous utility function. It turns out that the assumption of homothetic preferences enables the price statistician to deal with product substitution in a very straightforward way. In section 4, two results from microeconomic theory are discussed: Wold's Identity and Shephard's Lemma. These two results will be used in sections 5-7 where certain formulae or functional forms for price and quantity indexes are introduced and their connection to the economic approach to index number theory will be established. Section 5 introduces the concept of a flexible functional form for a utility function. A flexible functional form can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order around any given point. Thus it is useful to have index number formulae that are exactly consistent with preferences that can be represented by a flexible functional form since these functions can accommodate a wide variety of substitution responses on the part of consumers to changes in prices. Sections 5, 6 and 7 show that there are flexible functional forms for consumer utility functions that are exactly consistent with three well known index number formulae: the Fisher, Walsh and Törnqvist Theil indexes. An index number formula that is exactly consistent with a flexible functional form is called a superlative index. In section 8, it is shown that the superlative indexes studied in sections 5-7 all approximate each other to the second order around an equal price and quantity point and so in general, it will not matter too much which one of these three formulae is chosen in practice. The first eight sections of this chapter are the most important ones. The remaining sections deal with specific measurement topics that extend the basic theory in various directions.

In sections 9 and 10, index number formulae are given that are exact for two functional forms for the consumer's utility function that are not flexible. These two functional forms are the Cobb Douglas and Constant Elasticity of Substitution (CES) functions. Since they are widely used by economists and statisticians, it is useful to study these two functional forms and their corresponding exact index number formulae.

In section 11, the Allen quantity index is introduced. In the previous sections, quantity indexes that were exact for homothetic preferences were defined. The Allen quantity index is well defined even if preferences are not homothetic. It turns out that various Allen indexes match up with various Konüs cost of living indexes. The Törnqvist Theil price and quantity indexes turn out to be very useful in this context. They are also very useful in the following two sections which show how changes in tastes can be accomodated (section 12) and how price indexes that are conditional on environmental factors can be defined (section 13).

In section 14, the concept of a Hicksian reservation price is introduced. A reservation price is an imputed price that is just high enough to induce consumers to not purchase a product. It turns out that this concept is useful in the context of dealing with the problems that arise when new products are introduced and old, obsolete products disappear.

In section 15, it is noted that consumers face not only a budget constraint, but they also face a time constraint. The consumer's allocation of time interacts with his or her budget constraint and this interaction leads to difficult measurement problems when constructing consumer price indexes. An introduction to some of these problems is provided in this section.

Sections 16 and 17 generalize the single household Konüs price index and Allen quantity index concepts to many households. Fisher indexes play a large role in these sections.

There are demands on statistical agencies to produce price and volume indexes that take into account changes in the distribution of income over households. Section 18 provides the reader with an introduction to this topic.

Finally section 19 discusses the matching problem. If we attempt to construct a cost of living index for a single household, then due to the fact that many household purchases are made infrequently, it proves to be difficult to match the prices and quantities of purchased products over consecutive periods. For example, a seasonal product may be purchased only during certain seasons. Or a big discounted price may induce a household to stock up on a product this month and not purchase the product again for several months. This leads to a lack of matching of products problem that makes the construction of price indexes difficult. Section 19 offers some possible solutions to this problem.

An Appendix has proofs of various theoretical results that are stated in the main text.

2. The Konüs Cost of Living Index for a Single Consumer

In this section, we outline the theory of the cost of living index for a single consumer (or household)[2] that was first developed by the Russian economist, Konüs (1924). This theory relies on the assumption of *optimizing behavior* on the part of households. Given an observed vector of commodity or input prices p^t that the household faces in a given time period t, it is assumed that the corresponding observed quantity vector q^t is a solution to a cost (or expenditure) minimization problem that involves the consumer's preference or utility function f(q). Thus in contrast to the axiomatic approach to index number theory, the economic approach does *not* assume that the two quantity vectors q^0 and q^1 discussed in previous chapters are independent of the two price vectors p^0 and p^1 that the household faces in periods 0 and 1. In the economic approach, the period t quantity vector q^t is determined by the consumer's preference function f and the period t vector of prices p^t that the consumer faces in period t.[3]

We assume that the consumer (or household) has well defined *preferences* over different combinations of the N consumer commodities or items.[4] Each combination of items can be represented by a nonnegative vector $q \equiv [q_1,\ldots,q_N]$. The consumer's preferences over alternative possible consumption vectors q are assumed to be representable by a continuous, increasing[5] and concave[6] utility function f.[7] Thus if $f(q^1) >$

[2] A household may consist of more than one individual. Our exposition ignores the complications that can arise in multi-person households; i.e., we assume that the household has consistent preferences of the type explained below.
[3] In principle, the price p_n^t is a *period t unit value price for product n* for the household under consideration. The corresponding q_n^t is equal to the total purchases of product n by the household in period t. Thus the product $p_n^tq_n^t$ is the total expenditure of the household on product n during period t.
[4] In this section, these preferences are assumed to be invariant over time. Changing preferences and the complications that arise when the number of available products changes over time will be postponed to sections 12 and 14 and subsequent chapters.
[5] f(q) is *increasing* in q if $q^2 >> q^1 \geq 0_N$ implies $f(q^2) > f(q^1)$.

$f(q^0)$, then the consumer prefers the consumption vector q^1 to q^0. It is further assumed that *the consumer minimizes the cost of achieving the observed period t utility level* $u^t \equiv f(q^t)$ for periods t = 0,1. Thus the economic approach to index number theory assumes that the observed period t consumption vector $q^t >> 0_N$ solves the following *period t cost minimization problem*:[8]

(1) $C(u^t,p^t) \equiv \min_q \{p^t \cdot q : f(q) \geq u^t ; q \geq 0_N\} = p^t \cdot q^t$; t = 0,1.

The consumer's cost minimization problem for period 0 is to choose a consumption vector $q \equiv [q_1,...,q_N]$ which will minimize the cost $p^0 \cdot q \equiv \Sigma_{n=1}^N p_n^0 q_n$ of achieving at least the given utility level u^0, given that the consumer's preferences can be represented by the function f(q). The period 0 observed consumption vector for the consumer is $q^0 \equiv [q_1^0,...,q_N^0]$ where it is assumed that each q_n^0 is positive. An assumption which is imbedded in the above definition for the period 0 cost minimization problem is that the period 0 reference utility level is u^0 defined as $f(q^0)$. The final assumption which is imbedded in the period 0 cost minimization problem defined by (1) above is that the consumer's observed period 0 quantity vector is a solution to the period 0 cost minimization problem. A similar interpretation applies to the period 1 cost minimization problem. We also assume that the period t price vector for the N commodities under consideration that the consumer faces in each period is strictly positive; i.e., we assume that $p^t >> 0_N$ for t = 0,1. Thus there is a fair amount of complexity hidden behind the cost minimization problems (and their solutions) defined by (1).

Note that the solution to the cost or expenditure minimization problem (1) for a general utility level u and general vector of commodity prices p defines the *consumer's cost function*, C(u,p). This cost function will be used in order to define the consumer's cost of living price index. It can be shown that C(u,p) has the following mathematical properties under our regularity conditions on f(q): (i) C(u,p) is nonnegative for all $u \geq 0$ and $p >> 0_N$; (ii) for each $p >> 0_N$, C(u,p) is an increasing continuous function of u and (iii) for each $u \geq 0$, C(u,p) is a continuous, concave and linearly homogeneous function of p [9] that is also increasing[10] if all components of p increase.[11]

The Konüs (1924) family of *true cost of living indexes* pertaining to two periods, $P_K(p^0,p^1,q)$, where the consumer faces the strictly positive price vectors $p^0 \equiv (p_1^0,\ldots,p_N^0)$ and $p^1 \equiv (p_1^1,\ldots,p_N^1)$ in periods 0 and 1 respectively, is defined as *the ratio of the minimum costs of achieving the same utility level* $u \equiv f(q)$ where $q \equiv (q_1,\ldots,q_N) >> 0_N$ is a positive reference quantity vector:

(2) $P_K(p^0,p^1,q) \equiv C[f(q),p^1]/C[f(q),p^0]$.

Definition (2) defines a *family* of price indexes because there is one such index for each reference quantity vector q chosen.

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[6] f is *concave* over the set of nonnegative q if $f(\lambda q^1 + (1-\lambda)q^2) \geq \lambda f(q^1) + (1-\lambda)f(q^2)$ for all $0 \leq \lambda \leq 1$ and all $q^1 \geq 0_N$ and $q^2 \geq 0_N$. Note that $q \geq 0_N$ means that each component of the N dimensional vector q is nonnegative, $q >> 0_N$ means that each component of q is positive and $q > 0_N$ means that $q \geq 0_N$ but $q \neq 0_N$; i.e., q is nonnegative but at least one component is positive.

[7] For convenience, we assume that $f(0_N) = 0$ and f(q) tends to plus infinity as all components of q tend to plus infinity.

[8] Notation: $p^t \equiv [p_1^t,...,p_N^t]$, $q^t \equiv [q_1^t,...,q_N^t]$ and $p^t \cdot q^t \equiv \Sigma_{n=1}^N p_n^t q_n^t$ for t = 0,1. Note that we are assuming that all prices and quantities are positive. Thus $C(f(q^t),p^t) > 0$ for t = 0,1.

[9] This property is the following one: let $u \geq 0$, $p >> 0_N$ and $\lambda \geq 0$; then $C(u,\lambda p) = \lambda C(u,p)$.

[10] This property is the following one: let $u > 0$ and $0_N << p^1 << p^2$; then $C(u,p^1) < C(u,p^2)$.

[11] For additional materials on these properties of cost functions and references to the literature, see Diewert (1993a). The restriction that f(q) be a concave function is not the usual assumption in the economics literature but drawing on the work of Afriat (1967) and Diewert (1973), it can be shown that this assumption is not restrictive in practice.

It is natural to choose two specific reference quantity vectors q in definition (2): the observed base period quantity vector q^0 and the current period quantity vector q^1. The first of these two choices leads to the *following Laspeyres-Konüs true cost of living index*:

$$
\begin{aligned}
(3)\ P_K(p^0,p^1,q^0) &\equiv C[f(q^0),p^1]/C[f(q^0),p^0] \\
&= C[f(q^0),p^1]/p^0\cdot q^0 && \text{using (1) for } t = 0 \\
&= \min{}_q \{p^1\cdot q : f(q) \geq f(q^0) ;\ q \geq 0_N\}/p^0\cdot q^0 && \text{using the definition of } C[f(q^0),p^1] \\
&\leq p^1\cdot q^0/p^0\cdot q^0 && \text{since } q^0 \text{ is feasible for the minimization problem} \\
&= P_L(p^0,p^1,q^0,q^1)
\end{aligned}
$$

where P_L is the Laspeyres price index defined in earlier chapters. *Thus the (unobservable) Laspeyres-Konüs true cost of living index is bounded from above by the observable Laspeyres price index.*[12]

The second of the two natural choices for a reference quantity vector q in definition (2) leads to the *following Paasche-Konüs true cost of living index*:

$$
\begin{aligned}
(4)\ P_K(p^0,p^1,q^1) &\equiv C[f(q^1),p^1]/C[f(q^1),p^0] \\
&= p^1\cdot q^1/C[f(q^1),p^0] && \text{using (1) for } t = 1 \\
&= p^1\cdot q^1/\min{}_q\{p^0\cdot q : f(q) \geq f(q^1) ;\ q \geq 0_N\} && \text{using the definition of } C[f(q^1),p^0] \\
&\geq p^1\cdot q^1/p^0\cdot q^1 && \text{since } q^1 \text{ is feasible for the minimization problem and thus} \\
& && \min{}_q\{p^0\cdot q : f(q) \geq f(q^1)\} \leq p^0\cdot q^1 \text{ and hence } 1/C[f(q^1),p^0] \geq 1/p^0\cdot q^1 \\
&= P_P(p^0,p^1,q^0,q^1)
\end{aligned}
$$

where P_P is the Paasche price index defined in earlier chapters. *Thus the (unobservable) Paasche-Konüs true cost of living index is bounded from below by the observable Paasche price index.*[13]

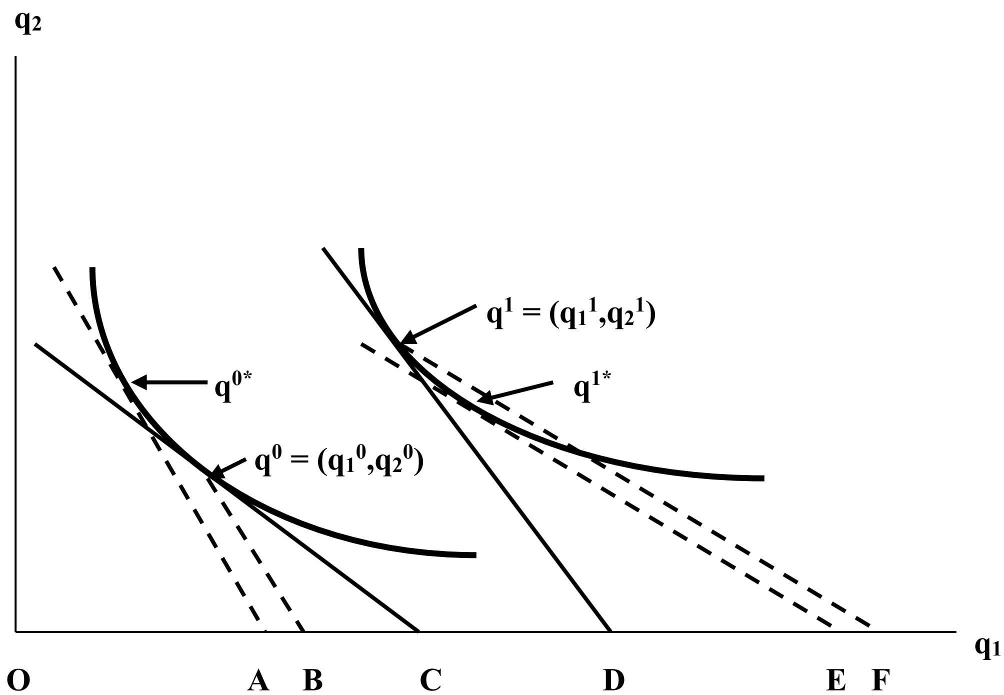
Figure 1 illustrates the bounds given by (3) and (4) for the case of two commodities.

The solution to the period 0 cost minimization problem is the vector q^0 and the straight line through C represents the consumer's period 0 budget constraint, the set of quantity points q_1,q_2 such that $p_1^0q_1+p_2^0q_2 = p_1^0q_1^0+p_2^0q_2^0$. The curved line through q^0 is the consumer's period 0 indifference curve, the set of points q_1,q_2 such that $f(q_1,q_2) = f(q_1^0,q_2^0)$; i.e., it is the set of consumption vectors that give the same utility as the observed period 0 consumption vector q^0. The solution to the period 1 cost minimization problem is the vector q^1 and the straight line through D represents the consumer's period 1 budget constraint, the set of quantity points q_1,q_2 such that $p_1^1q_1+p_2^1q_2 = p_1^1q_1^1+p_2^1q_2^1$. The curved line through q^1 is the consumer's period 1 indifference curve, the set of points q_1,q_2 such that $f(q_1,q_2) = f(q_1^1,q_2^1)$; i.e., it is the set of consumption vectors that give the same utility as the observed period 1 consumption vector q^1. The point q^{0*} solves the hypothetical cost minimization problem of minimizing the cost of achieving the base period utility level $u^0 \equiv f(q^0)$ when facing the period 1 price vector $p^1 = (p_1^1,p_2^1)$. Thus we have $C[u^0,p^1] = p_1^1q_1^{0*}+p_2^1q_2^{0*}$ and the dashed line through A is the corresponding isocost line $p_1^1q_1+p_2^1q_2 = C[u^0,p^1]$.

Note that the hypothetical cost line through A is parallel to the actual period 1 cost line through D. From (3), the Laspeyres-Konüs true index is $C[u^0,p^1]/[p_1^0q_1^0+p_2^0q_2^0]$ while the ordinary Laspeyres index is $[p_1^1q_1^0+p_2^1q_2^0]/[p_1^0q_1^0+p_2^0q_2^0]$. Since the denominators for these two indexes are the same, the difference between the indexes is due to the differences in their numerators. In Figure 1, this difference in the numerators is expressed by the fact that the cost line through A lies *below* the parallel cost line through B.

[12] This inequality was first obtained by Konüs (1924) (1939; 17). See also Pollak (1983).
[13] This inequality is also due to Konüs (1924) (1939; 19).

Figure 1: The Laspeyres and Paasche Bounds to the True Cost of Living



If the consumer's indifference curve through the observed period 0 consumption vector q^0 were L shaped with vertex at q^0, then the consumer would not change his or her consumption pattern in response to a change in the relative prices of the two commodities while keeping a fixed standard of living. In this case, the hypothetical vector q^{0*} would coincide with q^0, the dashed line through A would coincide with the dashed line through B and the true Laspeyres-Konüs index would *coincide* with the ordinary Laspeyres index. However, L shaped indifference curves are not generally consistent with consumer behavior; i.e., when the price of a commodity decreases, consumers generally demand more of it. Thus in the general case, there will be a gap between the points A and B. The magnitude of this gap represents the amount of *substitution bias* between the true index and the corresponding Laspeyres index; i.e., the Laspeyres index will generally be *greater* than the corresponding true cost of living index, $P_K(p^0,p^1,q^0)$.

Figure 1 can also be used to illustrate the inequality (4). First note that the dashed lines through E and F are parallel to the period 0 isocost line through C. The point q^{1*} solves the hypothetical cost minimization problem of minimizing the cost of achieving the current period utility level $u^1 \equiv f(q^1)$ when facing the period 0 price vector $p^0 = (p_1^0,p_2^0)$. Thus we have $C[u^1,p^0] = p_1^0q_1^{1*}+p_2^0q_2^{1*}$ and the dashed line through E is the corresponding isocost line $p_1^1q_1+p_2^1q_2 = C[u^0,p^1]$. From (4), the Paasche-Konüs true index is $[p_1^1q_1^1+p_2^1q_2^1]/C[u^1,p^0]$ while the ordinary Paasche index is $[p_1^1q_1^1+p_2^1q_2^1]/[p_1^0q_1^1+p_2^0q_2^1]$. Since the numerators for these two indexes are the same, the difference between the indexes is due to the differences in their denominators. In Figure 1, this difference in the denominators is expressed by the fact that the cost line through E lies *below* the parallel cost line through F. The magnitude of this difference represents the amount of *substitution bias* between the true index and the corresponding Paasche index; i.e., the Paasche index will generally be *less* than the corresponding true cost of living index, $P_K(p^0,p^1,q^1)$. Note that this inequality goes in the opposite direction to the previous inequality between the two Laspeyres indexes. The reason for this change in direction is due to the fact that one set of differences between the two indexes takes place in the numerators of the two indexes (the Laspeyres inequalities) while the other set takes place in the denominators of the two indexes (the Paasche inequalities).

The bound (3) on the Laspeyres-Konüs true cost of living $P_K(p^0,p^1,q^0)$ using the base period level of utility as the living standard is *one sided* as is the bound (4) on the Paasche-Konüs true cost of living $P_K(p^0,p^1,q^1)$ using the *current period* level of utility as the living standard. In a remarkable result, Konüs (1924; 20) showed that there exists an intermediate consumption vector q^* that is on the straight line joining the base period consumption vector q^0 and the current period consumption vector q^1 such that the corresponding (unobservable) true cost of living index $P_K(p^0,p^1,q^*)$ is between the observable Laspeyres and Paasche indexes, P_L and P_P.[14] The Konüs result is the following Proposition:

Proposition 1: There exists a number λ^* between 0 and 1 such that

(5) $P_L \leq P_K(p^0,p^1, \lambda^*q^0 + (1-\lambda^*)q^1) \leq P_P$ or $P_P \leq P_K(p^0,p^1, \lambda^*q^0 + (1-\lambda^*)q^1) \leq P_L$.

The first set of inequalities holds when $P_L \leq P_P$ and the second holds when $P_P \leq P_L$. For a proof of this result, see the Appendix.

The above inequalities are of some practical importance. If the observable (in principle) Paasche and Laspeyres indexes are not too far apart, then taking a symmetric average of these indexes should provide a good approximation to a true cost of living index where the reference standard of living is somewhere between the base and current period living standards. To determine the precise symmetric average of the Paasche and Laspeyres indexes, we can appeal to the results in Chapter 2 above and take the geometric mean, which is the Fisher price index. Thus the Fisher ideal price index receives a fairly strong justification as a good approximation to an unobservable theoretical cost of living index.

The bounds (3)-(5) are the best bounds that we can obtain on the true cost of living indexes without making further assumptions. In subsequent sections, we will make further assumptions on the class of utility functions that describe the consumer's tastes for the N commodities under consideration. By making specific functional form assumptions about the utility function f(q) or about the corresponding cost function C(u,p), it will be possible to determine the functional form for the consumer's true cost of living index.

Before proceeding further, it may be useful to discuss some problems with the economic approach to index number theory. A major objection to this approach is the assumption of cost minimizing (or equivalently of utility maximizing) behavior on the part of households. Do households even have consistent preferences over alternative combinations of goods and services, let alone minimize the cost of achieving a given level of utility or welfare? Even if households do not have perfectly consistent preferences, experience has shown that when the price of a product is significantly decreased, households will buy more of it and conversely, if the price of a product rises significantly, households will tend to purchase less of it. The economic approach to index number theory simply formalizes this behavior and at the same time, it is able to generate measures of possible changes in consumer welfare along with measures of changes in the cost of living. These measures are imperfect but they are valued by economists and policy makers. Thus it is useful to take an economic approach to index number theory. Moreover, government statisticians are obliged to produce *economic statistics*. It seems sensible for official statisticians to be at least aware of economic approaches to index number theory while producing economic statistics. Finally, as will be seen in later sections, the economic approach to index number theory provides useful insights into difficult measurement problems that other approaches to index number theory are unable to address.

Some of the limitations of the present framework will be relaxed in subsequent sections; i.e., the assumption that all prices and quantities are positive will be relaxed, the assumption of constant preferences will also be relaxed and the problems associated with the appearance of new products and the disappearance of existing products will be addressed. However, one problem that will not be addressed is *the stock piling problem*;

[14] See Diewert (1983;191).

i.e., when storable products go on sale, households may purchase large amounts of the products so that the period of consumption of these products does not coincide with the period of purchase. These left over stocks will affect demand for the products in subsequent periods and the model of economic behavior used in this section does not take this possibility into account.[15] The problem of storable goods not being consumed in the period of purchase suggests that the Konüs true cost of living index should not be implemented if the length of the period is very short. Thus *daily economic price indexes* for *individual households* may be more or less meaningless from the viewpoint of the economic approach to index numbers. The length of the accounting period for individual households should be a longer period, such as a month or a quarter.

3. The Cost of Living Index when Preferences are Homothetic

Up to now, the consumer's preference function f did not have to satisfy any particular homogeneity assumption. In this section, we assume that f is (positively) *linearly homogeneous*[16]; i.e., we assume that the consumer's utility function has the following property:

(6) $f(\lambda q) = \lambda f(q)$ for all $\lambda > 0$ and all $q \geq 0_N$.

Given the continuity of f, it can be seen that property (6) implies that $f(0_N) = 0$. Furthermore, f also satisfies $f(q) > 0$ if $q >> 0_N$.

In the economics literature, assumption (6) is known as the assumption of *homothetic preferences*.[17] This assumption is not strictly justified from the viewpoint of actual economic behavior, but, as will be seen below, it leads to economic price indexes that do not depend on the consumer's standard of living; i.e., the resulting aggregate prices do not depend on quantities.[18] Under this assumption, the consumer's expenditure or cost function, C(u,p) defined by (1) above, decomposes into the product of two terms. For positive commodity prices $p >> 0_N$ and a positive utility level u, we have the following decomposition of the cost function:

(7) $C(u,p) \equiv \min_q\{p\cdot q : f(q) \geq u;\ q \geq 0_N\}$

$= \min_q\{p\cdot q : (1/u)f(q) \geq 1;\ q \geq 0_N\}$ dividing both sides of the constraint by $u > 0$

$= \min_q\{p\cdot q : f(q/u) \geq 1;\ q \geq 0_N\}$ using the linear homogeneity of f

$= u \min_q\{p\cdot q/u : f(q/u) \geq 1;\ q \geq 0_N\}$ using the assumption that u is positive

$= u \min_z\{p\cdot z : f(z) \geq 1;\ z \geq 0_N\}$ defining $z \equiv q/u$

$= uC(1,p)$ using definition (1) with $u = 1$

$= uc(p)$

[15] The treatment of purchases of durable goods will be addressed in Chapter 10. A *durable good* (e.g., an automobile or a house) is able to provide a stream of services over its useful lifetime; a *storable good* (e.g., a can of beans) can only be used once but its consumption can be postponed from its period of purchase to a later period of consumption.

[16] This assumption is fairly restrictive in the consumer context. It implies that each indifference curve or surface is a radial projection of the unit utility indifference curve or surface. It also implies that all income elasticities of demand are unity, which is contradicted by empirical evidence. However, at lower levels of aggregation, the homotheticity assumption for the relevant subutility function is probably an acceptable approximation to reality.

[17] More precisely, Shephard (1953) defined a homothetic function to be a monotonic transformation of a linearly homogeneous function. However, if a consumer's utility function is homothetic, we can always rescale it to be linearly homogeneous without changing consumer behavior. Hence, we simply identify the homothetic preferences assumption with the linear homogeneity assumption.

[18] This particular branch of the economic approach to index number theory is due to Shephard (1953) (1970) and Samuelson and Swamy (1974). Shephard in particular realized the importance of the homotheticity assumption in conjunction with separability assumptions in justifying the existence of subindexes of the overall cost of living index.

where $c(p) \equiv C(1,p)$ is the *unit cost function* that corresponds to f.[19] It can be shown that the unit cost function c(p) satisfies the same regularity conditions that f satisfies; i.e., c(p) is positive, concave and (positively) linearly homogeneous for positive price vectors.[20] Substituting (7) into (1) and using $u^t = f(q^t)$ leads to the following equations:

$$(8)\ p^t \cdot q^t = c(p^t)f(q^t) \qquad \text{for } t = 0,1.$$

Thus under the linear homogeneity assumption on the utility function f, observed period t expenditure on the N commodities (the left hand side of (8) above) is equal to the period t unit cost $c(p^t)$ of achieving one unit of utility times the period t utility level, $f(q^t)$, (the right hand side of (8) above). Obviously, we can identify the period t unit cost, $c(p^t)$, as the *period t price level* P^t and the period t level of utility, $f(q^t)$, as the *period t quantity level* Q^t. Note that P^t does not depend on q^t and Q^t does not depend on p^t. This is the main advantage of assuming homothetic preferences when we use the economic approach to index number theory: we can decompose period t aggregate value, $p^t \cdot q^t$, into the product of an aggregate period t price level, $P^t \equiv c(p^t)$, which just depends on the vector of period t commodity prices p^t, times an aggregate period t quantity level, $Q^t \equiv f(q^t)$, which just depends on the period t quantity vector q^t.

The linear homogeneity assumption on the consumer's preference function f leads to a simplification for the family of Konüs true cost of living indexes, $P_K(p^0,p^1,q)$, defined by (2) above. Using definition (2) for an arbitrary reference quantity vector q, we have:[21]

$$\begin{aligned}(9)\ P_K(p^0,p^1,q) &\equiv C[f(q),p^1]/C[f(q),p^0] \\ &= c(p^1)f(q)/c(p^0)f(q) \qquad \text{using (8) twice} \\ &= c(p^1)/c(p^0).\end{aligned}$$

Thus under the homothetic preferences assumption, the entire family of Konüs true cost of living indexes collapses to a single index, $c(p^1)/c(p^0)$, the ratio of the minimum costs of achieving unit utility level when the consumer faces period 1 and 0 prices respectively. Put another way, *under the homothetic preferences assumption,* $P_K(p^0,p^1,q)$ *does not depend on the reference quantity vector q.*

Substitute (9) into the inequalities (3) and (4), which, of course, are still valid under the homothetic preferences assumption. The resulting two inequalities simplify into the following two inequalities:

$$(10)\ p^1 \cdot q^1/p^0 \cdot q^1 \equiv P_P(p^0,p^1,q^0,q^1) \le c(p^1)/c(p^0) = P_K(p^0,p^1,q) \le P_L(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^0/p^0 \cdot q^0.$$

Thus under the homothetic preferences assumption, every Konüs true cost of living index $P_K(p^0,p^1,q)$ is bounded from above by the ordinary Laspeyres price index and bounded from below by the ordinary Paasche price index. Moreover, if we can observe the quantity vectors for periods 0 and 1 that are generated by a cost minimizing consumer that has homothetic preferences, then we can calculate the Laspeyres and Paasche indexes for this consumer and it must be the case that not only will the consumer's true cost of

[19] Economists will recognize the producer theory counterpart to the result $C(u,p) = uc(p)$: if a producer's production function f is subject to constant returns to scale, then the corresponding total cost function C(u,p) (where $u > 0$ is output and p is a vector of input prices) is equal to the product of the output level u times the unit cost c(p).

[20] Obviously, the utility function f determines the consumer's cost function C(u,p) as the solution to the cost minimization problem defined by (1). Then the unit cost function c(p) is defined as C(1,p). Thus f determines c. But we can also use c to determine f under appropriate regularity conditions. In the economics literature, this is known as *duality theory.* For additional material on duality theory and the properties of f and c, see Samuelson (1953), Shephard (1953), McFadden (1966) (1978) and Diewert (1974a; 110-113) (1993a; 107-123).

[21] Konus and Byushgens (1926; 168) were the first to establish this result. Pollak (1971) (1983) independently established this result later.

living index be bounded by these two indexes, it must also be the case that the Paasche index is equal to or less than the corresponding Laspeyres index.[22]

If we use the Konüs true cost of living index defined by the right hand side of (9) as our price index concept, then the corresponding *implicit quantity index* defined by deflating the value ratio by this price index is the following index:[23]

$$\begin{aligned}(11)\ Q(p^0,p^1,q^0,q^1,q) &\equiv p^1\cdot q^1/\{p^0\cdot q^0 P_K(p^0,p^1,q)\} \\ &= c(p^1)f(q^1)/\{c(p^0)f(q^0)P_K(p^0,p^1,q)\} && \text{using (8) twice} \\ &= c(p^1)f(q^1)/\{c(p^0)f(q^0)[c(p^1)/c(p^0)]\} && \text{using (9)} \\ &= f(q^1)/f(q^0).\end{aligned}$$

Thus under the homothetic preferences assumption, the *implicit quantity index* that corresponds to the true cost of living price index $c(p^1)/c(p^0)$ is the *utility ratio* $f(q^1)/f(q^0)$. Since the utility function is assumed to be homogeneous of degree one, this is a natural definition for a quantity index.

The bounds (3), (4) and (10) are the best nonparametric bounds that we can obtain on the Konüs true cost of living index $P_K(p^0,p^1,q)$. In subsequent sections, we will assume specific functional forms for f(q) or c(p) and find price indexes that are consistent with the chosen functional forms. Before this is done, we will require two additional results from microeconomic theory: Wold's Identity and Shephard's Lemma.

4. Wold's Identity and Shephard's Lemma

Instead of using the assumption that a household minimizes the cost of achieving a given utility level, one can use the assumption that the household maximizes utility subject to a budget constraint. Thus let $p^t >> 0_N$ and $q^t >> 0_N$ be the household's observed period t price and quantity vectors for t = 0,1. Define the household's *period t observed expenditure* e^t as

$$(12)\ e^t \equiv p^t\cdot q^t\ ; \qquad t = 0,1.$$

The household's *period t utility maximization problem* is defined as the following constrained maximization problem:

$$(13)\ \max_q \{f(q) : p^t\cdot q \le e^t\ ;\ q \ge 0_N\} \equiv g(e^t,p^t)\ ; \qquad t = 0,1.$$

Instead of assuming that the household's observed consumption vector q^t is a solution to the period t cost minimization problem defined earlier by (1), an equivalent assumption (under the section 2 regularity conditions on f) is that the observed q^t solves the period t utility maximization problem defined by (13). The period t optimized objective function in (13) is defined as the consumer's *indirect utility function*, $g(e^t,p^t)$.[24] This function is the maximum utility that the consumer can achieve given that he or she faces the period t price vector p^t and has "income" e^t to spend on the N commodities under consideration.

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[22] This result is due to Konüs and Byushgens (1926; 168).

[23] The Product Test from Chapter 2 is used to define the implicit quantity index that corresponds to the price index defined by (9).

[24] When the consumer's utility function f(q) is linearly homogeneous, concave and increasing in q, then the corresponding indirect utility function defined by (13) is equal to $u^t \equiv g(e^t,p^t) = e^t/c(p^t)$ since $e^t = u^t c(p^t)$. Thus if we set $e^t = 1$ in (13), we obtain the following explicit formula for calculating the unit cost function from a knowledge of f: $c(p^t) = 1/\max_q \{f(q) : p^t\cdot q \le 1\ ;\ q \ge 0_N\}$. Alternatively, we can define $c(p^t)$ in the usual way as $c(p^t) \equiv \min_q \{p^t\cdot q\ ;\ f(q) \ge 1;\ q \ge 0_N\}$.

If we assume that the observed period t consumption vector q^t is a solution to (13) for t = 0,1 and, in addition, f(q) has partial derivatives at q^0 and q^1, then it is possible to establish the following connection of these partial derivatives to the observed period 0 and 1 price vectors, p^0 and p^1.

Proposition 2 (Wold's (1944; 69-71) (1953; 145) Identity): Suppose that: (i) $p^0 \gg 0_N$, $p^1 \gg 0_N$; (ii) the consumer's utility function f(q) is increasing, continuous and concave for all $q \geq 0_N$; (iii) f(q) has first order partial derivatives at the points q^0 and q^1 and (iv) $q^t \gg 0_N$ is a solution to the household's period t utility maximization problem (13) for t = 0,1. Then the following equations hold:

$$(14)\quad p_i^t/p^t \cdot q^t = [\partial f(q^t)/\partial q_i]/\sum_{k=1}^{N} q_k^t \partial f(q^t)/\partial q_k ; \qquad t = 0,1 ; i = 1,\ldots,N$$

where $\partial f(q^t)/\partial q_i$ denotes the partial derivative of the utility function f with respect to the ith quantity q_i evaluated at the period t quantity vector q^t.

A proof of Proposition 2 may be found in the Appendix.

It is useful to express equations (14) using some alternative notation. Denote the N dimensional vector of first order partial derivatives of $f(q^t)$ as $\nabla f(q^t) \equiv [\partial f(q^t)/\partial q_1,\ldots,\partial f(q^t)/\partial q_N]$ for t = 0,1. Using this notation, equations (14) can be rewritten more succinctly as follows:

$$(15)\quad p^t/p^t \cdot q^t = \nabla f(q^t)/q^t \cdot \nabla f(q^t); \qquad t = 0,1.$$

If in addition to the assumptions made for Proposition 2, the utility function f(q) is linearly homogeneous, then it turns out that the terms $q^t \cdot \nabla f(q^t) = \Sigma_{n=1}^{N} q_n^t \partial f(q^t)/\partial q_n$ are equal to $f(q^t)$ for t = 0,1; i.e., if $f(\lambda q) = \lambda f(q)$ for all $\lambda > 0$, then we have the following identities: [25]

$$(16)\quad f(q^t) = q^t \cdot \nabla f(q^t) ; \qquad t = 0,1.$$

Substituting (16) into (15) leads to the following very useful equations:

$$(17)\quad p^t/p^t \cdot q^t = \nabla f(q^t)/f(q^t); \qquad t = 0,1.$$

We turn now to the implications of differentiability of the consumer's cost function, C(u,p), with respect to components of the commodity price vector p. If $C(f(q^t),p^t)$ has first order partial derivatives $\partial C(u^t,p^t)/\partial p_n$ for n = 1,...,N and t = 0,1 where $u^t = f(q^t)$, then we have the following Proposition:

Proposition 3 (Shephard's (1953; 11) Lemma): Suppose: (i) the utility function f(q) is increasing, continuous and concave in q; (ii) $p^t \gg 0_N$ for t = 0,1; (iii) $q^t \equiv [q_1^t,\ldots,q_N^t] > 0_N$ is a solution to the cost minimization problem defined by (1) for t = 0,1 and (iv) for $u^t \equiv f(q^t)$, the first order partial derivatives of $C(u^t,p^t)$ with respect to the components of p exist for t = 0,1, then:

$$(18)\quad q_n^t = \partial C(u^t,p^t)/\partial p_n ; \qquad n = 1,\ldots,N; t = 0,1.$$

Moreover, q^t is the unique solution to the cost minimization problem defined by (1) for t = 0,1.

A proof of Proposition 3 can be found in the Appendix.

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[25] Proof: partially differentiate both sides of $f(\lambda q) = \lambda f(q)$ with respect to λ and evaluate the resulting partial derivatives at $\lambda = 1$ and $q = q^t$. This is Euler's Theorem on linearly homogeneous functions.

Let the vector of first order partial derivatives of $C(u^t,p^t)$ with respect to the components of the price vector p be denoted as $\nabla_p C(u^t,p^t) \equiv [\partial C(u^t,p^t)/\partial p_1,...,\partial C(u^t,p^t)/\partial p_N]$ for t = 0,1. Using this notation, equations (18) can be written more succinctly as follows:

$$(19)\ q^t = \nabla_p C(u^t,p^t)\ ; \qquad t = 0,1.$$

The above result has the following implication: postulate a differentiable functional form for the cost function C(u,p) that satisfies the appropriate regularity conditions on the cost function listed below definitions (1) above. Then differentiating C(u,p) with respect to the components of the product price vector p generates the consumer's system of Hicksian cost minimizing input demand functions,[26] $x(u,p) \equiv \nabla_p C(u,p)$.

If we make the *homothetic preferences assumption* and assume that the utility function is linearly homogeneous, then using (7), we have $C(u^t,p^t) = u^t c(p^t) = p^t \cdot q^t$ where $u^t \equiv f(q^t)$ for t = 0,1. Shephard's Lemma (19) becomes $q^t = \nabla_p C(u^t,p^t) = u^t \nabla c(p^t)$ for t = 0,1. Using these equations, we find that:

$$(20)\ q^t/p^t \cdot q^t = \nabla c(p^t)/c(p^t)\ ; \qquad t = 0,1.$$

These equations will be very useful in subsequent sections of this chapter. Note the nice symmetry between the Shephard's Lemma equations (20) and the Wold Identity equations (17). In the following sections, specific functional forms for a linearly homogeneous utility function f(q) or for a unit cost function c(p) will be made and index number formulae that are exactly correct for these specific functional forms will be derived. Thus for the next two sections, we assume that the consumer's preference function is linearly homogeneous.

5. Superlative Indexes: The Fisher Ideal Index

Suppose the consumer has the following utility function:[27]

$$(21)\ f(q_1,\ldots,q_N) \equiv [\sum_{i=1}^N \sum_{k=1}^N a_{ik} q_i q_k]^{1/2}$$

where the N^2 parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all indices i and k. Thus there are only N(N+1)/2 independent parameters in this functional form. Note that the f(q) defined by (21) is linearly homogeneous.

Differentiating f(q) defined by (21) with respect to q_i yields the following equations:

$$(22)\ \partial f(q)/\partial q_i = (1/2)[\sum_{j=1}^N \sum_{k=1}^N a_{jk} q_j q_k]^{-1/2}\ 2\sum_{k=1}^N a_{ik} q_k\ ; \qquad i = 1,\ldots,N$$
$$= \sum_{k=1}^N a_{ik} q_k / f(q)$$

where it is necessary to use the symmetry conditions, $a_{ik} = a_{ki}$ for $1 \le i,k \le N$ in order to derive the first set of equations in (22) and the second set of equations follows using definition (21). Now evaluate the second set of equations in (22) at the observed period t quantity vector $q^t \equiv (q_1^t,\ldots,q_N^t)$ and divide both sides of the resulting equations by $f(q^t)$. We obtain the following equations:

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[26] Hicks (1946; 311-331) introduced this type of demand function into the economics literature.

[27] This functional form was indirectly introduced into the economics literature by Konüs and Byushgens (1926; 171) and Diewert (1974b; 123) (1976; 116). Pollak (1971), Afriat (1972; 45) and others also considered this functional form but did not work out the region where the utility function was well behaved.

$$(23)\quad [\partial f(q^t)/\partial q_i]/f(q^t) = \sum_{k=1}^{N} a_{ik}q_k^t/[f(q^t)]^2 \qquad t = 0,1\ ;\ i = 1,\ldots,N.$$

At this point, it is convenient to rewrite equations (23) using matrix notation. Thus in what follows, we interpret the vectors p^t and q^t for $t = 0,1$ as column vectors. Denote the transpose of a column vector x by x^T which is the row vector $[x_1,...,x_N]$. Define $A \equiv [a_{ik}]$ as the N by N matrix that has component a_{ik} in row i and column k of A. We assume that A is a symmetric matrix so that its transpose A^T is equal to the original matrix A. Thus using matrix notation, $f(q) \equiv [q^TAq]^{1/2}$ where $A = A^T$.

Using the above matrix notation, equations (22) can be written as the following vector equation:

$$(24)\quad \nabla f(q) = Aq/[q^TAq]^{1/2}$$
$$= Aq/f(q)$$

where the second equation in (24) follows because $f(q) \equiv [q^TAq]^{1/2}$. Using matrix notation, equations (23) become the following equations:

$$(25)\quad \nabla f(q^t)/f(q^t) = Aq^t/[f(q^t)]^2\ ; \qquad t = 0,1.$$

The f(q) defined by (21) is obviously linearly homogeneous. But we also need it to be positive (if $q > 0_N$), nondecreasing and concave in q over at least a subset of the nonnegative orthant. Suppose the symmetric matrix A has one positive eigenvalue with a corresponding strictly positive eigenvector and the remaining $N-1$ eigenvalues of A are either 0 or negative.[28] Then the f(q) defined by (21) will be positive, nondecreasing and concave over the region of regularity S defined as follows:[29]

$$(26)\quad S \equiv \{q : q \geq 0_N;\ Aq \geq 0_N;\ q^TAq > 0\}.$$

Now assume utility maximizing behavior for the consumer in periods 0 and 1; i.e., assume that $q^t >> 0_N$ is a solution to the period t utility maximization problem defined by (13) where $p^t >> 0_N$ and $e^t \equiv p^t\cdot q^t$ for $t = 0,1$ and the utility function f(q) is defined by (21) where the matrix A satisfies the above regularity conditions. Assume that q^0 and q^1 are both in the regularity region defined by (26). Since the utility function f defined by (21) is linearly homogeneous and differentiable over S, equations (17) (Wold's Identity) will hold for periods 0 and 1. Thus using (17), we have:

$$(27)\quad p^t/p^t\cdot q^t = \nabla f(q^t)/f(q^t); \qquad t = 0,1$$
$$= Aq^t/[f(q^t)]^2$$

where the second set of equations in (27) follows using equations (25).

As usual, the *Fisher* (1922) *ideal quantity index*, Q_F, is defined as $Q_F(p^0,p^1,q^0,q^1) \equiv [p^0\cdot q^1p^1\cdot q^1/p^0\cdot q^0p^1\cdot q^0]^{1/2}$.

Thus the square of the Fisher quantity index is equal to:

$$(28)\quad Q_F(p^0,p^1,q^0,q^1)^2 = p^0\cdot q^1p^1\cdot q^1/p^0\cdot q^0p^1\cdot q^0$$
$$= [p^0/p^0\cdot q^0]^Tq^1/[p^1/p^1\cdot q^1]^Tq^0$$
$$= \{q^{0T}A^Tq^1/f(q^0)^2\}/\{q^{1T}A^Tq^0/f(q^1)^2\} \qquad \text{using (27)}$$

[28] These conditions were imposed on A by Diewert (1976; 116).

[29] See Diewert and Hill (2010; 272-274) for a proof of this result. It turns out that $f(q) \equiv (q^TAq)^{1/2}$ is a concave function over the regularity region $S \equiv \{q\text{: } Aq \geq 0_N;\ q \geq 0_N \text{ and } q^TAq > 0\}$ if A has a positive eigenvalue with a corresponding strictly positive eigenvector and the other eigenvalues of A are negative or zero.

$= \{1/f(q^0)^2\}/\{1/f(q^1)^2\}$ since $q^{0T}A^Tq^1 = q^{1T}A^Tq^0$ using $A = A^T$
$= [f(q^1)/f(q^0)]^2$.

Taking positive square roots of both sides of (28) shows that, under the above hypotheses, the Fisher quantity index is *exactly* equal to the utility ratio, which is the *consumer's true volume index*; i.e., we have

(29) $Q_F(p^0,p^1,q^0,q^1) = f(q^1)/f(q^0)$.

Finally, use the following *Product Test* to define the price index that corresponds to the Fisher volume index:

(30) $P_F(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/\{p^0\cdot q^0 Q_F(p^0,p^1,q^0,q^1)\}$
$= [p^1\cdot q^0 p^1\cdot q^1/p^0\cdot q^0 p^0\cdot q^1]^{1/2}$ using definition (27).

Let c(p) be the unit cost function that corresponds to the utility function f(q) defined by (21).[30] Then for this c(p), equations (8) will hold; i.e., we will have $p^t\cdot q^t = f(q^t)c(p^t)$ for t = 0,1. Substitute these equations into the first line of (30) and we obtain the following equation:

(31) $P_F(p^0,p^1,q^0,q^1) \equiv c(p^1)f(q^1)/\{c(p^0)f(q^0)Q_F(p^0,p^1,q^0,q^1)\}$
$= c(p^1)f(q^1)/\{c(p^0)f(q^0)[f(q^1)/f(q^0)]\}$ using (29)
$= c(p^1)/c(p^0)$

which is the Konüs true cost of living index defined by (9) when preferences are homothetic. Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function defined by (21), the *Fisher ideal price index* P_F *is exactly equal to the true cost of living index*, $c(p^1)/c(p^0)$.

What is useful about the above results is that *it is not necessary to estimate econometrically the N(N+1)/2 parameters in the A matrix in order to find an estimator for the consumer's true cost of living index and the corresponding true volume index*.

There is another useful property of the utility function f(q) that is defined by (21) above: this function is a flexible functional form. Diewert (1974a; 113) defined a twice continuously differentiable linearly homogeneous function of N variables, f(q), to be a *flexible functional form* if it could approximate an arbitrary twice continuously differentiable linearly homogeneous function of N variables, say $f^*(q)$, to the second order around an arbitrary positive vector $q^* >> 0_N$. Thus if $f^*(q)$ is an arbitrary linearly homogeneous function that is twice continuously differentiable at the given arbitrary point $q^* >> 0_N$, then the linearly homogeneous twice continuously differentiable function f(q) is a *flexible functional form* if it has a sufficient number of free parameters so that the following $1 + N + N^2$ equations can be satisfied:

(32) $f(q^*) = f^*(q^*)$;
(33) $\nabla f(q^*) = \nabla f^*(q^*)$;
(34) $\nabla^2 f(q^*) = \nabla^2 f^*(q^*)$

[30] It may not be easy to find an explicit formula for c(p) in terms of the A matrix. If the matrix A has an inverse, then it can be shown that the unit cost function that corresponds to the utility function f(q) defined by (21) is $c(p) \equiv (p^TA^{-1}p)^{1/2}$ for price vectors p belonging to the region of prices defined by $S^* \equiv \{p: A^{-1}p \geq 0_N ; p \geq 0_N \text{ and } p^TA^{-1}p > 0_N\}$.

where $\nabla f(q^*) \equiv [\partial f(q^*)/\partial q_1,..., \partial f(q^*)/\partial q_N]^T$ is the vector of first order partial derivatives of f(q) evaluated at the point q^* and $\nabla^2 f(q^*) \equiv [\partial^2 f(q^*)/\partial q_i \partial q_k]$ is the N by N matrix of second order partial derivatives of f(q) evaluated at the point q^* where the element in row i and column k is $\partial^2 f(q^*)/\partial q_i \partial q_k$ for i,k = 1,...,N.

If f(q) is a flexible functional form, then it can approximate an arbitrary twice differentiable linearly homogeneous utility function very closely in a neighbourhood of any arbitrarily chosen point q^*. Thus if q^0 and q^1, the consumer's observed quantity choices for periods 0 and 1, are fairly close to each other, then a flexible utility function f(q) can approximate the consumer's true utility function $f^*(q)$ reasonably closely and so index numbers based on the assumption that the consumer maximizes utility using the utility function f(q) instead of the true one $f^*(q)$ will be able to provide a good approximation to the consumer's behavior.[31]

Proposition 4: The utility function defined as $f(q) \equiv (q^T A q)^{1/2}$ over the region S defined by (27) where $A = A^T$ is a flexible functional form.

For a proof, see the Appendix.

Diewert (1976; 117) termed an index number formula $Q_F(p^0,p^1,q^0,q^1)$ that was *exactly* equal to the true quantity index $f(q^1)/f(q^0)$ (where f is a flexible functional form) *a superlative index number formula*.[32] Equation (29) plus the fact that the homogeneous quadratic function f(q) defined by (21) is a flexible functional form shows that the Fisher ideal quantity index Q_F defined (27) is a superlative index number formula. Since the corresponding implicit Fisher ideal price index P_F satisfies (31) where c(p) is the unit cost function that is generated by the homogeneous quadratic utility function, we also call P_F a superlative index number formula.

There is a special case of the homogeneous quadratic preferences that will play an important role in later chapters and that is the case of *linear preferences*. Thus suppose that the consumer has the following linear utility function:

(35) $f(q) = \Sigma_{n=1}^N a_n q_n$

where the parameters a_n are positive. If N = 2, the indifference curves for a consumer with linear preferences are a family of parallel straight lines. The parameters a_n are *quality adjustment parameters*; i.e., a_n is the marginal increment to the consumer's welfare due to the consumption of an extra unit of the nth commodity. The absolute magnitudes of the a_n are not meaningful (since the units of measurement for utility are not observable) but the relative valuations a_n/a_k are meaningful. If a consumer has linear preferences, then we say that the N products are *perfect substitutes*.

To see that the preferences defined by (35) are a special case of the preferences defined by $f(q) = (q^T A q)^{1/2}$, let the matrix A be defined as the following rank 1 matrix:

(36) $A \equiv aa^T$

where the row vector a^T is defined as $a^T \equiv [a_1,...,a_N]$. Thus if f(q) is defined as $(q^T A q)^{1/2}$, then using the A defined by (36), we have $f(q) = (q^T A q)^{1/2} = (q^T aa^T q)^{1/2} = ([a^T q]^2)^{1/2} = a^T q = \Sigma_{n=1}^N a_n q_n$. With linear

[31] A first order approximation to a consumer's utility function will not be able to provide a first order approximation to the consumer's system of consumer demand functions. A first order approximation will not be able to adequately describe a consumer's reactions to changes in relative prices.
[32] Fisher (1922; 247) used the term superlative to describe the Fisher ideal price index. Thus Diewert adopted Fisher's terminology but attempted to give some precision to Fisher's definition of superlativeness. Fisher defined an index number formula to be superlative if it approximated the corresponding Fisher ideal results using his data set.

preferences, the consumer's utility maximization problem (13) becomes the following linear programming problem:

(37) $\max_q \{a^Tq : p^t\cdot q \leq e^t ; q \geq 0_N\} = \max_n \{e^ta_n/p_n^t ; n = 1,...,N\}$.

Thus if a consumer has linear preferences, then he or she will usually end up at a corner solution where one or more commodities are not consumed at all. However, if a utility maximizing consumer with linear preferences ends up choosing a positive amount of each commodity for period t, then it must be the case that $a_n/p_n^t = \lambda_t$ for n = 1,...,N. Thus if a utility maximizing consumer with linear preferences consumes positive amounts of all N products in periods 0 and 1, then it must be the case that prices are varying in a proportional manner over periods 0 and 1; i.e., the period t price vector p^t must be equal to $\lambda_t a$ where $\lambda_t > 0$ for t = 0,1.[33] It is not realistic to assume that prices vary in strict proportion over time but if the variation in prices is approximately proportional, then it is not unrealistic to assume that a utility maximizing consumer's preferences can be adequately approximated by a linear utility function. The assumption of linear preferences will play a large role in our treatment of quality change in Chapter 8. The important point to take away from this discussion of utility maximizing behavior where the consumer has a linear utility function is that the use of the Fisher quantity index to measure quantity change (and hence to measure welfare change) is perfectly consistent with the assumption of linear preferences.

It is possible to show that the Fisher ideal price index is a superlative index number formula by a different route. Instead of starting with the assumption that the consumer's utility function is the homogeneous quadratic function defined by (21), we can start with the assumption that the consumer's unit cost function is a homogeneous quadratic. Thus suppose that the consumer has the following unit cost function:

(38) $c(p_1,\ldots,p_N) \equiv [\sum_{i=1}^N\sum_{k=1}^N b_{ik}p_ip_k]^{1/2}$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all $1 \leq i, k \leq N$. Thus there are N(N+1)/2 independent parameters in the functional form for c(p) defined by (38).[34] Let $B \equiv [b_{ik}]$ be the N by N matrix that has b_{ik} in row i and column k of B. Then $c(p_1,...,p_N) = c(p)$ can be defined as follows:

(39) $c(p) = (p^TBp)^{1/2} ; B = B^T$.

Using the above matrix notation, the vector of first order partial derivatives of the unit cost function defined by (39) is equal to the following expression:

(40) $\nabla c(p) = Bp/[p^TBp]^{1/2}$
$= Bp/c(p)$

where the second equation in (40) follows because $c(p) \equiv [p^TBp]^{1/2}$. Now evaluate (40) when $p = p^t$ for t = 0,1, where $p^t >> 0_N$ is the positive period t price vector facing the consumer. Divide the resulting equation t by $c(p^t)$ for t = 0,1 and we obtain the following equations:

(41) $\nabla c(p^t)/c(p^t) = Bp^t/[c(p^t)]^2$; t = 0,1.

[33] In this case, the solution set to the period t utility maximization problem defined by (37) is the set $\{q : p^t\cdot q = e^t ; q \geq 0_N\}$. This analysis for the case of a linear utility function follows that of Pollak (1971) (1983).
[34] This functional form for a unit cost function is essentially due to Konüs and Byushgens (1926; 168) and they showed the relationship of this functional form to the Fisher ideal price index. See also Diewert (1976) and Diewert and Hill (2010).

The c(p) defined by (39) is obviously linearly homogeneous. But we also need it to be positive (if $p > 0_N$), nondecreasing and concave in p over at least a subset of the nonnegative orthant. Suppose the symmetric matrix B has one positive eigenvalue with a corresponding strictly positive eigenvector and the remaining N−1 eigenvalues of B are either 0 or negative.[35] Then c(p) defined by (39) will be positive, nondecreasing and concave over the region of regularity S^* defined as follows:[36]

(42) $S^* \equiv \{p : p \geq 0_N;\ Bp \geq 0_N;\ p^TBp > 0\}$.

Now assume cost minimizing behavior for the consumer in periods 0 and 1; i.e., assume that $q^t >> 0_N$ is a solution to the consumer's period t cost minimization problem when the consumer faces the price vector $p^t >> 0_N$ for t = 0,1. Assume that the consumer has homothetic preferences and the consumer's unit cost function is c(p) defined by (39). Finally assume that p^t belongs to the regularity region for prices S^* defined by (42) for t = 0,1. Shephard's Lemma (20) applied to the c(p) defined by (39) gives us the following equations:

$$
\begin{aligned}
(43)\ q^t/p^t\cdot q^t &= \nabla c(p^t)/c(p^t) && t = 0,1 \\
&= Bp^t/[c(p^t)]^2 && \text{using (41).}
\end{aligned}
$$

Recall that the Fisher (1922) ideal price index was defined earlier by (30); i.e., $P_F(p^0,p^1,q^0,q^1)$ was defined as $[p^1\cdot q^0p^1\cdot q^1/p^0\cdot q^0p^0\cdot q^1]^{1/2}$. Thus the square of the Fisher price index is equal to:

$$
\begin{aligned}
(44)\ [P_F(p^0,p^1,q^0,q^1)]^2 &= p^1\cdot q^0p^1\cdot q^1/p^0\cdot q^0p^0\cdot q^1 \\
&= p^{1T}[q^0/p^0\cdot q^0]/p^{0T}[q^1/p^1\cdot q^1] \\
&= p^{1T}\{Bp^0/[c(p^0)]^2\}/p^{0T}\{Bp^1/[c(p^1)]^2\} \\
&= \{1/c(p^0)^2\}/\{1/c(p^1)^2\} && \text{since } p^{1T}Bp^0 = p^{0T}Bp^1 \text{ using } B = B^T \\
&= [c(p^1)/c(p^0)]^2.
\end{aligned}
$$

Taking positive square roots of both sides of (44) shows that, under the above hypotheses, the Fisher price index is *exactly* equal to the unit cost ratio, which is the *consumer's true cost of living index* in the case of homothetic preferences i.e., we have

(45) $P_F(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0)$.

Finally, use the *Product Test* to define a quantity index Q_F^* that corresponds to the Fisher price index defined by (30):

$$
\begin{aligned}
(46)\ Q_F^*(p^0,p^1,q^0,q^1) &\equiv p^1\cdot q^1/\{p^0\cdot q^0P_F(p^0,p^1,q^0,q^1)\} \\
&= [p^0\cdot q^1p^1\cdot q^1/p^0\cdot q^0p^1\cdot q^0]^{1/2} && \text{using definition (30)} \\
&= Q_F(p^0,p^1,q^0,q^1)
\end{aligned}
$$

where $Q_F(p^0,p^1,q^0,q^1)$ was defined earlier by (27). Thus the implicit quantity index that corresponds to the Fisher price index defined by (30) is the Fisher quantity index defined by (27) and the implicit price index that corresponds to the Fisher quantity index defined by (27) is the Fisher price index.

Since preferences are homothetic, equations (8) will hold; i.e., we have $p^t\cdot q^t = c(p^t)f(q^t)$ for t = 0,1. From (46), we have

[35] These regularity conditions on B are counterparts to our earlier regularity conditions that were imposed on A.
[36] Again see Diewert and Hill (2010; 272-274) for a proof of this result.

$$(47)\ Q_F(p^0,p^1,q^0,q^1) = p^1\cdot q^1/\{p^0\cdot q^0 P_F(p^0,p^1,q^0,q^1)\}$$
$$= c(p^1)f(q^1)/\{c(p^0)f(q^0)P_F(p^0,p^1,q^0,q^1)\} \qquad \text{using (8)}$$
$$= c(p^1)f(q^1)/\{c(p^0)f(q^0)[c(p^1)/c(p^0)]\} \qquad \text{using (45)}$$
$$= f(q^1)/f(q^0).$$

Again, the Fisher quantity index is equal to the utility ratio under our assumptions on consumer behavior.

The proof of Proposition 4 can be adapted to show that $c(p) \equiv (p^TBp)^{1/2}$ is a flexible functional form. Thus we have again shown that the Fisher ideal price index is a *superlative index*; i.e., it is exact for a flexible functional form for the unit cost function.

An important special case of this functional form is the case where the matrix B is equal to a rank 1 matrix; i.e., suppose B is given by:

$$(48)\ B = bb^T$$

where $b^T \equiv [b_1,...,b_N]$ where the $b_n > 0$ for n = 1,...,N. Using Shephard's Lemma (19) for the cost function $C(u^t,p^t) \equiv u^tc(p^t) = u^t(p^Tbb^Tp)^{1/2} = u^tb^Tp$ for periods t = 0,1 leads to the following equations to describe the period t demand vectors, q^t:

$$(49)\ q^t = u^t\nabla c(p^t) = u^tb\ ; \qquad t = 0,1.$$

Thus $q_n^1/q_n^0 = u^1/u^0$ for n = 1,...,N and the demand for each commodity moves in a proportional manner over the two periods. Note also that *changes in commodity prices do not change the demands*. Thus preferences are such that the consumer will not substitute cheaper products for more expensive products as prices change over time. For N = 2, the consumer's family of indifference curves are L shaped. The preferences that are represented by the cost function $uc(p) = ub\cdot p = u\ \Sigma_{n=1}^N b_np_n$ are called *no substitution* or *Leontief preferences* in the economic literature.[37] These preferences are completely opposite to linear preferences where products were perfect substitutes. What is interesting is that the Fisher ideal price and quantity indexes are completely consistent with utility maximizing behavior for both types of preferences.

We conclude this section by showing how a linearly homogeneous utility function f(q) can be derived from its dual unit cost function c(p). Suppose that the unit cost function c(p) is given and it is nonnegative, increasing, linearly homogeneous, concave and continuous for $q \geq 0_N$. Let $q^* >> 0_N$. The utility level $u \equiv f(q)$ that corresponds to c(p) must satisfy the inequality $c(p)u \leq p\cdot q^*$ for all $p > 0_N$. Since c(p) and $p\cdot q^*$ are linearly homogeneous in p, we can replace the set of p such that $p > 0_N$ by the set $\{p : p \geq 0_N;\ p\cdot q^* = 1\}$. Thus the inequalities $c(p)u \leq p\cdot q^*$ for all $p > 0_N$ are equivalent to the inequalities $c(p)u \leq 1$ for all $p \geq 0_N$; $p\cdot q^* = 1$. Since c(p) will be positive for all such p vectors, this last set of inequalities can be replaced by $u \leq 1/c(p)$ for all $p \geq 0_N$; $p\cdot q^* = 1$. The biggest such $u = f(q^*)$ that will satisfy all of the inequalities is given by $1/c(p^*)$ where p^* solves the concave programming problem: $\max_p \{c(p) : p\cdot q^* = 1;\ p \geq 0_N\}$. Thus we have the following representation for $f(q^*)$ in terms of c(p):[38]

$$(50)\ f(q^*) = 1/\max_p \{c(p) : p\cdot q^* = 1;\ p \geq 0_N\}.$$

We can use the above formula in order to calculate the utility function that corresponds to the no substitution unit cost function defined as $c(p) \equiv b\cdot p$. The constrained maximization problem that appears in (50) for this unit cost function is:

---

[37] See Diewert (1971).

[38] This formula may be found in Diewert (1974a; 112).

(51) $\max_p \{\Sigma_{n=1}^N b_np_n : \Sigma_{n=1}^N q_n^*p_n = 1; p \geq 0_N\} = \max_n \{b_n/q_n^* : n = 1,...,N\}$.

Since all of the numbers b_n and q_n^* are assumed to be positive, $1/\max_n \{b_n/q_n^* : n = 1,...,N\}$ will equal $\min_n \{q_n^*/b_n : n = 1,...,N\}$. Using this equality and (51), (50) becomes the following explicit representation for the *no substitution preference function*:

(52) $f(q^*) = \min_n \{q_n^*/b_n : n = 1,...,N\}$.

Another special case of the homogeneous quadratic unit cost function defined by (39) is the case where the B matrix has an inverse.[39] Let $c(p) = (p^TBp)^{1/2}$ where $B = B^T$ and B has one positive eigenvalue with a strictly positive eigenvector and the remaining N− 1 eigenvalues of B are negative. In this case, B has full rank and so B^{-1} exists. We show in the Appendix how a modification of formula (50) can be used to calculate $f(q^*)$ for some $q^* >> 0_N$.

Proposition 5: Let $c(p) = (p^TBp)^{1/2}$ where $B = B^T$ and B has one positive eigenvalue with a strictly positive eigenvector and the remaining N− 1 eigenvalues of B are negative. Let $q^* >> 0_N$ and suppose also that $B^{-1}q^* >> 0_N$. Let f(q) be the utility function that is dual to c(p). Then $f(q^*) = (q^{*T}B^{-1}q^*)^{1/2}$.

In the following sections, we will exhibit some additional exact index number formulae.

6. Quadratic Means of Order r and the Walsh Index

It turns out that there are many other superlative index number formulae; i.e., there exist many quantity indexes $Q(p^0,p^1,q^0,q^1)$ that are exactly equal to $f(q^1)/f(q^0)$ and many price indexes $P(p^0,p^1,q^0,q^1)$ that are exactly equal to $c(p^1)/c(p^0)$ where the aggregator function f or the unit cost function c is a flexible functional form. We will define two families of superlative indexes in this section.

Suppose the consumer has the *following quadratic mean of order r utility function*:[40]

(53) $f^r(q_1,\ldots,q_N) \equiv [\sum_{i=1}^N \sum_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}]^{1/r}$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. It turns out that $f^r(q)$ is a flexible functional form.

Proposition 6: For each $r \neq 0$, $f^r(q)$ defined by (53) is a flexible functional form.

See the Appendix for a proof. From the proof of Proposition 6, it can be seen that the quadratic mean of order r utility function defined by (53) can adequately represent the preferences for a utility maximizing consumer for quantity vectors q in a neighbourhood around any strictly positive q^* since there will be a neighbourhood around q^* where $f^r(q)$ will be concave and increasing. Hence for this region, $f^r(q)$ can provide an adequate approximation to arbitrary differentiable homothetic preferences. However, this neighbourhood may not be very large and this point should be kept in mind.[41]

Let $r \neq 0$ and define the *quadratic mean of order r quantity index* Q^r by:

---

[39] This is the model of consumer behavior considered by Konüs and Byushgens (1926; 168).

[40] This terminology is due to Diewert (1976; 129). When r = 1, $f^r(q)$ simplifies into the Generalized Linear Utility Function; see Diewert (1971).

[41] This index number formula is due to Diewert (1976; 130).

(54) $Q^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^N s_i^0(q_i^1/q_i^0)^{r/2}\}^{1/r}\{\sum_{i=1}^N s_i^1(q_i^1/q_i^0)^{-r/2}\}^{-1/r}$

where $s_i^t \equiv p_i^t q_i^t/\sum_{k=1}^N p_k^t q_k^t$ is the period t expenditure share for commodity i for i = 1,...,N and t = 0,1. It can be verified that when r = 2, Q^r simplifies into Q_F, the Fisher ideal quantity index.

Proposition 7: Let $r \neq 0$ and define $f^r(q)$ by (53) over an open convex set S of positive quantity vectors q. We assume that $f^r(q)$ defined by (53) is positive, increasing and concave over S.[42] Finally assume that q^t solves the following period t local utility maximization problem where $p^t >> 0_N$ and $e^t > 0$ for t = 0,1:

(55) $\max_q \{f^r(q) : p^t \cdot q \leq e^t \,;\, q \in S\}$.

Then $Q^r(p^0,p^1,q^0,q^1)$ defined by (54) is exact for $f^r(q)$ defined by (53); i.e., we have

(56) $Q^r(p^0,p^1,q^0,q^1) = f^r(q^1)/f^r(q^0)$.

See the Appendix for a proof of Proposition 7.

Thus under the assumption that the consumer engages in utility maximizing behavior during periods 0 and 1 and has local preferences over the N commodities that correspond to the utility function defined by (53) for a region that includes q^0 and q^1, then the quadratic mean of order r quantity index Q^r is *exactly* equal to the true quantity index, $f^r(q^1)/f^r(q^0)$.[43] Since Q^r is exact for f^r and f^r is a flexible functional form, we see that the quadratic mean of order r quantity index Q^r is a *superlative index* for each $r \neq 0$. Thus there are an infinite number of superlative quantity indexes.[44]

For each quantity index Q^r, we can use the product test in order to define the corresponding *implicit quadratic mean of order r price index* P^{r*}:

(57) $P^{r*}(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/\{p^0\cdot q^0 Q^r(p^0,p^1,q^0,q^1)\}$
$= c^{r*}(p^1)/c^{r*}(p^0)$

where c^{r*} is the unit cost function that corresponds to the aggregator function f^r defined by (53) above. For each $r \neq 0$, the implicit quadratic mean of order r price index P^{r*} is also a superlative index.

When r = 2, as noted earlier, Q^r defined by (54) simplifies to Q_F, the Fisher ideal quantity index and P^{r*} defined by (57) simplifies to P_F, the Fisher ideal price index. When r = 1, Q^r defined by (54) simplifies to:

(58) $Q^1(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^N s_i^0(q_i^1/q_i^0)^{1/2}\}/\{\sum_{i=1}^N s_i^1(q_i^1/q_i^0)^{-1/2}\}$
$= \{[\sum_{i=1}^N p_i^0 q_i^0/\sum_{i=1}^N p_i^0 q_i^0](q_i^1/q_i^0)^{1/2}\}/\{[\sum_{i=1}^N p_i^1 q_i^1/\sum_{i=1}^N p_i^1 q_i^1](q_i^1/q_i^0)^{-1/2}\}$
$= \{\sum_{i=1}^N p_i^0(q_i^0 q_i^1)^{1/2}/p^0\cdot q^0\}/\{\sum_{i=1}^N p_i^1(q_i^0 q_i^1)^{1/2}/p^1\cdot q^1\}$
$= [p^1\cdot q^1/p^0\cdot q^0]/P_W(p^0,p^1,q^0,q^1)$

[42] Using the techniques described in Blackorby and Diewert (1979), the utility function $f^r(q)$ which satisfies the appropriate regularity conditions over the set S can be extended to preferences that are defined over $q \geq 0_N$.
[43] See Diewert (1976; 130).
[44] However, as r becomes large in magnitude, the region where $f^r(q)$ can approximate a well behaved utility function will tend to shrink. In the limiting cases where r tends to plus or minus infinity, Hill (2006) showed that $f^r(q)$ loses its flexibility property. Thus it is recommended that $Q^r(p^0,p^1,q^0,q^1)$ only be used for r small in magnitude.

where P_W is the *Walsh* (1901) (1921) *price index* defined in Chapter 2. Thus P^{1*} is equal to P_W, the *Walsh price index*, and hence it is also a superlative price index.[45]

Suppose the consumer has the *following quadratic mean of order r unit cost function*:[46]

(59) $c^r(p_1,\ldots,p_N) \equiv [\sum_{i=1}^N \sum_{k=1}^N b_{ik} p_i^{r/2} p_k^{r/2}]^{1/r}$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Note that when $r = 2$, c^r equals the homogeneous quadratic unit cost function defined by (39) above.[47]

Proposition 8: For each $r \neq 0$, $c^r(p)$ defined by (59) is a flexible functional form.[48]

The proof of this proposition is analogous to the proof of Proposition 6: just replace q by p and replace $f^r(q)$ by $c^r(p)$.

Since $c^r(p)$ is unlikely to be a well behaved unit cost function over the entire set of positive price vectors, we need a method for recovering preferences defined by a unit cost function defined over a smaller set of prices where $c^r(p)$ satisfies the necessary conditions for unit cost function; i.e., where it is increasing and concave.[49] Thus let S^* be a set of prices that satisfies the following conditions:[50]

(60) S^* is a set of N dimensional vectors that has the following properties: (i) if $p \in S^*$, then $p >> 0_N$; (ii) S^* is an open set [51]; (iii) S^* is a convex set [52]; (iv) S^* is a cone [53]; (v) if p belongs to S^*, then $\nabla c^r(p) >> 0_N$ and (vi) $c^r(p)$ is a concave function over S^*.

We need to find the utility function $f^{r*}(q)$ that is consistent with the unit cost function $c^r(p)$ defined by (59) over S^*. We can find this corresponding utility function but it will not be defined over all nonnegative quantity vectors, $q \geq 0_N$. It will be defined over the set S defined as follows:

(61) $S \equiv \{q: q = \lambda \nabla c^r(p); \lambda > 0; p \in S^*\}$.

It can be seen using property (v) in (60) that S will also be a cone and moreover, if $q \in S$, then $q >> 0_N$.

If S^* turns out to be the interior of the nonnegative orthant, then the $f^{r*}(q^*)$ that is generated by the unit cost function $c^r(p)$ for $q^* >> 0_N$ can be defined as follows:

---

[45] For r = 1, the utility function defined by (53) turns out to be the Generalized Linear function that was introduced to the economics literature by Diewert (1971).
[46] This terminology is due to Diewert (1976; 130). This unit cost function was first defined by Denny (1974). We restrict p to belong to a set of prices S^* that is defined in Proposition 8.
[47] When r = 1, $c^r(p)$ defined by (59) becomes the Generalized Leontief functional form for a cost function; see Diewert (1971).
[48] See Diewert (1976; 130).
[49] The $c^r(p)$ defined by (59) is automatically linearly homogeneous over the set of prices where it is positive, increasing and concave since linear homogeneity is imposed on the functional form by its definition.
[50] Using the techniques described in Blackorby and Diewert (1979), if $c^r(p)$ is linearly homogeneous, increasing and concave over S^*, then the domain of definition of $c^r(p)$ can be extended to all $p \geq 0_N$.
[51] This means: if $p \in S^*$, then there exists a $\delta > 0$ such that the open ball of radius δ, $B_\delta(p)$, also belongs to S^* where $B_\delta(p) \equiv \{x: (x-p)\cdot(x-p) < \delta^2\}$.
[52] This means: if p^1 and p^2 belong to S^* and $0 < \lambda < 1$, then $\lambda p^1 + (1-\lambda)p^2$ also belongs to S^*.
[53] If p belongs to S^* then λp also belongs to S^* for all $\lambda > 0$.

(62) $f^{r*}(q^*) \equiv \max_{u>0,p} \{u: c^r(p)u \leq p \cdot q^*;\ p > 0_N\}$
$= \max_{u>0,p} \{u: c^r(p)u \leq e;\ e = p \cdot q^*;\ p > 0_N\}$ where e > 0 is an arbitrary positive number[54]
$= \max_{u>0,p} \{u: u \leq e/c^r(p);\ e = p \cdot q^*;\ p \geq 0_N\}$[55]
$= e/\max_{p} \{c^r(p);\ e = p \cdot q^*;\ p \geq 0_N\}$.

However, in general, $c^r(p)$ will not be a well behaved unit cost function for all $p > 0_N$. Thus in the following definition for $f^{r*}(q^*)$, we restrict p to belong to the set S^* that has the properties listed in (60) above and we restrict q^* to belong to S where S is defined by (61). Thus let q^* belong to S and define $f^{r*}(q^*)$ as follows:[56]

(63) $f^{r*}(q^*) \equiv \max_{u>0,p} \{u: c^r(p)u \leq p \cdot q^*;\ p \in S^*\}$
$= \max_{u>0,p} \{u: c^r(p)u \leq e;\ e = p \cdot q^*;\ p \in S^*\}$ where e > 0 is an arbitrary positive number
$= \max_{u>0,p} \{u: u \leq e/c^r(p);\ e = p \cdot q^*;\ p \in S^*\}$
$= e/\max_{p} \{c^r(p);\ e = p \cdot q^*;\ p \in S^*\}$.

The above representation for $f^{r*}(q^*)$ will be used in the proof of the following Proposition:

Proposition 9: Let $c^r(p)$ be defined by (59) for $p \in S^*$ where S^* is defined by (60). Let $e^t > 0$ and $p^t \in S^*$. Define q^t as

(64) $q^t \equiv e^t \nabla c^r(p^t)/c^r(p^t)$.

Then p^t is a solution to $\max_{p} \{c^r(p);\ e = p \cdot q^t;\ p \in S^*\}$. Define $f^{r*}(q)$ by (63) (with $e = e^t$) for $q \in S$ where S is defined by (61). Then $f^{r*}(q^t)$ is equal to the following expression:

(65) $f^{r*}(q^t) = e^t/c^r(p^t)$.

Finally, the q^t defined by (64) is a solution to the consumer's local utility maximization problem defined as follows:

(66) $\max_{q} \{f^{r*}(q) : p^t \cdot q = e^t;\ q \in S\}$.

For a proof of Proposition 9, see the Appendix.

Note that using (64), we have:

(67) $p^t \cdot q^t = p^t \cdot e^t \nabla c^r(p^t)/c^r(p^t)$
$= e^t c^r(p^t)/c^r(p^t)$ since $c^r(p^t) = p^t \cdot e^t \nabla c^r(p^t)$
$= e^t$
$= f^{r*}(q^t)c^r(p^t)$ using (65).

Using (64) and $p^t \cdot q^t = e^t$, we also have the Shephard's Lemma equality:[57]

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[54] The number e is a fixed positive number. In order to justify moving from the first equality in (62) to the second equality, we need to use the fact that $c^r(p)$ is linearly homogeneous.
[55] Since $e > 0$, $p \geq 0_N$, $q^* >> 0_N$ and $p \cdot q^* = e$, we can replace the constraints $p > 0_N$ by $p \geq 0_N$.
[56] Again, using the methods described in Blackorby and Diewert (1979), the domain of definition for $f^{r*}(q)$ can be extended to $q \geq 0_N$. However, for $q > 0_N$ but $q \notin S$, the extended $f^{r*}(q)$ may not represent the true preferences of the consumer.
[57] Recall (20) above.

(68) $q^t/p^t \cdot q^t = \nabla c^r(p^t)/c^r(p^t)$.

We will relate the preferences defined by $c^r(p)$ to the following price index formula. Let $r \neq 0$ and define the *quadratic mean of order r price index* P^r by:

$$(69)\quad P^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^N s_i^0(p_i^1/p_i^0)^{r/2}\}^{1/r}\{\sum_{i=1}^N s_i^1(p_i^1/p_i^0)^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^t q_i^t/p^t \cdot q^t$ is the period t expenditure share for commodity i for i = 1,...,N and t = 0,1. It can be verified that when r = 2, P^r simplifies into P_F, the Fisher ideal price index.

Proposition 10: Let $r \neq 0$ and assume that $c^r(p)$ given by (59) is defined over a set S^* which satisfies conditions (60). Define the set S by (61) and define the locally dual utility function $f^{r*}(q^*)$ for $q^* \in S$ by (63) for any e > 0. Let e^t equal the consumer's "income" in period t that is allocated to spending on the N commodities for t = 0,1. Let p^0 and p^1 belong to S^* and define q^0 and q^1 by:

$$(70)\quad q^t \equiv e^t \nabla c^r(p^t)/c^r(p^t)\ ; \qquad t = 0,1.$$

Then q^t solves the local utility maximization problem, $\max_q \{f^{r*}(q)\ ;\ p^t \cdot q = e^t;\ q \in S\}$, for t = 0,1. Moreover, P^r defined by (69) is exact for the preferences defined by $f^{r*}(q)$ over the set S; i.e., we have

$$(71)\quad P^r(p^0,p^1,q^0,q^1) = c^r(p^1)/c^r(p^0).$$

See the Appendix for a proof.

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the unit cost function defined by (59), the quadratic mean of order r price index P^r is *exactly* equal to the true price index, $c^r(p^1)/c^r(p^0)$.[58] Since P^r is exact for c^r and c^r is a flexible functional form, we see that the quadratic mean of order r price index P^r is a *superlative index* for each $r \neq 0$. Thus there is an infinite number of superlative price indexes.

For each price index P^r, we can use the product test in order to define the corresponding *implicit quadratic mean of order r quantity index* Q^{r*}:

$$(72)\quad Q^{r*}(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1/\{p^1 \cdot q^1 P^r(p^0,p^1,q^0,q^1)\} = f^{r*}(q^1)/f^{r*}(q^0)$$

where f^{r*} is the utility function that corresponds to the unit cost function c^r defined by (53) above. For each $r \neq 0$, the implicit quadratic mean of order r quantity index Q^{r*} is also a superlative index.

When r = 2, P^r defined by (69) simplifies to P_F, the Fisher ideal price index and Q^{r*} defined by (72) simplifies to Q_F, the Fisher ideal quantity index. When r = 1, P^r simplifies to:

$$\begin{aligned}(73)\quad P^1(p^0,p^1,q^0,q^1) &\equiv \{\sum_{i=1}^N s_i^0\, (p_i^1/p_i^0)^{1/2}\}/\{\sum_{i=1}^N s_i^1\, (p_i^1/p_i^0)^{-1/2}\} \\ &= \{[\sum_{i=1}^N p_i^0 q_i^0/\, p^0 \cdot q^0](p_i^1/p_i^0)^{1/2}\}/\{[\, p^1 \cdot q^1/\sum_{i=1}^N p_i^1 q_i^1](p_i^1/p_i^0)^{-1/2}\} \\ &= \{\sum_{i=1}^N q_i^0(p_i^0 p_i^1)^{1/2}/p^0 \cdot q^0\}/\{\sum_{i=1}^N q_i^1(p_i^0 p_i^1)^{1/2}/p^1 \cdot q^1\} \\ &= [p^1 \cdot q^1/p^0 \cdot q^0]/Q_W(p^0,p^1,q^0,q^1)\end{aligned}$$

[58] See Diewert (1976; 133-134).

where Q_W is the *Walsh quantity index*. Thus Q^{1*} is equal to Q_W, the Walsh quantity index, and hence it is also a superlative quantity index.[59]

The results in this section can be summed up as follows:

- Superlative indexes are nice in theory since they enable statisticians to compute price and volume indexes that are consistent with the economic approach to index number theory where the underlying preference functions and their corresponding unit cost functions can approximate arbitrary differentiable preferences to the second order around an arbitrary point. These superlative indexes do not require econometric estimation in order to be implemented.
- These indexes are consistent with a wide range of substitution responses on the part of consumers to changes in prices.
- However, superlative indexes have the disadvantage that the quantity and price regions where the underlying preferences are well behaved is generally not known to the statistician. If there are large fluctuations in prices and quantities across periods, then the various exact indexes may no longer be exact![60]
- It is of some comfort that the Fisher and Walsh indexes that have been recommended as “best” from the approaches to index number theory that were described in previous chapters emerge as being “best” from the economic approach as well.

We turn our attention to yet another superlative index number formula.

7. Superlative Indexes: The Törnqvist Theil Index

In this section, we will revert to the assumptions made on the consumer in section 2 above. In particular, we do not assume that the consumer’s utility function f is necessarily linearly homogeneous as in sections 3-6 above.

Before we derive our main result, we require a preliminary result. Suppose the function of N variables, $f(z_1,\ldots,z_N) \equiv f(z)$, is quadratic; i.e.,

(74) $f(z_1,\ldots,z_N) \equiv a_0 + \sum_{i=1}^N a_i z_i + (1/2) \sum_{i=1}^N \sum_{k=1}^N a_{ik}z_iz_k$; $a_{ik} = a_{ki}$ for all i and k,

where the a_i and the a_{ik} are constants. Let $f_i(z)$ denote the first order partial derivative of f evaluated at z with respect to the ith component of z, z_i. Let $f_{ik}(z)$ denote the second order partial derivative of f with respect to z_i and z_k. Then it is well known that the second order Taylor series approximation to a quadratic function is *exact*; i.e., if f is defined by (74) above, then for any two points, z^0 and z^1, we have

$$\begin{aligned}(75)\quad f(z^1) - f(z^0) &= \sum_{i=1}^N f_i(z^0)[z_i^1-z_i^0] + (1/2) \sum_{i=1}^N \sum_{k=1}^N f_{ik}(z^0)[z_i^1-z_i^0][z_k^1-z_k^0] \\ &= \nabla f(z^0)\cdot[z^1 - z^0] + (1/2)[z^1 - z^0]^T\nabla^2 f(z^0)^T[z^1 - z^0].\end{aligned}$$

[59] The Walsh quantity index is a useful one for national income accountants since it is a superlative index but it is also an index that defines real output for periods 0 and 1 as $Q^t \equiv \Sigma_{n=1}^N (p_n^0p_n^1)^{1/2}q_n^t$ for t = 0,1. Thus the price weights are *constant* over the two periods and the quantity aggregate Q^t for period t is *linear* in the period t quantities, q_n^t. See Diewert (1996).

[60] This warning is particularly relevant for the use of the quadratic mean of order r functional forms where r is large in magnitude. The regularity regions for these functions will tend to shrink as r approaches plus or minus infinity.

It is less well known that *an average of two first order Taylor series approximations* to a quadratic function is also *exact*; i.e., if f is defined by (74) above, then for any two points, z^0 and z^1, we have[61]

(76) $f(z^1) - f(z^0) = (1/2)\sum_{i=1}^N [f_i(z^0) + f_i(z^1)][z_i^1 - z_i^0] = (1/2)[\nabla f(z^0) + \nabla f(z^1)]^T[z^1 - z^0]$.

Diewert (1976; 118) and Lau (1979) showed that equation (76) characterized a quadratic function and called the equation the *quadratic approximation lemma*. We will refer to (76) as the *quadratic identity*.

We now suppose that the consumer's *cost function*,[62] C(u,p), has the following *translog functional form*:[63]

(77) $\ln C(u,p) \equiv a_0 + \sum_{i=1}^N a_i \ln p_i + (1/2) \sum_{i=1}^N \sum_{k=1}^N a_{ik} \ln p_i \ln p_k$
$+ b_0 \ln u + \sum_{i=1}^N b_i \ln p_i \ln u + (1/2)\, b_{00}[\ln u]^2$

where ln is the natural logarithm function and the parameters a_i, a_{ik}, and b_i satisfy the following restrictions:

(78) $a_{ik} = a_{ki}$; $i,k = 1,\ldots,N$;
(79) $\sum_{i=1}^N a_i = 1$;
(80) $\sum_{i=1}^N b_i = 0$;
(81) $\sum_{k=1}^N a_{ik} = 0$; $i = 1,\ldots,N$.

The parameter restrictions (78)-(81) ensure that C(u,p) defined by (77) is linearly homogeneous in p, a property that a cost function must have. It can be shown that the translog cost function defined by (77)-(81) can provide a second order Taylor series approximation to an arbitrary cost function.[64]

We assume that the consumer has preferences that correspond to the translog cost function and that the consumer engages in cost minimizing behavior during periods 0 and 1. Let p^0 and p^1 be the period 0 and 1 observed price vectors[65] and let q^0 and q^1 be the period 0 and 1 observed quantity vectors. Using the assumption of cost minimizing behavior, we have:

(82) $C(u^0,p^0) = p^0 \cdot q^0$ and $C(u^1,p^1) = p^1 \cdot q^1$

where C is the translog cost function defined above. We can also apply Shephard's Lemma[66] to $C(u^t,p^t)$ defined by (77):

(83) $q_i^t = \partial C(u^t,p^t)/\partial p_i$; $i = 1,\ldots,N$; $t = 0,1$
$= [C(u^t,p^t)/p_i^t]\partial \ln C(u^t,p^t)/\partial \ln p_i$.

---

[61] To prove that (75) and (76) are true, use definition (74) and substitute into the left hand sides of (75) and (76). Then calculate the partial derivatives of the quadratic function defined by (74) and substitute these derivatives into the right hand side of (75) and (76).

[62] The consumer's cost function was defined by (1) above.

[63] Christensen, Jorgenson and Lau (1971) (1975) introduced this function into the economics literature.

[64] It can also be shown that if $b_0 = 1$ and all of the $b_i = 0$ for $i = 1,...,N$ and $b_{00} = 0$, then $C(u,p) = uC(1,p) \equiv uc(p)$; i.e., with these additional restrictions on the parameters of the general translog cost function, we have homothetic preferences. Note that we also assume that utility u is scaled so that u is always positive.

[65] We need to assume that (u^0,p^0) and (u^1,p^1) belong to the region of prices S^* where the translog C(u,p) satisfies the regularity conditions that a cost function must satisfy. If we think of C(u,p) as an approximation to an arbitrary differentiable cost function, then because of the flexibility property of the translog cost function, it is not a problem to assume that (u^0,p^0) belongs to S^* but if the vector (u^1,p^1) is not close to (u^0,p^0), then (u^1,p^1) may not belong to the regularity region so that equation (83) for t = 1 may not hold and hence equation (87) may not be valid.

[66] See (18) above.

Now use (82) to replace $C(u^t,p^t)$ in (83). After some cross multiplication, equations (83) become the following system of equations:

(84) $p_i^t q_i^t / \sum_{k=1}^N p_k^t q_k^t \equiv s_i^t = \partial \ln C(u^t,p^t)/\partial \ln p_i$; $i = 1,\ldots,N$; $t = 0,1$ or
(85) $s_i^t = a_i + \sum_{k=1}^N a_{ik} \ln p_k^t + b_i \ln u^t$; $i = 1,\ldots,N$; $t = 0,1$

where s_i^t is the period t expenditure share on commodity i and (85) follows from (84) by differentiating (77) with respect to $\ln p_i$ for $t = 0,1$ and $i = 1,\ldots,N$.

Define the geometric average of the period 0 and 1 utility levels as u^*; i.e., define

(86) $u^* \equiv [u^0 u^1]^{1/2}$.

Now observe that the right hand side of the equation that defines the natural logarithm of the translog cost function, equation (77), is a quadratic function of the variables $z_i \equiv \ln p_i$ if we hold utility constant at the level u^*. Hence we can apply the quadratic identity, (76), and get the following equation:

(87) $\ln C(u^*,p^1) - \ln C(u^*,p^0)$
$= (1/2)\sum_{i=1}^N [\partial \ln C(u^*,p^0)/\partial \ln p_i + \partial \ln C(u^*,p^1)/\partial \ln p_i][\ln p_i^1 - \ln p_i^0]$
$= (1/2)\sum_{i=1}^N [a_i + \sum_{k=1}^N a_{ik} \ln p_k^0 + b_i \ln u^* + a_i + \sum_{k=1}^N a_{ik} \ln p_k^1 + b_i \ln u^*][\ln p_i^1 - \ln p_i^0]$
differentiating (77) at the points (u^*,p^0) and (u^*,p^1)
$= (1/2)\sum_{i=1}^N [a_i + \sum_{k=1}^N a_{ik} \ln p_k^0 + b_i \ln[u^0 u^1]^{1/2} + a_i + \sum_{k=1}^N a_{ik} \ln p_k^1 + b_i \ln[u^0 u^1]^{1/2}][\ln p_i^1 - \ln p_i^0]$
using definition (86) for u^*
$= (1/2)\sum_{i=1}^N [a_i + \sum_{k=1}^N a_{ik} \ln p_k^0 + b_i \ln u^0 + a_i + \sum_{k=1}^N a_{ik} \ln p_k^1 + b_i \ln u^1][\ln p_i^1 - \ln p_i^0]$ rearranging terms
$= (1/2)\sum_{i=1}^N [\partial \ln C(u^0,p^0)/\partial \ln p_i + \partial \ln C(u^1,p^1)/\partial \ln p_i][\ln p_i^1 - \ln p_i^0]$
differentiating (77) at the points (u^0,p^0) and (u^1,p^1)
$= (1/2)\sum_{i=1}^N [s_i^0 + s_i^1][\ln p_i^1 - \ln p_i^0]$ using equations (85).

The last equation in (87) can be recognized as the logarithm of the Törnqvist[67] Theil (1967) index number formula P_T defined in Chapter 4. Hence exponentiating both sides of (87) yields the following equality between the true cost of living between periods 0 and 1, evaluated at the intermediate utility level u^* and the observable Törnqvist Theil index P_T:[68]

(88) $C(u^*,p^1)/C(u^*,p^0) = P_T(p^0,p^1,q^0,q^1)$.

Since the translog cost function which appears on the left hand side of (88) is a flexible functional form, the Törnqvist Theil price index P_T is also a *superlative index*. Note that it is not necessary to assume homothetic preferences to derive this result.

It is somewhat mysterious how a ratio of *unobservable* cost functions of the form appearing on the left hand side of the above equation can be *exactly* estimated by an *observable* index number formula but the key to this mystery is the assumption of cost minimizing behavior and the quadratic identity (76) along with the fact that derivatives of cost functions are equal to quantities, as specified by Shephard's lemma, (18). In fact, all of the exact index number results derived in this section and the previous section can be derived using

[67] See Törnqvist and Törnqvist (1937).
[68] This result is due to Diewert (1976; 122).

transformations of the quadratic identity along with Shephard's lemma (or Wold's identity (15) above).[69] Fortunately, for most empirical applications, assuming that the consumer has (transformed) quadratic preferences will be an adequate assumption so the results presented in this section and the previous section are quite useful to index number practitioners who are willing to adopt the economic approach to index number theory. Essentially, the economic approach to index number theory provides a strong justification for the use of the Fisher price index P_F, the Törnqvist Theil price index P_T, the implicit quadratic mean of order r price indexes P^{r*} defined by (57) (when r = 1, this index is the Walsh price index P_W) and the quadratic mean of order r price indexes P^r defined by (69), provided that r is a number that is small in magnitude.

8. The Numerical Approximation Properties of Superlative Indexes

In the previous section, we have exhibited two families of superlative price and quantity indexes, Q^r and P^{r*} defined by (54) and (57), and P^r and Q^{r*} defined by (69) and (72) for each $r \neq 0$. The Fisher index P_F was a special case of P^r with r = 2 and the Walsh index P_W was a special case of P^{r*} with r = 1. Another superlative index was the Törnqvist Theil index P_T. A natural question to ask at this point is: how different will these indexes be? It is possible to show that all of the price indexes P^r approximate each other to the second order around any point where the price vectors p^0 and p^1 are equal and where the quantity vectors q^0 and q^1 are equal; i.e., we have the following equalities if the first and second order partial derivatives are evaluated at $p^0 = p^1 = p >> 0_N$ and $q^0 = q^1 = q >> 0_N$ for any $r \neq 0$:[70]

(89) $P_F(p^0,p^1,q^0,q^1) = P_T(p^0,p^1,q^0,q^1) = P_W(p^0,p^1,q^0,q^1) = P^r(p^0,p^1,q^0,q^1) = P^{r*}(p^0,p^1,q^0,q^1)$;
(90) $\nabla P_F(p^0,p^1,q^0,q^1) = \nabla P_T(p^0,p^1,q^0,q^1) = \nabla P_W(p^0,p^1,q^0,q^1) = \nabla P^r(p^0,p^1,q^0,q^1) = \nabla P^{r*}(p^0,p^1,q^0,q^1)$;
(91) $\nabla^2 P_F(p^0,p^1,q^0,q^1) = \nabla^2 P_T(p^0,p^1,q^0,q^1) = \nabla^2 P_W(p^0,p^1,q^0,q^1) = \nabla^2 P^r(p^0,p^1,q^0,q^1) = \nabla^2 P^{r*}(p^0,p^1,q^0,q^1)$.

The vector of first order partial derivatives of the function of 4N variables $P_F(p^0,p^1,q^0,q^1)$ is the vector of dimension 4N denoted by $\nabla P_F(p^0,p^1,q^0,q^1)$ and the matrix of second order partial derivatives of $P_F(p^0,p^1,q^0,q^1)$ is a 4N by 4N matrix denoted by $\nabla^2 P_F(p^0,p^1,q^0,q^1)$ and so on. A similar set of equalities holds for the companion quantity indexes that match up to P_F, P_T, P_W, P^r and P^{r*} using the product test, $Q(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1/p^0 \cdot q^0 P(p^0,p^1,q^0,q^1)$. The implication of the above equalities is that if prices and quantities do not change much over the two periods being compared, then all of above price indexes will give much the same answer.

For empirical comparisons of some of the above indexes, see Diewert (1978; 894-895) and Hill (2006). Hill (2006) showed that the second order approximation property of the mean of order r indexes breaks down as r approaches plus or minus infinity. However, in most empirical applications, we generally choose r equal to 2 (the Fisher case) or 1 (the Walsh case) or 0 (the Törnqvist Theil case). For these cases, the resulting indexes generally approximate each other very closely.[71]

It turns out that the Laspeyres and Paasche price indexes approximate each other (and superlative indexes like the Fisher index) to the first order around an equal price and quantity point *but not to the second order*;

[69] See Diewert (2002). However, when applying Wold's Identity or Shephard's Lemma to observed price and quantity data, we need the assumption of optimizing behavior on the part of the consumer and we need the observed data to be in the regions of regularity for the utility function or cost function that we are working with.
[70] The proof is a straightforward differentiation exercise; see Diewert (1978; 889). In fact, the equalities in (89)-(91) are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$.
[71] The approximations will be close if we are using annual time series data where price and quantity changes are generally smooth. However, if we are making international comparisons or using panel data or using subannual time series data, then the approximations may not be close.

i.e., we have the following equalities if the first order partial derivatives are evaluated at $p^0 = p^1 = p >> 0_N$ and $q^0 = q^1 = q >> 0_N$:

$$(92)\quad P_F(p^0,p^1,q^0,q^1) = P_L(p^0,p^1,q^0,q^1) = P_P(p^0,p^1,q^0,q^1)\ ;$$
$$(93)\quad \nabla P_F(p^0,p^1,q^0,q^1) = \nabla P_L(p^0,p^1,q^0,q^1) = \nabla P_P(p^0,p^1,q^0,q^1).$$

Up to this point, we have considered four different approaches to index number theory:

- Fixed basket approaches and averages of baskets;
- Test approaches to index number theory;
- Stochastic or descriptive statistics approaches to index number theory and
- Economic approaches.

The first approach led to the Fisher and Walsh indexes as being "best", the second approach led to the Fisher and Törnqvist Theil indexes as being "best", the third approach led to the Törnqvist Theil index as "best" and the economic approach led to the Fisher, Walsh and Törnqvist Theil indexes as being among the "best" indexes. Thus P_F, P_W and P_T keep emerging as "best' indexes. The results in this section tell us that if prices and quantities do not change all that much going from the first period to the second period, then all three of these indexes will give us more or less the same answer.

9. The Cobb Douglas Price Index

Suppose that the consumer's utility function is defined as follows for all $q \geq 0_N$:

$$(94)\ f(q) \equiv \alpha_0 \Pi_{n=1}^{N}\ q_n^{\alpha_n}$$

where the $\alpha_n > 0$ for n = 0,1,...,N and in addition satisfy the following constraint:

$$(95)\ \Sigma_{n=1}^{N}\ \alpha_n = 1.$$

This is the Cobb Douglas functional form.[72] It can be seen that f(q) defined by (94) is linearly homogeneous. It is also positive, concave and increasing over the set of strictly positive quantity vectors.

Let the consumer's preferences be represented by f(q) and suppose that the commodity price vector $p >> 0_N$ and is given. The *consumer's unit cost minimization problem* is defined as follows:

$$(96)\ \min_q \{p \cdot q : f(q) \geq 1\ ;\ q \geq 0_N\} \equiv c(p).$$

Proposition 11: The solution to the unit cost minimization problem defined by (96) when f(q) is the Cobb Douglas utility function defined by (94) and (95) is the *Cobb Douglas unit cost function defined* as follows for $p >> 0_N$:

$$(97)\ c(p) \equiv \kappa \Pi_{n=1}^{N} (p_n)^{\alpha_n}\ ;\ \kappa \equiv [\alpha_0 \Pi_{n=1}^{N} \alpha_n^{\alpha_n}]^{-1}.$$

[72] This functional form was used as a production function for the case N = 2 by Cobb and Douglas (1928). It was also used by Knut Wicksell as a production function much earlier in 1916; see Olsson (1971). This functional form was first used as a utility function for the N commodity case in section 8 of Konüs and Byushgens (1926). Our algebra in this section was more or less worked out by Konüs and Byushgens. In particular, these authors realized that the assumption of Cobb Douglas preferences implied that commodity expenditure shares must be constant over time. See also Pollak (1971) (1983) for his analysis of Cobb Douglas preferences, which is followed in the present section.

See the Appendix for a proof.

It can be seen that the Cobb Douglas unit cost function has more or less the same functional form as the Cobb Douglas utility function: p replaces q when we move from the utility function to the unit cost function.

Let $p^t >> 0_N$ for t = 0,1. Suppose the consumer has Cobb Douglas preferences and faces the prices p^t in period t for t = 0,1. The observed period t quantity vector is $q^t >> 0_N$. Assume that the consumer minimizes the cost of achieving the utility level $u^t \equiv f(q^t)$ for each period. Then the components of $q^t \equiv [q_1^t,...,q_N^t]$ must satisfy the following equations which follow by using (97) and Shephard's Lemma:

$$(98)\ q_n^t = [\partial c(p^t)/\partial p_n]f(q^t)\ ; \qquad n = 1,...,N;\ t = 0,1$$
$$= \alpha_n c(p^t)[p_n^t]^{-1} f(q^t).$$

Multiply both sides of equation n in period t by p_n^t and we obtain the following equations:

$$(99)\ p_n^t q_n^t = \alpha_n c(p^t) f(q^t)\ ; \qquad n = 1,...,N;\ t = 0,1.$$

Summing equations (99) for each period t gives us the following equations, making use of $\Sigma_{n=1}^N \alpha_n = 1$:

$$(100)\ p^t \cdot q^t = c(p^t) f(q^t)\ ; \qquad t = 0,1.$$

Using equations (99) and (100), we see that the following equations hold:

$$(101)\ s_n^t \equiv p_n^t q_n^t / p^t \cdot q^t = \alpha_n c(p^t) f(q^t)/c(p^t) f(q^t) = \alpha_n\ ; \qquad n = 1,...,N;\ t = 0,1.$$

Equations (101) are important: they tell us that a utility maximizing consumer that has Cobb Douglas preferences will have expenditure shares on each commodity that will remain constant across all time periods. This assumption is unlikely to be satisfied in practice. Nevertheless, equations (101) lead to an exact Konüs true cost of living index as will be seen below.

Since Cobb Douglas preferences are homothetic, the true cost of living index going from period 0 to 1 is $c(p^1)/c(p^0)$ where c(p) is defined by (97). Thus we have the following *exact index number formula* for a Cobb Douglas consumer:

$$(102)\ c(p^1)/c(p^0) = \kappa \Pi_{n=1}^N (p_n^1)^{\alpha_n} / \kappa \Pi_{n=1}^N (p_n^0)^{\alpha_n}$$
$$= \Pi_{n=1}^N (p_n^1/p_n^0)^{\alpha_n}$$
$$= \Pi_{n=1}^N (p_n^1/p_n^0)^{s_n^0} \qquad \text{using (101) for } t = 0$$
$$\equiv P_{KB}(p^0,p^1,q^0,q^1)$$

where $P_{KB}(p^0,p^1,q^0,q^1)$ is the Konus Byushgens or Cobb Douglas price index. This formula is a handy one for price statisticians: the price index for a current period can be evaluated using only the prices p_n^0 and expenditure shares s_n^0 for a past period 0 and prices p_n^1 for the current period 1.

We turn now to a functional form for the utility function that is more flexible than the Cobb Douglas utility function but is still not completely flexible.

10. Constant Elasticity of Substitution (CES) Preferences

It is useful to introduce a family of functions that calculate an *average* of N positive numbers, $x \equiv [x_1,...,x_N]$. Assume that the number r is not equal to zero and the positive weights α_n sum to 1 so that $\alpha \equiv [\alpha_1,...,\alpha_N]$ satisfies conditions (95). Define the *weighted mean of order r* of the N components of the x vector as follows:[73]

(103) $M_r(x) \equiv [\sum_{n=1}^N \alpha_n x_n^r]^{1/r}$.

The functional form defined by (103) occurs frequently in the economics literature. If r = 1, then $M_r(x)$ equals $\alpha\cdot x$, a linear function of x. As r tends to plus infinity, $M_r(x)$ tends to $\max_n \{x_n: n = 1,...,N\}$. As r tends to minus infinity, $M_r(x)$ tends to $\min_n \{x_n: n = 1,...,N\}$. As r tends to 0, $M_r(x)$ tends to the Cobb Douglas functional form which is the weighted geometric mean, $\Pi_{n=1}^N (x_n)^{\alpha_n}$. It is readily verified that $M_r(\lambda x) = \lambda M_r(x)$ for all $\lambda > 0$ and $x >> 0_N$. If we multiply $M_r(x)$ by a constant, then we obtain the CES (*Constant Elasticity of Substitution*) functional form popularized by Arrow, Chenery, Minhas and Solow (1961) in the context of production theory (where x is an input vector and $\alpha_0 M_r(x)$ is the output produced by the input vector x). This functional form is also widely used as a utility function and it also used extensively when measures of income inequality are constructed.[74] We note that the function $M_r(x)$ is flexible if $r \neq 0$ and N = 2. It is not flexible if N > 2.

For future reference, the first and second order partial derivatives of $M_r(x)$ for $x >> 0_N$ are as follows:

(104) $\partial M_r(x)/\partial x_i = (1/r)[\sum_{n=1}^N \alpha_n x_n^r]^{(1/r)-1}\alpha_i\, r\, x_i^{r-1} = [\sum_{n=1}^N \alpha_n x_n^r]^{(1/r)-1}\alpha_i x_i^{r-1}$; $i = 1,...,N$.

Differentiating (104) again with respect to x_i yields the following second order partial derivatives: for i = 1,...,N:

$$\begin{aligned}(105)\ \partial^2 M_r(x)/\partial x_i^2 &= [(1/r) -1][\textstyle\sum_{n=1}^N \alpha_n x_n^r]^{(1/r)-2}\alpha_i\, r x_i^{r-1}\alpha_i x_i^{r-1} + [\sum_{n=1}^N \alpha_n x_n^r]^{(1/r)-1}\alpha_i\ (r-1)x_i^{r-2}\ ; \qquad i = 1,...,N \\ &= [r -1][\textstyle\sum_{n=1}^N \alpha_n x_n^r]^{(1/r)-2}\{[\sum_{n=1}^N \alpha_n x_n^r]\alpha_i x_i^{r-2} - \alpha_i^2 x_i^{2r-2}\}.\end{aligned}$$

Differentiating (104) with respect to x_k for $k \neq i$ yields:

$$\begin{aligned}(106)\ \partial^2 M_r(x)/\partial x_i\partial x_k &= [(1/r) -1][\textstyle\sum_{n=1}^N \alpha_n x_n^r]^{(1/r)-2}\alpha_k\, r x_k^{r-1}\alpha_i x_i^{r-1}\ ; \qquad k \neq i \\ &= (1 - r)[\textstyle\sum_{n=1}^N \alpha_n x_n^r]^{(1/r)-2}\,\alpha_i\alpha_k x_i^{r-1}x_k^{r-1}.\end{aligned}$$

It can be shown if $r \leq 1$, then the matrix of second order partial derivatives of $M_r(x)$, $\nabla^2 M_r(x)$, is a negative semidefinite matrix for all $x >> 0_N$ and this property in turn implies that $M_r(x)$ is a concave function over the set of positive x vectors.[75] Hence $M_r(q)$ is a suitable functional form for a utility function and $M_r(p)$ is a suitable functional form for a unit cost function if $r \leq 1$. These functions satisfy the required regularity conditions over the entire positive orthant. For future reference, the derivatives defined by (104)-(106) can be used in order to establish the following equalities:

(107) $M_r(x)[\partial^2 M_r(x)/\partial x_i\partial x_k]/[\partial M_r(x)/\partial x_i][\partial M_r(x)/\partial x_k] = (1 - r)$; $x >> 0_N;\ 1 \leq i \neq k \leq N$.

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[73] Hardy, Littlewood and Polya (1934; 12-14) refer to this family of means or averages as elementary weighted mean values and study their properties in great detail. $M_r(x)$ has the following properties where $x >> 0_N$: (i) $M_r(\lambda 1_N) = \lambda$ for any $\lambda > 0$; (ii) $\nabla M_r(x) >> 0_N$ so that $M_r(x)$ is increasing in x; (iii) $\min \{x_n ; n =1,...,N\} \leq M_r(x) \leq \max \{x_n ; n =1,...,N\}$ and (iv) $M_r(\lambda x) = \lambda M_r(x)$. Thus $M_r(x)$ is a homogeneous mean. See Diewert (1993b) for materials on mean functions and their application to economics.

[74] See Diewert (1993b).

[75] The definition of $M_r(x)$ can be extended to the set $x \geq 0_N$; see Hardy, Littlewood and Polya (1934).

Suppose that the unit cost function has the following CES functional form for $r \le 1$:

$$(108)\ c(p) \equiv \alpha_0 [\sum_{n=1}^{N} \alpha_n (p_n)^r]^{1/r} \text{ if } r \neq 0;$$
$$\equiv \alpha_0 \prod_{n=1}^{N} p_n^{\alpha_n} \qquad \text{if } r = 0$$

where $\alpha_0 > 0$ and $\alpha \equiv [\alpha_1,...,\alpha_N]$ satisfies conditions (95).

Under the assumption of cost minimizing behavior on the part of the consumer in period t, Shephard's Lemma, (18) above, tells us that the observed period t consumption of commodity i, q_i^t, will be equal to $u^t\partial c(p^t)/\partial p_i$ where $\partial c(p^t)/\partial p_i$ is the first order partial derivative of the unit cost function with respect to the ith commodity price evaluated at the period t prices and $u^t = f(q^t)$ is the aggregate (unobservable) level of period t utility. Using the CES unit cost function defined by (108) and assuming that $r \neq 0$, the following equations are obtained that express the components of the consumer's observed consumption vector q^t in terms of the period t prices p^t facing the consumer and either the period t utility level for the consumer u^t or the observed period t expenditure for the consumer, $e^t \equiv p^t \cdot q^t$:

$$(109)\ q_i^t = u^t\alpha_0 [\sum_{n=1}^{N} \alpha_n (p_n^t)^r]^{(1/r)-1}\alpha_i (p_i^t)^{r-1}; \qquad t = 0,1;\ i = 1,\ldots,N$$
$$= u^t c(p^t)\, \alpha_i(p_i^t)^{r-1}/\sum_{n=1}^{N} \alpha_n(p_n^t)^r$$
$$= e^t\alpha_i(p_i^t)^{r-1}/\sum_{n=1}^{N} \alpha_n(p_n^t)^r$$

where the last equation in (109) follows because observed period t expenditure, e^t, is equal to $p^t \cdot q^t$ which in turn is equal to $u^tc(p^t)$. The last set of equations in (109) could be used to estimate the unknown parameters r and α that appear in definition (108).[76]

Equations (109) can be rewritten as

$$(110)\ s_i^t \equiv p_i^tq_i^t/p^t \cdot q^t = p_i^tq_i^t/u^tc(p^t) = \alpha_i(p_i^t)^r/\sum_{n=1}^{N} \alpha_n(p_n^t)^r\ ; \qquad t = 0,1;\ i = 1,\ldots,N.$$

Equations (110) give observed expenditure shares s^t as functions of consumer prices p^t and the unknown parameters r and $\alpha_1,...,\alpha_N$. These equations could also be used as estimating equations for the unknown parameters in an econometric model.[77]

Recall the definition of the consumer's cost function (1), which we repeat here for convenience for some positive level of utility u, given that the consumer is facing the positive vector of consumer prices $p >> 0_N$:

$$(111)\ C(u,p) \equiv \min_q \{p \cdot q : f(q) \ge u\ ; q \ge 0_N\}.$$

If the cost function C(u,p) is differentiable with respect to the components of the commodity price vector p, then Shephard's Lemma (18) applies and the consumer's system of commodity demand functions as

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[76] Note that the parameter α_0 cannot be identified using observable data. This makes sense since the scale of utility cannot be observed and so some arbitrary decision will have to be made in order to determine the utility scale. Usually, we normalize period 0 utility u^0 (which is equal to the period 0 volume level Q^0) to equal period 0 observed expenditure $e^0 = p^0 \cdot q^0$. This normalization determines the units of measurement for utility.

[77] Note that the right hand sides of equations (110) are homogeneous of degree 0 in the α_n parameters. However, the normalization $\Sigma_{n=1}^{N} \alpha_n = 1$ can be used to solve for say $\alpha_N = 1 - \Sigma_{n=1}^{N-1} \alpha_n$ which will allow all of the parameters to be identified. Because $\Sigma_{n=1}^{N} s_n^t = 1$ for t = 0,1, the N share equations for period t are not statistically independent and hence one of these estimating equations should be dropped from the estimation procedure. Similar adjustments need to be made to the system of estimating equations defined by (109) since the equations $p^t \cdot q^t = e^t$ hold without error for t = 0,1.

functions of the chosen utility level u and the commodity price vector p, q(u,p), is equal to the vector of first order partial derivatives of the cost or expenditure function with respect to the components of p:

$$(112)\ q(u,p) = \nabla_p C(u,p)$$

where $q(u,p) \equiv [q_1(u,p),...,q_N(u,p)]$.The demand functions $q_n(u,p) \equiv \partial C(u,p)/\partial p_n$ are known as *Hicksian*[78] *demand functions*. We expect that the demand for commodity i will increase if the price of commodity k (not equal to i) increases if i and k are substitutes in consumption; i.e., we expect $\partial q_i(u,p)/\partial p_k > 0$ if i and k are substitutes. Note that $q_i(u,p) = \partial C(u,p)/\partial p_i$ so that $\partial q_i(u,p)/\partial p_k = \partial^2 C(u,p)/\partial p_i \partial p_k$. A unit free measure of the magnitude of the response of the demand for product i due to an increase in the price of product k is the *elasticity function* $\varepsilon_{ik}(u,p)$ defined as:

$$(113)\ \varepsilon_{ik}(u,p) \equiv [\partial q_i(u,p)/\partial p_k][p_k/q_i(u,p)] = p_k[\partial^2 C(u,p)/\partial p_i \partial p_k]/\partial C(u,p)/\partial p_i = p_k[\partial^2 C(u,p)/\partial p_i \partial p_k]/q_i(u,p).$$

Allen (1938; 504) and Uzawa (1962)[79] suggested the following measure of the response of product i to a change in the price of product k:

$$(114)\ \sigma_{ik}(u,p) \equiv C(u,p)[\partial^2 C(u,p)/\partial p_i \partial p_k]/[\partial C(u,p)/\partial p_i][\partial C(u,p)/\partial p_k]\ ; \qquad i \neq k.$$

The Allen Uzawa measure is also independent of the units of measurement, but their measure converted the response into a measure that applied to both i and k. The bigger are $\varepsilon_{ik}(u,p)$ and $\sigma_{ik}(u,p)$, the more *substitutable* are the products.[80] Thus $\sigma_{ik}(u,p)$ defined by (114) is called the *elasticity of substitution* between products i and k. Note that $\sigma_{ik}(u,p) = \sigma_{ki}(u,p)$.

Define the cost function to be $C(u,p) = uc(p)$ where c(p) is defined by (108). Using equations (104)-(106), which apply to the CES functional form, it can be verified that the $\sigma_{ik}(u,p)$ defined by (114) simplify to the following equations:

$$(115)\ \sigma_{ik}(u,p) = 1 - r \equiv \sigma \geq 0\ ; \qquad i \neq k$$

where we have defined $\sigma \equiv 1 - r$. Thus if the consumer has CES preferences, which are dual to the unit cost function defined by (108), then *the elasticity of substitution between every pair of products is equal to the same number*, $1 - r \equiv \sigma$ which is equal to or greater than 0, since in order for c(p) to be a concave function, we required $r \leq 1$. Thus the CES functional form rules out complementary commodities and is far from being able to model arbitrary preferences if $N \geq 3$. However, the CES functional form is still a useful one, since it can model both Leontief and Cobb Douglas preferences: simply set r = 1 or r = 0 to get these two special cases.[81]

We turn now to the problem of finding exact index number formulae for preferences that are defined by the CES unit cost function. Our first exact index number formula requires an estimate for the elasticity of substitution, $\sigma \equiv 1 - r$. For $\sigma \neq 1$, define the *Lloyd* (1975) *Moulton* (1996) *price index* $P_{LM}(p^0,p^1,q^0,q^1)$ as follows for $p^t >> 0_N$ and $q^t >> 0_N$, t = 0,1:

$$(116)\ P_{LM}(p^0,p^1,q^0,q^1) \equiv [\textstyle\sum_{i=1}^N s_i^0 (p_i^1/p_i^0)^{1-\sigma}]^{1/(1-\sigma)}\ ; \qquad \sigma \neq 1$$

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[78] See Hicks (1946; 311-331).
[79] They suggested their measure in the context of production theory but it carries over to Hicksian demand functions.
[80] Hicks (1946) showed that if N = 2, then $\varepsilon_{12}(u,p)$ and $\sigma_{12}(u,p)$ must be nonnegative. However, if $N \geq 3$, then $\varepsilon_{12}(u,p)$ and $\sigma_{12}(u,p)$ could be negative. In this case, products 1 and 2 are called *complements*.
[81] It can also model linear preferences by letting r tend to plus infinity.

where s_i^0 is the period 0 expenditure share of commodity i as usual. Substitute equations (110) for s_i^0 into the right hand side of (116) and we obtain the following equation:

$$(117)\ P_{LM}(p^0,p^1,q^0,q^1) \equiv [\sum_{i=1}^N s_i^0 (p_i^1/p_i^0)^r]^{1/r} \qquad \text{letting } r = 1 - \sigma$$
$$= [\sum_{i=1}^N \{\alpha_i(p_i^0)^r/\sum_{n=1}^N \alpha_n(p_n^0)^r\}(p_i^1/p_i^0)^r]^{1/r} \qquad \text{using (110)}$$
$$= [\sum_{i=1}^N \alpha_i(p_i^1)^r/\sum_{n=1}^N \alpha_n(p_n^0)^r]^{1/r}$$
$$= [\sum_{i=1}^N \alpha_i (p_i^1)^r]^{1/r}/[\sum_{n=1}^N \alpha_n(p_n^0)^r]^{1/r}$$
$$= \alpha_0[\sum_{i=1}^N \alpha_i (p_i^1)^r]^{1/r}/\alpha_0[\sum_{n=1}^N \alpha_n (p_n^0)^r]^{1/r}$$
$$= c(p^1)/c(p^0) \qquad \text{using definition (108) for } p = p^0 \text{ and } p = p^1.$$

Equation (117) shows that the Lloyd Moulton index number formula P_{LM} is *exact* for CES preferences. Lloyd (1975) and Moulton (1996) independently derived this result but it was Moulton who appreciated the significance of the formula (117) for statistical agency purposes. Note that in order to evaluate (116) numerically, it is necessary to have information on:

- base period expenditure shares s_i^0;
- the price relatives p_i^1/p_i^0 between the base period and the current period and
- an estimate of the elasticity of substitution between the commodities in the aggregate, σ.

The first two pieces of information are the standard information sets that statistical agencies use to evaluate the Laspeyres price index P_L (note that P_{LM} reduces to P_L if $\sigma = 0$ or $r = 1$). Hence, if the statistical agency is able to estimate the elasticity of substitution σ based on past experience[82], *then the Lloyd Moulton price index can be evaluated using essentially the same information set that is used in order to evaluate the traditional Laspeyres index*. Moreover, the resulting consumer price index may be free of *substitution bias* to a reasonable degree of approximation.[83] Of course, the practical problem with implementing this methodology is that estimates of the elasticity of substitution parameter σ are bound to be somewhat uncertain and hence the resulting Lloyd Moulton index may be subject to charges that it is not *objective* or *reproducible*. The statistical agency will have to balance the benefits of reducing substitution bias with these possible costs.

Our second index number formula that is exact for CES preferences does not require an estimate for the elasticity of substitution. Suppose that $p^t >> 0_N$ and $q^t >> 0_N$ for t = 0,1. The logarithm of the *Sato* (1976) *Vartia* (1976) price index $P_{SV}(p^0,p^1,q^0,q^1)$ is defined by the following equation:[84]

$$(118)\ \ln P_{SV}(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^N w_n \ln(p_n^1/p_n^0)$$

[82] For the first application of this methodology (in the context of the consumer price index), see Shapiro and Wilcox (1997; 121-123). They calculated superlative Törnqvist indexes for the U.S. for the years 1986-1995 and then calculated the Lloyd Moulton CES index for the same period using various values of σ. They then chose the value of σ (which was 0.7) that caused the CES index to most closely approximate the Törnqvist index. Essentially the same methodology was used by Alterman, Diewert and Feenstra (1999) in their study of U.S. import and export price indexes. Alternative methods for estimating σ will be considered below.

[83] What is a "reasonable" degree of approximation depends on the context. Assuming that consumers have CES preferences may not be a reasonable assumption in the context of forming index numbers over an aggregate that contains heterogeneous products where elasticities of demand for the various products are very different. However, if the aggregate consists of fairly similar products, then it may be adequate to assume a CES approximation to preferences over these relatively homogeneous products, which are presumably highly substitutable with each other.

[84] Sato and Vartia both defined P_{SV} independently. Sato (1976; 225) showed that P_{SV} was exact for CES preferences; i.e., Sato provided a (somewhat sketchy) proof of a dual version of Proposition 12.

where the weights w_n that appear in equations (118) are calculated in two stages. The first stage set of weights is defined as $w_n^* \equiv (s_n^1 - s_n^0)/(\ln s_n^1 - \ln s_n^0)$ for n = 1,...,N provided that $s_n^1 \neq s_n^0$. If $s_n^1 = s_n^0$, then define $w_n^* \equiv s_n^1 = s_n^0$. The second stage weights are defined as $w_n \equiv w_n^*/\Sigma_{i=1}^N w_i^*$ for n = 1,...,N. Note that in order for the logarithm of $P_{SV}(p^0,p^1,q^0,q^1)$ to be well defined, we require that $s_n^0 > 0$, $s_n^1 > 0$, $p_n^0 > 0$ and $p_n^1 > 0$ for all n = 1,...,N; i.e., all prices and quantities must be positive for all products in both periods.

Proposition 12: The Sato Vartia price index is exact for CES preferences; i.e., if the consumer faces the positive prices $p^0 >> 0_N$ and $p^1 >> 0_N$ in periods 0 and 1, has CES preferences dual to the unit cost function defined by (108) and maximizes utility in periods 0 and 1 with solution vectors $q^0 >> 0_N$ and $q^1 >> 0_N$ for periods 0 and 1, then we have:

$$(119)\ P_{SV}(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0)$$
$$= [\sum_{n=1}^N \alpha_n (p_n^1)^r]^{1/r}/[\sum_{n=1}^N \alpha_n (p_n^0)^r]^{1/r} \qquad \text{if } r \neq 0$$
$$= \prod_{n=1}^N (p_n^1)^{\alpha_n} / \prod_{n=1}^N (p_n^0)^{\alpha_n} \qquad \text{if } r = 0.$$

For a proof of this proposition, see the Appendix.

We noted above that equations (109) and (110) could be used to estimate the unknown parameters $r = 1 - \sigma$ and the α_n that characterize the CES unit cost function defined by (108). However, if our focus is on obtaining an estimate for r (or equivalently for the elasticity of substitution $\sigma \equiv 1 - r$), then much simpler systems of estimating equations can be derived as will be indicated below.

Recall the system of share equations defined by (110) that express cost minimizing expenditure shares as functions of prices. Extend this system of equations to period T, take logarithms of both sides of the resulting equations and add error terms η_i^t.[85] The following system of estimating equations is obtained:

$$(120)\ \ln s_n^t = \ln\alpha_n + r\ln(p_n^t) - \ln[\sum_{i=1}^N \alpha_i(p_i^t)^r] + \eta_n^t\ ; \qquad t = 0,1,...,T;\ n = 1,\ldots,N.$$

Now difference the *logarithms* of the s_n^t with respect to time; i.e., define Δs_n^t as follows:

$$(121)\ \Delta s_n^t \equiv \ln(s_n^t) - \ln(s_n^{t-1})\ ; \qquad n = 1,...,N;\ t = 1,...,T.$$

Now pick product N as the numeraire product[86] and difference the Δs_n^t with respect to product N, giving rise to the following *double differenced log variable*, ds_n^t:

$$(122)\ ds_n^t \equiv \Delta s_n^t - \Delta s_N^t\ ; \qquad n = 1,...,N{-}1;\ t = 1,...,T$$
$$= \ln s_n^t - \ln s_n^{t-1} - [\ln s_N^t - \ln s_N^{t-1}].$$

Define the *double differenced log price variables* in a similar manner:

$$(123)\ dp_n^t \equiv \Delta p_n^t - \Delta p_N^t\ ; \qquad n = 1,...,N{-}1;\ t = 1,...,T.$$
$$= \ln p_n^t - \ln p_n^{t-1} - [\ln p_N^t - \ln p_N^{t-1}].$$

Finally, define the *double differenced error variables* $d\eta_n^t$ as follows:

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[85] A standard specification for the error terms η_n^t is that they have 0 means, a constant variance-covariance matrix for the error terms belonging to the same period t and zero covariances across time periods.

[86] In practice, the numeraire commodity should be chosen to be a commodity that has a small predicted variance and a large expenditure share. However, it is not straightforward to find such a commodity. Below, an alternative method of estimation will be suggested that avoids the need to choose a numeraire commodity.

$$(124)\ d\eta_n^t \equiv \eta_n^t - \eta_n^{t-1} - \eta_N^t + \eta_N^{t-1} \equiv \varepsilon_n^t\,; \qquad n = 1,\ldots,N-1;\ t = 1,\ldots,T.$$

Using definitions (121)-(124) and equations (120), it can be verified that the double differenced log shares ds_n^t satisfy the following system of (N−1)T estimating equations under our assumptions:

$$(125)\ ds_n^t = rdp_n^t + \varepsilon_n^t\,; \qquad n = 1,\ldots,N-1;\ t = 1,\ldots,T$$

where the new residuals, ε_n^t, have means 0 and a constant covariance matrix with 0 covariances for observations that are separated by two or more time periods. Thus we have a system of linear estimating equations with only one unknown parameter across all equations, namely the parameter r. This is almost the simplest possible system of estimating equations that one could imagine. This *double differencing method* for estimating the elasticity of substitution when consumers have CES preferences was suggested by Feenstra (1994).[87]

Instead of starting with the share equations (110), one could start with the demand functions defined by equations (109). Extend this system of equations to period T, take logarithms of both sides of the resulting equations and add error terms η_i^t. The following system of estimating equations is obtained:[88]

$$(126)\ \ln q_n^t = \ln e^t + \ln\alpha_n + (r-1)\ln p_n^t - \ln[\textstyle\sum_{i=1}^N \alpha_i(p_i^t)^r] + \eta_n^t\,; \qquad t = 0,1,\ldots,T;\ n = 1,\ldots,N.$$

Define Δq_n^t as the *time difference for the logarithms of quantities* as follows:

$$(127)\ \Delta q_n^t \equiv \ln q_n^t - \ln q_n^{t-1}\,; \qquad n = 1,\ldots,N;\ t = 1,,\ldots,T.$$

Again, pick product N as the numeraire product and difference the Δq_n^t with respect to product N, giving rise to the following *double differenced log variable*, dq_n^t:

$$\begin{aligned}(128)\ dq_n^t &\equiv \Delta q_n^t - \Delta q_N^t\,; \qquad n = 1,\ldots,N-1;\ t = 1,\ldots,T\\ &= \ln q_n^t - \ln q_n^{t-1} - (\ln q_N^t - \ln q_N^{t-1}).\end{aligned}$$

Define the double differenced price and error variables, dp_n^t and $d\eta_n^t$ by (123) and (124). Using these definitions and (126)-(128), it is straightforward to show that the following equations will hold:

$$\begin{aligned}(129)\ dq_n^t &= (r-1)dp_n^t + d\eta_n^t\,; \qquad n = 1,\ldots,N-1;\ t = 1,\ldots,T\\ &= -\,\sigma dp_n^t + \varepsilon_n^t\end{aligned}$$

since the elasticity of substitution σ is equal to 1 − r. Again, this is an extremely simple system of estimating equations.

The double differenced share equation specification given by (125) and the double difference quantity demanded specification given by (129) both depend on the choice of the numeraire commodity. This dependence could be a problem for statistical agencies in that the estimation procedure is not completely reproducible: different statisticians could pick different commodities as the numeraire commodity and get

[87] For an empirical application of the method, see Diewert and Feenstra (2019). The variance covariance structure is not quite classical due to the correlation of residuals between adjacent time periods. Another problem with the method is that the estimates for r will generally depend on the choice of the numeraire commodity.

[88] The error terms in (126) are different from the error terms in (120). For convenience, we did not introduce a new notation for the error terms in (126).

different estimates for the elasticity of substitution. It is possible to modify the double difference method so that it is not dependent on the choice of a numeraire commodity.

For each time period t, define the geometric average of the s_n^t and p_n^t as $s_\bullet^t$ and $p_\bullet^t$ respectively for t = 0,1,...,T. For each time period t, define the arithmetic average of the η_n^t as $\eta_\bullet^t$ for t = 0,1,...,T. Finally define the geometric average of the α_n as $\alpha_\bullet$. Recall equations (120). For each time period t, take the arithmetic average of both sides of equations (120) for all N observations in period t. The following equations are the result of these operations:

(130) $\ln s_\bullet^t = \ln\alpha_\bullet + r\ln p_\bullet^t - \ln[\sum_{i=1}^N \alpha_i(p_i^t)^r] + \eta_\bullet^t$; t = 0,1,...,T.

Now difference the $\ln s_n^t$ defined by equations (120) with the $\ln s_\bullet^t$ defined by (130); i.e., essentially we are choosing the *average* (over commodities n) *log shares* in place of the log shares of a numeraire commodity. The following equations are obtained:

(131) $\ln s_n^t - \ln s_\bullet^t = \ln\alpha_n - \ln\alpha_\bullet + r\ln p_n^t - r\ln p_\bullet^t + \eta_n^t - \eta_\bullet^t$; t = 0,1,...,T; n = 1,…,N.

Now difference the variables $\ln s_n^t - \ln s_\bullet^t$ with respect to time and we obtain the following estimating equations:[89]

(132) $\ln s_n^t - \ln s_n^{t-1} - \ln s_\bullet^t + \ln s_\bullet^{t-1} = r[\ln p_n^t - \ln p_n^{t-1} - \ln p_\bullet^t + \ln p_\bullet^{t-1}] + \varepsilon_n^t$; t = 1,...,T; n = 1,…,N

where $\varepsilon_n^t \equiv \eta_n^t - \eta_n^{t-1} - \eta_\bullet^t + \eta_\bullet^{t-1}$. Again, we have a system of estimating equations that is linear in the single parameter r.

Instead of starting with the share equations (110), one could start with the demand functions defined by equations (109). Extend this system of equations to period T, take logarithms of both sides of the resulting equations and add error terms η_n^t. The system of estimating equations defined by (126) is obtained. Now define the geometric average of the q_n^t for period t as $q_\bullet^t$ for t = 0,1,...,T. Apply the same definitions and techniques that led to equations (130)-(132) and we obtain the following system of estimating equations:

(133) $\ln q_n^t - \ln q_n^{t-1} - \ln q_\bullet^t + \ln q_\bullet^{t-1} = (r-1)[\ln p_n^t - \ln p_n^{t-1} - \ln p_\bullet^t + \ln p_\bullet^{t-1}] + \varepsilon_n^t$; t = 1,...,T; n = 1,…,N
$= -\sigma[\ln p_n^t - \ln p_n^{t-1} - \ln p_\bullet^t + \ln p_\bullet^{t-1}] + \varepsilon_n^t$

where $\varepsilon_n^t \equiv \eta_n^t - \eta_n^{t-1} - \eta_\bullet^t + \eta_\bullet^{t-1}$. Equations (133) are a system of estimating equations that is linear in the single parameter σ, which is the elasticity of substitution between all pairs of commodities.

It turns out that estimating the consumer's utility function directly (rather than estimating the dual unit cost function) is advantageous when estimates of reservation prices[90] for products that are not available are required. In the case of CES preferences, this advantage is not immediately apparent since the CES reservation prices are automatically set equal to infinity. But it turns out that there may be advantages in estimating the CES utility function directly because of econometric considerations as we shall see below.

[89] Note that for each t, we have the following equalities: $0 = \Sigma_{n=1}^N [\ln s_n^t - \ln s_\bullet^t] = \Sigma_{n=1}^N [\ln\alpha_n - \ln\alpha_\bullet] = \Sigma_{n=1}^N [\ln p_n^t - \ln p_\bullet^t] = \Sigma_{n=1}^N [\eta_n^t - \eta_\bullet^t]$. Thus for each t, the N equations for $\ln s_n^t - \ln s_\bullet^t$ for n = 1,...,N are linearly dependent and hence any one of these N equations can be dropped. If the commodity N equations are dropped, then we use equations (132) as estimating equations only for t = 1,...,T and n = 1,...,N−1. Under an appropriate stochastic specification, the estimate for r will not depend on which equation is dropped.

[90] Reservation prices will be discussed in section 14 below and in Chapter 8.

Thus we will conclude this section by deriving the consumer demand functions that are consistent with the maximization of a CES utility function.

We now assume that the utility function f(q) is defined directly as the following *CES utility function*:

(134) $f(q_1,...,q_N) \equiv [\Sigma_{n=1}^N \beta_n q_n^s]^{1/s}$

where the parameters β_n are positive and sum to 1 and s is a parameter that satisfies $s \leq 1$ (so that f(q) will be a concave function of q and $s \neq 0$ (in which case f(q) is a Cobb Douglas utility function). Thus f(q) is a mean of order s.

Suppose s = 1 and let $p >> 0_N$. In this case, the utility function is the *linear function* $f(q) \equiv \beta\cdot q = \Sigma_{n=1}^N \beta_n q_n$. The cost minimization problem that defines the dual unit cost function for this case is the following linear programming problem:

(135) $\min_q \{p\cdot q: \beta\cdot q \geq 1; q \geq 0_N\} = \min_n \{p_n/\beta_n: n = 1,...,N\} \equiv c(p)$.

The unit cost function c(p) defined by the solution to (135) is not differentiable but it is a well defined continuous, increasing, linearly homogeneous and concave function of p. If the minimum over n is unique and attained for say product 1, then the solution q^* to (135) is unique and is given by $q_1^* = 1/\beta_1$ with $q_i^* = 0$ for i = 2,3,...,N. If p happens to equal $\lambda\beta$ for some $\lambda > 0$, then the solution set of q vectors that solve (135) is the set $\{q: \beta\cdot q = 1/\lambda; q \geq 0_N\}$.

We turn our attention to the case where s satisfies s < 1 and $s \neq 0$. Suppose $p \equiv (p_1,...,p_N) >> 0_N$. Ignoring the constraints $q \geq 0_N$ for the moment, the first order necessary (and sufficient) conditions that can be used to solve the unit cost minimization problem defined by (96) when f(q) is defined by (134) are the following conditions:

(136) $p_n = \lambda^*\beta_n q_n^{s-1}$; $\qquad$ n = 1,...,N;
(137) $1 = [\Sigma_{n=1}^N \beta_n q_n^s]^{1/s}$.

Equations (136) are equivalent to the equations $q_n = [p_n/\lambda^*\beta_n]^{1/(s-1)}$ for n = 1,...,N. Substitute these equations into equation (137) and obtain the following equations: $1 = \Sigma_{n=1}^N \beta_n q_n^s = \Sigma_{n=1}^N \beta_n[p_n/\lambda^*\beta_n]^{1/(s-1)}$. This equation can be solved for $\lambda^* = [\Sigma_{n=1}^N \beta_n^{1/(1-s)} p_n^{s/(s-1)}]^{(s-1)/s}$.[91] The optimal q_n^* are defined as $q_n^* = [p_n/\lambda^*\beta_n]^{1/(s-1)}$ for n = 1,...,N. All of the equations in (136) and (137) will be satisfied by this λ^*, q^* solution.

Evaluate (136) and (137) at the optimal solution. Multiply both sides of equation n in (136) by q_n^* and sum the resulting N equations. This leads to the following equations:

(138) $c(p) \equiv \Sigma_{n=1}^N p_n q_n^*$
$= \lambda^*\Sigma_{n=1}^N \beta_n(q_n^*)^s$
$= \lambda^*$ $\qquad$ using (137)
$= [\Sigma_{n=1}^N \beta_n^{1/(1-s)} p_n^{s/(s-1)}]^{(s-1)/s}$.

It can be seen that the dual unit cost function c(p) that corresponds to the CES utility function defined by (134) for $s \neq 0$ and $s \neq 1$ is proportional to a mean of order r in prices where r = s/(s–1). Thus if f(q) is the CES utility function defined by (134), then the corresponding elasticity of substitution is:

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[91] Note that we require $s \neq 0$ and $s \neq 1$ in order for λ^* to be well defined.

(139) $\sigma = 1 - r = 1 - [s/(s-1)] = -1/(s-1) = 1/(1-s)$.

As s approaches 1 from below, σ approaches plus infinity. For s = 0, σ = 1 and we have Cobb Douglas preferences. As s approaches minus infinity, σ approaches 0 as a limiting case.[92]

In order to derive the system of inverse demand functions that correspond to the CES utility function f(q) defined by (134), we make use of Wold's Identity, equations (17) which were $p^t/p^t \cdot q^t = \nabla f(q^t)/f(q^t)$. Upon defining the consumer's period t "income" as $e^t \equiv p^t \cdot q^t$, the CES system of *inverse demand functions* for period t is given by:

(140) $p^t = e^t \nabla f(q^t)/f(q^t)$; $\quad t = 0,1,...,T.$

The system of inverse demand functions gives the period t price vector p^t as the prices that are consistent with q^t solving the consumer's period t utility maximization problem given that the consumer has "income" e^t to spend on the N commodities in the aggregate.

If consumers maximize the CES utility function defined by (134) when they face the positive period t price vector p^t and have $e^t > 0$ to spend on the N commodities, the utility maximizing q^t will satisfy equations (140). If we evaluate equations (140) using the period t price and quantity data for periods t = 0,1,...,T and add error terms, we obtain the following system of equations:

(141) $p_n^t = e^t \beta_n (q_n^t)^{s-1} / \Sigma_{i=1}^N \beta_i (q_i^t)^s$; $\quad t = 0,1,...,T;\ n = 1,...,N.$

Equations (141) is the consumer's *system of inverse demand functions*. Equations (141) are the counterparts to the consumer's system of (ordinary) demand functions defined earlier by equations (109). It can be seen that the expressions $\beta_n (q_n^t)^s / \Sigma_{i=1}^N \beta_i (q_i^t)^s$ are homogeneous of degree 0 in the parameters $\beta_1,...,\beta_N$, so a normalization of these parameters is required for the identification of the β_n parameters. The normalization $\Sigma_{n=1}^N \beta_n = 1$ can be replaced by an equivalent normalization such as $\beta_N = 1$.[93]

Equations (141) are perfectly symmetric with equations (109), which gave us estimating equations for the system of ordinary consumer demand functions for a utility maximizing consumer with CES preferences, except that the role of prices and quantities has been interchanged. Equations (109) gave consumer demands q_n^t as functions of e^t and p^t, whereas equations (140) give us prices p_n^t that are consistent with utility maximization for CES preferences that are consistent with total expenditure equal to e^t and the utility maximizing quantity vector q^t. If equations (109) fit the given price and quantity data perfectly, then equations (141) will also fit the given price and quantity data perfectly as well (and vice versa). However, with data that do not fit the CES model exactly, the two methods for fitting a CES utility function will usually not agree. We will discuss the problem of deciding which model is "best" later.

Equations (141) can be converted into a system of share equations where the period t expenditure shares s_n^t are functions of e^t and q^t: multiply both sides of equation n for period t by q_n^t/e^t to obtain the expenditure share s_n^t on the left hand side of the resulting equation. The following system of share equations is obtained:

[92] The limiting case is the case of Leontief preferences.

[93] The normalization on the β_n determines the units of measurement for utility. Since $\Sigma_{n=1}^N s_n^t = 1$ for t = 0,1,...,T, the error terms will satisfy the constraints $\Sigma_{n=1}^N \eta_n^t = 0$ t = 0,1,...,T and thus the error terms pertaining to each time period cannot be distributed independently and so the estimating equations for one commodity n should be dropped from equations (141).

(142) $s_n^t = \beta_n(q_n^t)^s/\Sigma_{i=1}^N\ \beta_i(q_i^t)^s$; $t = 0,1,...,T;\ n = 1,...,N.$

Equations (141) and (142) can be used as systems of estimating equations. Below, we will consider some alternative systems of estimating equations.

Take the logarithm of the s_n^t defined by (142) and add the error term η_n^t to the right hand side of equation n in period t. We obtain the following system of estimating equations:

(143) $\ln s_n^t = \ln\beta_n + s\ln q_n^t - \ln[\Sigma_{i=1}^N\ \beta_i(q_i^t)^s] + \eta_n^t$; $t = 0,1,...,T;\ n = 1,...,N$

Equations (142) (which express the logarithm of shares as functions of quantities) are the counterparts to equations (120) (which expressed the logarithms of shares as functions of prices instead of quantities).

We can now repeat the differencing methods explained earlier when the task was to find estimates for the elasticity of substitution using the CES unit cost function as the starting point. Thus the counterparts to the estimating equations defined earlier by (125) and (129) are now the following *double differenced systems of inverse demand estimating equations*:[94]

(144) $ds_n^t = sdq_n^t + \varepsilon_n^t$; $t = 1,...,T;\ n = 1,...,N-1;$

(145) $dp_n^t = (s-1)dq_n^t + \varepsilon_n^t$; $t = 1,...,T;\ n = 1,...,N-1$

$\quad = -\sigma^{-1}dq_n^t + \varepsilon_n^t$ using (139).

As was the case with the systems of estimating equations defined by (125) and (129), the systems of estimating equations defined by (144) and (145) will depend on the choice of a numeraire commodity. To avoid this problem, we can adapt the analysis surrounding equations (130)-(132) to the present situation. Thus for each time period t, define the geometric average of the s_n^t and q_n^t as $s_\bullet^t$ and $q_\bullet^t$ respectively for $t = 0,1,...,T$. For each time period t, define the arithmetic average of the η_n^t in equations (143) as $\eta_\bullet^t$ for $t = 0,1,...,T$. Finally define the geometric average of the β_n as $\beta_\bullet$ For each time period t, take the arithmetic average of both sides of equations (143) for all N observations in period t. The following equations are the result of these operations:

(146) $\ln s_\bullet^t = \ln\beta_\bullet + s\ln q_\bullet^t - \ln[\sum_{i=1}^N\ \beta_i(q_i^t)^s] + \eta_\bullet^t$; $t = 0,1,...,T.$

Difference the $\ln s_n^t$ defined by equations (143) with the $\ln s_\bullet^t$ defined by (146). The following equations are obtained:

(147) $\ln s_n^t - \ln s_\bullet^t = \ln\beta_n - \ln\beta_\bullet + s\ln q_n^t - s\ln q_\bullet^t + \eta_n^t - \eta_\bullet^t$; $t = 0,1,...,T;\ n = 1,\ldots,N.$

Now difference the variables $\ln s_n^t - \ln s_\bullet^t$ with respect to time and we obtain the following estimating equations:[95]

(148) $\ln s_n^t - \ln s_n^{t-1} - \ln s_\bullet^t + \ln s_\bullet^{t-1} = s[\ln q_n^t - \ln q_n^{t-1} - \ln q_\bullet^t + \ln q_\bullet^{t-1}] + \varepsilon_n^t$; $t = 1,...,T;\ n = 1,\ldots,N$

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[94] We require that $s \neq 0$ and $s \neq 1$.

[95] Note that for each t, we have the following equalities: $0 = \Sigma_{n=1}^N\ [\ln s_n^t - \ln s_\bullet^t] = \Sigma_{n=1}^N\ [\ln\beta_n - \ln\beta_\bullet] = \Sigma_{n=1}^N\ [\ln q_n^t - \ln q_\bullet^t] = \Sigma_{n=1}^N\ [\eta_n^t - \eta_\bullet^t]$. Thus for each t, the N equations for $\ln s_n^t - \ln s_\bullet^t$ for $n = 1,...,N$ are linearly dependent and hence any one of these N equations can be dropped. If the commodity N equations are dropped, then we use equations (148) as estimating equations only for $t = 1,...,T$ and $n = 1,...,N-1$. Under an appropriate stochastic specification, the estimate for s will not depend on which equation is dropped.

where $\varepsilon_n{}^t \equiv \eta_n{}^t - \eta_n{}^{t-1} - \eta_{\bullet}{}^t + \eta_{\bullet}{}^{t-1}$. Equations (148) are a system of estimating equations that is linear in the single parameter s.

Instead of starting with the share equations (142), one could start with the inverse demand functions defined by equations (141). Take logarithms of both sides of these equations and add error terms $\eta_n{}^t$. The following system of estimating equations is obtained:

$$(149)\ \ln p_n{}^t = \ln\beta_n + (s-1)\ln q_n{}^t - \ln[\Sigma_{i=1}{}^N \beta_i(q_i{}^t)^s] + \eta_n{}^t\ ; \qquad t = 0,1,...,T;\ n = 1,...,N$$

Define the geometric average of the $p_n{}^t$ for period t as $p_{\bullet}{}^t$ for t = 0,1,...,T. Apply the same definitions and techniques that led to equations (146)-(148) and we obtain the following system of estimating equations:

$$(150)\ \ln p_n{}^t - \ln p_n{}^{t-1} - \ln p_{\bullet}{}^t + \ln p_{\bullet}{}^{t-1} = (s-1)[\ln q_n{}^t - \ln q_n{}^{t-1} - \ln q_{\bullet}{}^t + \ln q_{\bullet}{}^{t-1}] + \varepsilon_n{}^t\ ; \qquad t = 1,...,T;\ n = 1,\ldots,N$$
$$= -\sigma^{-1}[\ln q_n{}^t - \ln q_n{}^{t-1} - \ln q_{\bullet}{}^t + \ln q_{\bullet}{}^{t-1}] + \varepsilon_n{}^t \qquad \text{using (139)}$$

where $\varepsilon_n{}^t \equiv \eta_n{}^t - \eta_n{}^{t-1} - \eta_{\bullet}{}^t + \eta_{\bullet}{}^{t-1}$. Equations (150) are a system of estimating equations that is linear in the single parameter σ^{-1}, which is the reciprocal of the elasticity of substitution between all pairs of commodities.[96]

From the above materials, it can be seen that there is a bewildering array of alternative methods for estimating CES preferences. We have considered in some detail 12 methods. Equations (109) and (141) estimate the consumer's CES demand system and inverse demand system. In equations (109), quantities q^t are functions of total expenditure e^t and prices p^t for each period t; in equations (141), prices p^t are functions of e^t and q^t. The parameters of the CES unit cost function c(p) defined by (108) are estimated using equations (109), while the parameters of the CES utility function f(q) defined by (134) are estimated using equations (141). Equations (109) and (141) are our preferred specifications. The problem with the econometric specifications that involve shares as dependent variables is that shares by their very nature combine price and quantity information and so errors in either prices or quantities will show up in shares. Thus a model involving shares as dependent variables could fit the data very well but the approximation errors or deviations from the CES exact model for either prices or quantities could offset each other in the shares. The model fits for (109) and (141) could be much worse than the model fits for any model involving shares. Thus the share models will tend to give us an incomplete view of how well the CES model describes the actual data. Put another way, the use of shares does not make use of all available information on prices and quantities, whereas the models based on using (109) and (141) as estimating equations do use all available information and thus these models are the best at showing us how well the CES model approximates reality. This observation means that the unit cost estimation models that use shares, (110), (125) and (132), are less preferred than (109) and the utility function estimation models that use shares, (142), (144) and (148) are less preferred than (141). Similarly, differencing the data throws out information on exactly how good the underlying CES model is at approximating the underlying price and quantity data. Thus the unit cost function models using differences, (125), (129), (132) and (133) are less preferred than (109) and the utility function models using differences, (144), (145), (148) and (150) are less preferred than (141). If we reject share models and differenced models, we are left with the models defined by (109) and (141).

How can a choice be made between (109) and (141)? The answer to this question depends on the purpose for estimating CES preferences. If we want to predict q^t given e^t and p^t, then the model defined by equations (109) is the best choice. If we want to predict p^t given e^t and q^t, then (141) is the best choice. If the goal is to

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[96] As usual, we need to drop the estimating equations for one of the N commodities since the error terms in (150) are not statistically independent because the data for each period satisfies the linear constraint $p^t \cdot q^t = e^t$ for t = 0,1,...,T.

get a good estimate for the elasticity of substitution to use in the Lloyd Moulton formula, then run both (109) and (141) and choose the model with the best fit. As was mentioned earlier, if (109) fits the data perfectly, then so will (141) (as well as the other 10 models under consideration). However, in reality, neither (109) nor (141) will fit the data perfectly. If the underlying preference function is approximately equal to a linear utility function (so that the products are highly substitutable), then the model defined by (141) will fit the data better than the model defined by (109). On the other hand, if preferences are close to being of the no substitution variable so that the unit cost function is almost linear, then the model defined by (109) will fit the data better than the model defined by (141). Choosing between (109) and (141) based on how well the two models fit the data seems to be a sensible strategy.[97]

11. The Allen Quantity Index

Make the same general assumptions on the utility function f(q) that we made at the beginning of section 2 so that f(q) is a nonnegative, increasing, continuous and concave function of q defined for $q \geq 0_N$.[98] Let $C(f(q),p)$ be the consumer's cost function that is dual to the aggregator function f(q).[99] Let $p^t >> 0_N$ be the vector of observed prices that the consumer faces in period t for t = 0,1. Let $q^t >> 0_N$ be the vector of observed consumer choices for periods t = 0,1. We assume cost minimizing behavior on the part of the consumer in periods 0 and 1 so that the following equations are satisfied:

(151) $C(f(q^t),p^t) \equiv \min_q \{p^t \cdot q : f(q) \geq f(q^t) ; q \geq 0_N\} = p^t \cdot q^t$; t = 0,1.

The *Allen* (1949) *family of quantity indexes*, $Q_A(q^0,q^1,p)$, is defined for an arbitrary positive reference price vector $p >> 0_N$ as follows:

(152) $Q_A(q^0,q^1,p) \equiv C(f(q^1),p)/C(f(q^0),p)$.

The basic idea of the Allen quantity index dates back to Hicks (1941-42), who observed that if the price vector p were held fixed and the quantity vector q is free to vary, then C(f(q),p) is a perfectly valid cardinal measure of utility.[100]

As was the case with the true cost of living, the Allen definition simplifies considerably if the utility function happens to be linearly homogeneous. In this case, for any $p >> 0_N$, (152) simplifies to:

(153) $Q_A(q^0,q^1,p) = f(q^1)C(1,p)/f(q^0)C(1,p) = f(q^1)/f(q^0)$.

However, in the general case where the consumer has nonhomothetic preferences, we do not obtain the nice simplification given by (153).

As usual, it is useful to specialize the general definition of the Allen quantity index and let the reference price vector equal either the period 0 price vector p^0 or the period 1 price vector p^1:

(154) $Q_A(q^0,q^1,p^0) \equiv C(f(q^1),p^0)/C(f(q^0),p^0)$;
(155) $Q_A(q^0,q^1,p^1) \equiv C(f(q^1),p^1)/C(f(q^0),p^1)$.

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97 A possible disadvantage of using (109) or (141) to estimate σ is that nonlinear regression techniques have to be used in the estimation procedure. Thus an attractive alternative would be to use either (133) or (150) to estimate σ. These models are linear in the single unknown parameter.
98 In this section, we no longer assume that f(q) is linearly homogeneous. The results in this section are due to Diewert (2009; 239-241).
99 Recall definition (1) in section 2.
100 Samuelson (1974) called this a money metric measure of utility.

Index number formulae that are exact for either of the theoretical indexes defined by (154) and (155) do not seem to exist, at least for the case of nonhomothetic preferences that can be represented by a flexible functional form. However, we can find an index number formula that is exactly equal to the geometric mean of the Allen indexes defined by (154) and (155), where the underlying preferences are represented by a flexible functional form. Before demonstrating this result, we need the following proposition:

Proposition 13: Let x and y be N and M dimensional vectors respectively and let F^0 and F^1 be two general quadratic functions defined as follows:

(156) $F^0(x,y) \equiv a_0^0 + a^{0T}x + b^{0T}y + (½)x^TA^0x +(½)y^TB^0y + x^TC^0y$; $A^{0T} = A^0$; $B^{0T} = B^0$;
(157) $F^1(x,y) \equiv a_0^1 + a^{1T}x + b^{1T}y + (½)x^TA^1x +(½)y^TB^1y + x^TC^1y$; $A^{1T} = A^1$; $B^{1T} = B^1$

where the a_0^i are scalar parameters, the a^i and b^i are parameter vectors and the A^i, B^i and C^i are parameter matrices for i = 0,1. Note that the A^i and B^i are symmetric matrices. If $A^0 = A^1$, then the following equation holds for all x^0, x^1, y^0 and y^1:[101]

(158) $F^0(x^1,y^0) - F^0(x^0,y^0) + F^1(x^1,y^1) - F^1(x^0,y^1) = [\nabla_xF^0(x^0,y^0) + \nabla_xF^1(x^1,y^1)]\cdot[x^1 - x^0]$.

Straightforward differentiation of the functions defined by (156) and (157) and substitution into (158) proves the proposition. The identity (158) is a *generalized quadratic identity*. This identity will prove to be useful as will be seen below.

Suppose that the consumer's preferences can be represented by the general translog cost function, C(u,p) defined by (77), with the restrictions (78)-(81). Shephard's Lemma implies that the period t expenditure shares, s_n^t, will satisfy the following equations:[102]

(159) $s_n^t = \partial \ln C(u^t,p^t)/\partial \ln p_n = a_n^t + b_n^t \ln u^t + \sum_{i=1}^N a_{ni}^t \ln p_i^t$; t = 0,1

where $u^t \equiv f(q^t)$ for t = 0,1. Note that lnC(u,p) is quadratic in the variables $x_1 \equiv \ln p_1,\ldots,x_N \equiv \ln p_N$ and $y_1 \equiv \ln u$. Thus we will be able to apply the generalized quadratic identity to lnC(u,p).

Recall that in section 2, the Konüs Laspeyres cost of living index was defined by $P_K(p^0,p^1,q^0) \equiv C[f(q^0),p^1]/C[f(q^0),p^0]$ and the Konüs Paasche cost of living index was defined by $P_K(p^0,p^1,q^1) \equiv C[f(q^1),p^1]/C[f(q^1),p^0]$. Assuming that C(u,p) is the translog cost function, we can obtain an exact formula for the geometric mean of $P_K(p^0,p^1,q^0)$ and $P_K(p^0,p^1,q^1)$. The logarithm of this geometric mean is:

(160) $\ln\{[P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}\} = (½)\ln P_K(p^0,p^1,q^0) + (½)\ln P_K(p^0,p^1,q^1)$
$= (½)\ln[C(f(q^0),p^1)/C(f(q^0),p^0)] + (½)\ln[C(f(q^1),p^1)/C(f(q^1),p^0)]$ using definitions (3) and (4)
$= (½)\ln[C(u^0,p^1)/C(u^0,p^0)] + (½)\ln[C(u^1,p^1)/C(u^1,p^0)]$
$= (½)\{\ln C(u^0,p^1) - \ln C(u^0,p^0) + \ln C(u^1,p^1) - \ln C(u^1,p^0)\}$
$= (½)\Sigma_{n=1}^N \{[\partial \ln C(u^0,p^0)/\partial \ln p_n] + [\partial \ln C(u^1,p^1)/\partial \ln p_n]\}[\ln p_n^1 - \ln p_n^0]$
using (158) with $F^0(x,y) = F^1(x,y) \equiv \ln C(y,x)$ with $y \equiv \ln u$ and $x_n \equiv \ln p_n$ for n = 1,...,N
$= (½)\Sigma_{n=1}^N [s_n^0 + s_n^1][\ln p_n^1 - \ln p_n^0]$ using (159)
$= \ln P_T(p^0,p^1,q^0,q^1)$

[101] Balk (1998; 225-226) established this result using the Translog Lemma in Caves, Christensen and Diewert (1982; 1412) which is simply a logarithmic version of (158).
[102] We need to assume that the points (u^0,p^0) and (u^1,p^1) are in the regularity region where the translog cost function C(u,p) is well behaved.

where $P_T(p^0,p^1,q^0,q^1)$ is the Törnqvist Theil index number formula P_T defined in Chapter 4. The exact index number formula given by (160) is different from our earlier exact index number formula for P_T which was given by (88). The earlier result was $C(u^*,p^1)/C(u^*,p^0) = P_T(p^0,p^1,q^0,q^1)$ where u^* was the geometric mean of u^0 and u^1. Our new result is:

$$(161)\ P_T(p^0,p^1,q^0,q^1) = [C(f(q^0),p^1)/C(f(q^0),p^0)]^{1/2}[C(f(q^1),p^1)/C(f(q^1),p^0)]^{1/2} .$$

Thus P_T is also equal to the geometric mean of $C(f(q^0),p^1)/C(f(q^0),p^0)$ and $C(f(q^1),p^1)/C(f(q^1),p^0)$.

The implicit quantity index, $Q_{T*}(p^0,p^1,q^0,q^1)$ that corresponds to the Törnqvist Theil price index P_T is defined as the value ratio, $p^1\cdot q^1/p^0\cdot q^0$, divided by P_T. Thus we have:

$$\begin{aligned}(162)\ Q_{T*}(p^0,p^1,q^0,q^1) &\equiv [p^1\cdot q^1/p^0\cdot q^0]/P_T(p^0,p^1,q^0,q^1) \\ &= [C(f(q^1),p^1)/C(f(q^0),p^0)]/P_T(p^0,p^1,q^0,q^1) && \text{using (151)} \\ &= [C(f(q^1),p^1)/C(f(q^0),p^0)]/[\{C(f(q^0),p^1)/C(f(q^0),p^0)\}\{C(f(q^1),p^1)/C(f(q^1),p^0)\}]^{1/2} && \text{using (161)} \\ &= [\{C(f(q^1),p^0)/C(f(q^0),p^0)\}\{C(f(q^1),p^1)/C(f(q^0),p^1)\}]^{1/2} \\ &= [Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2}\end{aligned}$$

where the last equality follows using definitions (154) and (155). Thus the observable implicit Törnqvist Theil quantity index, $Q_{T*}(p^0,p^1,q^0,q^1)$, is exactly equal to the geometric mean of the two Allen quantity indexes defined by (154) and (155).

Note that in general, the geometric mean of the two "natural" Allen quantity indexes defined by (154) and (155) matches up with the geometric mean of the two "natural" Konüs price indexes defined by (3) and (4); i.e., using these definitions, we have:

$$\begin{aligned}(163)\ [P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}[Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2} &= C(f(q^1),p^1)/C(f(q^0),p^0) \\ &= p^1\cdot q^1/p^0\cdot q^0.\end{aligned}$$

Thus in general, these two "natural" geometric mean price and quantity indexes satisfy the product test. Under our translog assumptions, we have a special case of (163) where $Q_{T*}(p^0,p^1,q^0,q^1)$ is equal to $[Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2}$ and $P_T(p^0,p^1,q^0,q^1)$ is equal to $[P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}$.[103] This result justifies the use of P_T and Q_{T*} even if the consumer does not have homothetic preferences. Hence, it indirectly justifies the use of the Fisher and Walsh price indexes if consumers do not have homothetic preferences since these indexes will approximate $P_T(p^0,p^1,q^0,q^1)$ to the second order around an equal price and quantity point.

12. Modeling Changes in Tastes

Suppose that the consumer's preference function changes going from period 0 to period 1. What is an appropriate theoretical concept for a price index under these conditions?

Suppose that the consumer's utility function is $f^0(q)$ in period 0 and $f^1(q)$ in period 1. Let $C^0(u,p)$ and $C^1(u,p)$ be the cost functions that correspond to these preferences for periods 0 and 1, respectively. A reasonable strategy under these circumstances is the following one:

- Calculate the Laspeyres Konüs cost of living index using the preferences of period 0. This is the index $P_K(p^0,p^1,q^0) \equiv C^0(u^0,p^1)/C^0(u^0,p^0)$ where $u^0 = f(q^0)$ and q^0 satisfies $p^0\cdot q^0 = C^0(u^0,p^0)$.

103 See Diewert (2009; 239-241).

- Calculate the Paasche Konüs cost of living index using the preferences of period 1. This is the index $P_K(p^0,p^1,q^1) \equiv C^1(u^1,p^1)/C^1(u^1,p^0)$ where $u^1 = f(q^1)$ and q^1 satisfies $p^1 \cdot q^1 = C^1(u^1,p^1)$.
- Take the geometric mean of $P_K(p^0,p^1,q^0)$ and $P_K(p^0,p^1,q^1)$ as the final measure of price change over the two periods under consideration.

Make the additional assumption that the consumer's preferences can be modeled by translog cost functions in a region of regularity that includes $u^0 > 0$, $p^0 >> 0_N$ and $u^1 > 0$, $p^1 >> 0_N$. In this regularity region, the logarithms of the period t cost functions $C^t(u,p)$ are defined as follows:

$$(164)\ \ln C^0(u,p) \equiv F^0(x,y_1) \equiv a_0^{\ 0} + \Sigma_{n=1}^N a_n^{\ 0}x_n + b_1^{\ 0}y_1 + (½)x^TAx + (½)b_{11}^{\ 0}(y_1)^2 + \Sigma_{n=1}^N c_n^{\ 0}x_ny_1;$$
$$(165)\ \ln C^1(u,p) \equiv F^1(x,y_1) \equiv a_0^{\ 1} + \Sigma_{n=1}^N a_n^{\ 1}x_n + b_1^{\ 1}y_1 + (½)x^TAx + (½)b_{11}^{\ 1}(y_1)^2 + \Sigma_{n=1}^N c_n^{\ 1}x_ny_1$$

where $A = A^T$, $x^T \equiv [x_1,...,x_N] \equiv [\ln p_1,...,\ln p_N]$ and $y_1 \equiv \ln u$. Note that the parameters in (164) can be quite different from the parameters in (165) except that we assume that the $N(N+1)/2$ a_{ik} parameters in the A matrix are the same in (164) and (165). It can be seen that the quadratic functions $F^0(x,y_1)$ and $F^1(x,y_1)$ are special cases of the functions $F^0(x,y)$ and $F^1(x,y)$ defined by (156) and (157) in the previous section. In order for $C^t(u,p)$ to be linearly homogeneous in p, we need to impose the restrictions $\Sigma_{n=1}^N a_n^{\ t} = 1$, $A1_N = 0_N$ and $\Sigma_{n=1}^N c_n^{\ t} = 0$ on the parameters for $t = 0,1$, where 1_N is a vector of ones of dimension N.

Shephard's Lemma implies that the period t expenditure shares, $s_n^{\ t}$, will satisfy the following equations:

$$(166)\ s_n^{\ t} = \partial \ln C(u^t,p^t)/\partial \ln p_n = a_n^{\ t} + c_n^{\ t}\ln u^t + \textstyle\sum_{k=1}^N a_{nk}\ln p_k^{\ t}; \qquad t = 0,1$$

The logarithm of the geometric mean of $P_K(p^0,p^1,q^0)$ and $P_K(p^0,p^1,q^1)$ is equal to the following expression:

$$\begin{aligned}(167)\ \ln\{[P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}\} &= (½)\ln P_K(p^0,p^1,q^0) + (½)\ln P_K(p^0,p^1,q^1)\\ &= (½)\ln[C^0(u^0,p^1)/C^0(u^0,p^0)] + (½)\ln[C^1(u^1,p^1)/C^1(u^1,p^0)]\\ &= (½)\{\ln C^0(u^0,p^1) - \ln C^0(u^0,p^0) + \ln C^1(u^1,p^1) - \ln C^1(u^1,p^0)\}\\ &= (½)\Sigma_{n=1}^N \{[\partial \ln C^0(u^0,p^0)/\partial \ln p_n] + [\partial \ln C^1(u^1,p^1)/\partial \ln p_n]\}[\ln p_n^{\ 1} - \ln p_n^{\ 0}] \quad \text{using (158)}\\ &= (½)\Sigma_{n=1}^N [s_n^{\ 0} + s_n^{\ 1}][\ln p_n^{\ 1} - \ln p_n^{\ 0}] \quad \text{using (166)}\\ &= \ln P_T(p^0,p^1,q^0,q^1)\end{aligned}$$

where $P_T(p^0,p^1,q^0,q^1)$ is the Törnqvist Theil index number formula P_T defined in Chapter 4 and $u^t \equiv f^t(q^t)$ for $t = 0,1$. Note that (167) implies the following equalities:

$$(168)\ P_T(p^0,p^1,q^0,q^1) = [P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2} = \{[C^0(u^0,p^1)/C^0(u^0,p^0)][C^1(u^1,p^1)/C^1(u^1,p^0)]\}^{1/2}$$

where $u^t \equiv f^t(q^t)$ for $t = 0,1$. Thus at least some forms of taste change can be accomodated by the use of the Törnqvist Theil price index.

13. Conditional Cost of Living Indexes

The models of consumer behavior considered in previous sections all assumed that the consumer maximized a utility function, $f(q)$, subject to a budget constraint of the form $p \cdot q = e$, where $e > 0$ is the total amount of "income" or expenditure that the consumer allocates to the purchase of the N goods and services under consideration. However, the utility of the consumer may be affected by other variables in addition to purchases of market goods and services that are represented by $q \equiv [q_1,...,q_N]$. Thus we now assume that utility is affected by an M dimensional vector of nonmarket *environmental*[104] *or demographic*[105] *variables or*

[104] This is the terminology used by Pollak (1989; 181) in his model of the conditional cost of living concept.

public goods, $z \equiv (z_1,z_2,\ldots,z_M)$. We suppose that the preferences of the household over different combinations of market commodities q and nonmarket variables z can be represented by the continuous utility function f(q,z).[106] For periods t = 0,1, it is assumed that the observed household consumption vector $q^t \equiv (q_1^t,\ldots,q_N^t) > 0_N$ is a solution to the following household expenditure minimization problem:

$$(169)\quad \min_q \{p^t\cdot q: f(q,z^t) \geq u^t; q \geq 0_N\} \equiv C(p^t,u^t,z^t) = p^t\cdot q^t ; \qquad t = 0,1$$

where z^t is the environmental vector facing household h in period t, $u^t \equiv f(q^t,z^t)$ is the utility level achieved by household h during period t and C is the *conditional cost or expenditure function* that is dual to the utility function f.[107] Basically, these assumptions mean that the household has *stable preferences* over the same list of market commodities during the two periods under consideration and the household chooses its market consumption vector in the most cost efficient way during each period, conditional on the environmental vector z^t that it faces during each period t.

With the above assumptions in mind, the family of Pollak (1975; 142) *conditional cost of living index* between periods 0 and 1, conditional on the utility level u and the nonmarket vector z, is defined as follows:[108]

$$(170)\quad P_{Po}(p^0,p^1,u,z) \equiv C(p^1,u,z)/C(p^0,u,z).$$

In the above definition, the household utility level u and the vector of nonmarket or environmental variables z are held constant in the numerator and denominator of the right hand side of (170). Thus only the price variables are different, which is precisely what we want in a theoretical definition of a consumer price index. Note that if z does not enter the consumer's utility function so that f(q,z) is just f(q), then C(u,p,z) becomes C(u,p) and the Pollak conditional cost of living indexes collapses down to the Konüs family of true cost of living indexes, $P_K(p^0,p^1,q)$ where u = f(q).

The *Laspeyres Pollak conditional cost of living index* is defined by (169) when $(u,z) = (u^0,z^0)$. Using (169) for t = 0, a feasibility argument establishes the following upper bound to $P_{Po}(p^0,p^1,u^0,z^0)$; i.e., we have

$$(171)\quad P_{Po}(p^0,p^1,u^0,z^0) \leq p^1\cdot q^0/p^0\cdot q^0 = P_L(p^0,p^1,q^0,q^1)$$

where $P_L(p^0,p^1,q^0,q^1)$ is the ordinary Laspeyres price index for market commodities. The *Paasche Pollak conditional cost of living index* is defined by (169) when $(u,z) = (u^1,z^1)$. Using (169) for t = 1, a feasibility argument establishes the following lower bound to $P_{Po}(p^0,p^1,u^1,z^1)$; i.e., we have

$$(172)\quad P_{Po}(p^0,p^1,u^1,z^1) \geq p^1\cdot q^1/p^0\cdot q^1 = P_P(p^0,p^1,q^0,q^1)$$

where $P_P(p^0,p^1,q^0,q^1)$ is the ordinary Paasche price index for market commodities.[109]

It is possible to obtain two sided bounds to a Pollak conditional cost of living index; i.e., we have the following generalization of Proposition 1 above:

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105 Caves, Christensen and Diewert (1982; 1409) used the terms *demographic variables* or *public goods* to describe the vector of conditioning variables z in their generalized model of the Konüs price index or cost of living index. Weather variables could also be included in the z vector.

106 We initially assume that f(q,z) is jointly continuous in (q,z), increasing in the components of q and concave in the components of q.

107 Conditional cost functions were first defined by Pollak (1975; 142).

108 See also Caves, Christensen and Diewert (1982; 1409).

109 The bounds (171) and (172) can be found in Caves, Christensen and Diewert (1982; 1409-1410).

Proposition 14: There exists a number λ^* between 0 and 1 such that

(173) $P_L \le P_{Po}[p^0,p^1,\lambda^*(q^0,z^0) + (1-\lambda^*)(q^1,z^1)] \le P_P$ or $P_P \le P_P[p^0,p^1,\lambda^*(q^0,z^0) + (1-\lambda^*)(q^1,z^1)] \le P_L$.

The proof of Proposition 14 is similar to the proof of Proposition 1; see Diewert (2001) for the details.

There is one additional result on conditional cost of living indexes that is very useful and it involves the use of the generalized quadratic identity (158) and a generalized translog functional form for the conditional cost function C(u,p,z). Suppose that the logarithm of the consumer's conditional cost function is defined as follows:

(174) $\ln C(p,u,z) \equiv F(x,y) \equiv a_0 + a^Tx + b^Ty + (\tfrac{1}{2})x^TAx + (\tfrac{1}{2})y^TBy + x^TCy$; $\quad A^T = A; B^T = B$

where $x^T \equiv [\ln p_1,...,\ln p_N]$, $y^T \equiv [\ln u,z_1,...,z_M]$, a_0 is a scalar parameter, a and b are parameter vectors, A is an N by N symmetric matrix of parameters, B is an M+1 by M+1 symmetric matrix of parameters and C is an N by M+1 matrix of parameters. In order to impose linear homogeneity in prices on C(p,u,z), we require that the following restrictions on the parameters hold:

(175) $a^T1_N = 1$; $A1_N = 0_N$ and $C^T1_N = 0_{M+1}$.

Note that the demographic variables enter the right hand side of (174) in a linear and quadratic fashion; this allows for the z_m variables to be discrete variables that can take on the value 0.[110] We assume that the period 0 and 1 price vectors, p^0 and p^1, are strictly positive and we assume that $q^t >> 0_N$ solves the period t conditional cost minimization problem defined by (169) for t = 0,1. Thus we have the following equations:

(176) $p^t \cdot q^t = C(p^t,u^t,z^t)$; $\quad t = 0,1.$

Shephard's Lemma can be applied to these cost minimization problems, since the translog conditional cost function C(p,u,z) defined by (174) is differentiable with respect to the components of p. Thus we have the following equations:[111]

$$\begin{aligned}(177)\ q_n^t &= \partial C(p^t,u^t,z^t)/\partial p_n ; \qquad n = 1,\ldots,N ; t = 0,1\\ &= [C(p^t,u^t,z^t)/p_n^t]\partial \ln C(p^t,u^t,z^t)/\partial \ln p_n.\end{aligned}$$

Using definition (174), the above equations (177) can be rearranged to read as follows:

(178) $s_n^t = \partial \ln C(p^t,u^t,z^t)/\partial \ln p_n = a_n + \Sigma_{k=1}^N a_{nk}\ln p_k^t + \Sigma_{m=1}^M c_{nm}z_m^t$; $\quad n = 1,\ldots,N ; t = 0,1.$

Now take the logarithm of the geometric mean of the the two conditional indexes $P_{Po}(p^0,p^1,u^0,z^0)$ and $P_{Po}(p^0,p^1,u^1,z^1)$. We find that:

$$\begin{aligned}(179)\ &\ln\{[P_{Po}(p^0,p^1,u^0,z^0)P_{Po}(p^0,p^1,u^1,z^1)]^{1/2}\}\\ &= (\tfrac{1}{2})[\ln C(p^1,u^0,z^0) - \ln C(p^0,u^0,z^0) + \ln C(p^1,u^1,z^1) - \ln C(p^0,u^1,z^1)]\\ &= (\tfrac{1}{2})\Sigma_{n=1}^N [(\partial \ln C(p^0,u^0,z^0)/\partial \ln p_n) + (\partial \ln C(p^1,u^1,z^1)/\partial \ln p_n)][\ln p_n^1 - \ln p_n^0]\end{aligned}$$

[110]Thus the number of children in a household is a discrete variable that can take on the value 0. If we entered the corresponding variable as z_1 on the right hand side of (174) as the logarithm of the number of children, the definition of C(p,u,z) would break down.

[111] See equations (83) in section 7 above. We require that (p^t,u^t,z^t) be in the regularity set where C(p,u,z) is positive and increasing in the components of p and u and concave in p holding u and z fixed.

using definition (174) and the generalized quadratic identity (158)

$= (½)\Sigma_{n=1}^{N} [s_n^1 + s_n^0][\ln p_n^1 - \ln p_n^0]$ using (178)

$= \ln P_T(p^0,p^1,q^0,q^1)$

where $P_T(p^0,p^1,q^0,q^1)$ is the Törnqvist Theil index number formula P_T defined in Chapter 4. Note that (179) implies the following equalities:[112]

$$(180)\ P_T(p^0,p^1,q^0,q^1) = [P_{Po}(p^0,p^1,u^0,z^0)P_{Po}(p^0,p^1,u^1,z^1)]^{1/2}$$
$$= \{[C(p^1,u^0,z^0)/C(p^0,u^0,z^0)][C(p^1,u^1,z^1)/C(p^0,u^1,z^1)]\}^{1/2}.$$

Thus the Törnqvist Theil price index has many useful interpretations.

14. Reservation Prices and New and Disappearing Products

New products appear and old products disappear at substantial annual rates in most economies in the world today. This creates substantial problems for national statistical offices that are responsible for producing consumer price indexes, since traditional index number theory is based on matching prices for identical products over time. Thus up to now, our treatment of the different approaches to index number theory has assumed that the number of consumer goods and services available to the public has remained constant over the two periods being compared. This implicit assumption is not an accurate reflection of reality: in practice, perhaps one to two percent of all consumer products appear or disappear each month. The economic approach to index number theory can be helpful in providing a framework for treating this lack of matching problem.

The basic idea for the treatment of new products in a cost of living type price index is as follows. Assume that the consumer has the same preferences over continuing and new and disappearing products over periods 0 and 1. For a product that is not available in one of the two periods under consideration, the quantity consumed is obviously equal to zero units. The corresponding prices for these products that are present in only one of the two periods are missing. It turns out that if we can somehow estimate *reservation prices* for these missing products in the two periods under consideration, then normal index number theory using the economic approach to index number theory can be applied. The reservation price for a missing product is the price that is just high enough to induce purchasers of the product to demand zero units of it. This reservation price approach for the treatment of new goods is due to Hicks (1940; 114). Hofsten (1952; 95-97) extended the approach of Hicks to cover the case of disappearing goods as well.

In Chapter 8, we will consider several alternative methods that have been suggested in the literature to estimate reservation prices.[113] In the present section, we will use maximum overlap price indexes to form approximations to reservation prices and we will derive some theoretical bias estimates for these approximate reservation prices. A *maximum overlap index*[114] is one that constructs a price index using just the products that are present in the two periods under consideration. Typically, the maximum overlap price index will be biased compared to the "true" cost of living index, which uses reservation prices. This bias in

[112] This result is a special case of a more general result established by Caves, Christensen and Diewert (1982; 1410). Their result also allows for taste change between the periods.

[113] These methods include Feenstra's (1994) CES methodology, the Diewert and Feenstra (2019) methodology that involves the estimation of the preference function that is exact for the Fisher ideal index and methodologies based on experimental economics. See Brynjolfsson, Collis, Diewert, Eggers and Fox (2018) (2020) and Diewert, Fox and Schreyer (2019) on the experimental approach.

[114] This type of index dates back to Marshall (1887). Keynes (1930; 94) called it the highest common factor method while Triplett (2004; 18) called it the overlapping link method. See Diewert (1993c; 52-56) for additional material on the early history of the new goods problem.

the deflator translates into a corresponding bias in the real output aggregate. We will evaluate this bias in the context of a statistical agency that uses a maximum overlap Törnqvist Theil price index.[115]

Consider two periods, 0 and 1. There are three classes of commodities. Class 1 products are present in both periods with positive prices and quantities for all N products in this group. Denote the period t price and quantity vectors for this group of products as $p_1^t \equiv [p_{11}^t,...,p_{1N}^t] >> 0_N$ and $q_1^t \equiv [q_{11}^t,...,q_{1N}^t] >> 0_N$ for t = 0,1.

Class 2 products are the *new* goods and services that are not available in period 0 but are available in period 1. Denote the period 0 price and quantity vectors for this group of K products as $p_2^{0*} \equiv [p_{21}^{0*},...,p_{2K}^{0*}] >> 0_N$ and $q_2^0 \equiv [q_{11}^0,...,q_{1K}^0] = 0_N$. The prices in the vector p_2^{0*} are the positive reservation prices that make the demand for these products in period 0 equal to zero. These reservation prices have to be estimated somehow. The period 1 price and quantity vectors for these K products are $p_2^1 \equiv [p_{21}^1,...,p_{2K}^1] >> 0_N$ and $q_2^1 \equiv [q_{21}^1,...,q_{2K}^1] >> 0_N$ and these vectors are observable.

Class 3 products are the *disappearing* goods and services that were available in period 0 but are not available in period 1. Denote the period 0 price and quantity vectors for this group of M products as $p_3^0 \equiv [p_{31}^0,...,p_{3M}^0] >> 0_N$ and $q_3^0 \equiv [q_{31}^0,...,q_{3M}^0] >> 0_N$. The period 1 price and quantity vectors for these M products are $p_3^{1*} \equiv [p_{31}^{1*},...,p_{3M}^{1*}] >> 0_N$ and $q_3^1 \equiv [q_{31}^1,...,q_{3M}^1] = 0_N$. The prices in the vector p_3^{1*} are the positive reservation prices that make the demand for these products in period 1 equal to zero. Again, these reservation prices have to be estimated somehow.

Define the *true expenditure shares* for product n in Group 1 for periods 0 and 1, s_{1n}^0 and s_{1n}^1, as the following fractions of total expenditure in period 0 or 1:

$$(181)\ s_{1n}^0 \equiv p_{1n}^0 q_{1n}^0/[p_1^0\cdot q_1^0 + p_2^{0*}\cdot q_2^0 + p_3^0\cdot q_3^0]\ ; \qquad n = 1,...,N;$$
$$= p_{1n}^0 q_{1n}^0/[p_1^0\cdot q_1^0 + p_3^0\cdot q_3^0] \qquad \text{since } q_2^0 = 0_N;$$
$$(182)\ s_{1n}^1 \equiv p_{1n}^1 q_{1n}^1/[p_1^1\cdot q_1^1 + p_2^1\cdot q_2^1 + p_3^{1*}\cdot q_3^1]\ ; \qquad n = 1,...,N;$$
$$= p_{1n}^1 q_{1n}^1/[p_1^1\cdot q_1^1 + p_2^1\cdot q_2^1] \qquad \text{since } q_3^1 = 0_N.$$

Note that these shares can be calculated using observable data; i.e., these shares do not depend on the imputed prices p_2^{0*} and p_3^{1*}.

Define the *true expenditure shares* for product k in Group 2 for periods 0 and 1, s_{2k}^0 and s_{2k}^1, as follows:

$$(183)\ s_{2k}^0 \equiv p_{2k}^0 q_{2k}^0/[p_1^0\cdot q_1^0 + p_2^{0*}\cdot q_2^0 + p_3^0\cdot q_3^0]\ ; \qquad k = 1,...,K;$$
$$= p_{2k}^0 q_{2k}^0/[p_1^0\cdot q_1^0 + p_3^0\cdot q_3^0] \qquad \text{since } q_2^0 = 0_N;$$
$$= 0\ ; \qquad \text{since } q_{2k}^0 = 0;$$
$$(184)\ s_{2k}^1 \equiv p_{2k}^1 q_{2k}^1/[p_1^1\cdot q_1^1 + p_2^1\cdot q_2^1 + p_3^{1*}\cdot q_3^1]\ ; \qquad k = 1,...,K;$$
$$= p_{2k}^1 q_{2k}^1/[p_1^1\cdot q_1^1 + p_2^1\cdot q_2^1] \qquad \text{since } q_3^1 = 0_N.$$

Note that these shares can also be calculated using observable data.

Define the *true expenditure shares* for product m in Group 3 for periods 0 and 1, s_{3m}^0 and s_{3m}^1, as follows:

$$(185)\ s_{3m}^0 \equiv p_{3m}^0 q_{3m}^0/[p_1^0\cdot q_1^0 + p_2^{0*}\cdot q_2^0 + p_3^0\cdot q_3^0]\ ; \qquad m = 1,...,M;$$
$$= p_{3m}^0 q_{3m}^0/[p_1^0\cdot q_1^0 + p_3^0\cdot q_3^0] \qquad \text{since } q_2^0 = 0_N;$$

[115] The material in this section is mostly due to de Haan and Krsinich (2012) (2014). Diewert, Fox and Schreyer (2017b) extended the de Haan and Krisinich analysis to bias estimates if the Laspeyres, Paasche or Fisher maximum overlap indexes are used in place of the Törnqvist Theil price index.

$$(186)\ s_{3m}^{1} \equiv p_{3m}^{1}q_{3m}^{1}/[p_1^{1}\cdot q_1^{1} + p_2^{1}\cdot q_2^{1} + p_3^{1*}\cdot q_3^{1}]\ ; \qquad m = 1,\ldots,M;$$
$$= p_{3m}^{1}q_{3m}^{1}/[p_1^{1}\cdot q_1^{1} + p_2^{1}\cdot q_2^{1}] \qquad \text{since } q_3^{1} = 0_N;$$
$$= 0 \qquad \text{since } q_{3m}^{1} = 0.$$

Note that these shares can also be calculated using observable data.

Now define the expenditure shares for product Group 1 using just the products that are in Group 1. These are the shares that are relevant for the maximum overlap indexes which will be defined shortly. The *maximum overlap share* for product n in period t, s_{1nO}^{t}, is defined as follows:

$$(187)\ s_{1nO}^{t} \equiv p_{1n}^{t}q_{1n}^{t}/p_1^{t}\cdot q_1^{t}\ ; \qquad t = 0,1;\ n = 1,\ldots,N.$$

These maximum overlap shares are also observable. It can be seen that the following relationships hold between the true Group 1 shares and the maximum overlap Group 1 shares:[116]

$$(188)\ s_{1n}^{0} = s_{1nO}^{0}p_1^{0}\cdot q_1^{0}/[p_1^{0}\cdot q_1^{0} + p_3^{0}\cdot q_3^{0}]\ ; \qquad n = 1,\ldots,N;$$
$$= s_{1nO}^{0}[1 - \Sigma_{m=1}^{M} s_{3m}^{0}]\ ;$$
$$(189)\ s_{1n}^{1} = s_{1nO}^{1}p_1^{1}\cdot q_1^{1}/[p_1^{1}\cdot q_1^{1} + p_2^{1}\cdot q_2^{1}]\ ; \qquad n = 1,\ldots,N;$$
$$= s_{1nO}^{1}[1 - \Sigma_{k=1}^{K} s_{2k}^{1}]\ .$$

Let P_{TO} denote the *Törnqvist maximum overlap index*. The logarithm of this index is defined as follows:

$$(190)\ \ln P_{TO} \equiv \Sigma_{n=1}^{N}\ (1/2)(s_{1nO}^{0} + s_{1nO}^{1})\ln(p_{1n}^{1}/p_{1n}^{0}).$$

The logarithm of the *true Törnqvist index*, P_T, is defined as follows:

$$(191)\ \ln P_T \equiv \Sigma_{n=1}^{N}\ \tfrac{1}{2}(s_{1n}^{0} + s_{1n}^{1})\ln(p_{1n}^{1}/p_{1n}^{0}) + \Sigma_{k=1}^{K}\ \tfrac{1}{2}(s_{2k}^{0} + s_{2k}^{1})\ln(p_{2k}^{1}/p_{2k}^{0*})$$
$$+ \Sigma_{m=1}^{M}\ \tfrac{1}{2}(s_{3m}^{0} + s_{3m}^{1})\ln(p_{3m}^{1*}/p_{3m}^{0})$$
$$= \Sigma_{n=1}^{N}\ \tfrac{1}{2}(s_{1n}^{0} + s_{1n}^{1})\ln(p_{1n}^{1}/p_{1n}^{0}) + \Sigma_{k=1}^{K}\ \tfrac{1}{2}(0 + s_{2k}^{1})\ln(p_{2k}^{1}/p_{2k}^{0*})$$
$$+ \Sigma_{m=1}^{M}\ \tfrac{1}{2}(s_{3m}^{0} + 0)\ln(p_{3m}^{1*}/p_{3m}^{0}) \qquad \text{using (183) and (186)}$$
$$= \Sigma_{n=1}^{N}\ \tfrac{1}{2}\{s_{1nO}^{0}[1-\Sigma_{m=1}^{M} s_{3m}^{0}]+s_{1nO}^{1}[1 - \Sigma_{k=1}^{K} s_{2k}^{1}]\}\ln(p_{1n}^{1}/p_{1n}^{0})$$
$$+ \Sigma_{k=1}^{K}\ \tfrac{1}{2}(s_{2k}^{1})\ln(p_{2k}^{1}/p_{2k}^{0*})\ + \Sigma_{m=1}^{M}\ \tfrac{1}{2}(s_{3m}^{0})\ln(p_{3m}^{1*}/p_{3m}^{0}) \qquad \text{using (188) and (189)}$$
$$= \ln P_{TO} + \tfrac{1}{2}\Sigma_{k=1}^{K}\ s_{2k}^{1}[\ln(p_{2k}^{1}/p_{2k}^{0*}) - \Sigma_{n=1}^{N}\ s_{1nO}^{1}\ \ln(p_{1n}^{1}/p_{1n}^{0})]$$
$$+ \tfrac{1}{2}\Sigma_{m=1}^{M}\ s_{3m}^{0}[\ln(p_{3m}^{1*}/p_{3m}^{0}) - \Sigma_{n=1}^{N}\ s_{1nO}^{0}\ \ln(p_{1n}^{1}/p_{1n}^{0})] \qquad \text{using (190)}$$
$$= \ln P_{TO} + \ln\kappa + \ln\mu$$

where the logarithms of the terms κ and μ are defined as:

$$(192)\ \ln\kappa \equiv (1/2)\Sigma_{k=1}^{K}\ s_{2k}^{1}[\ln(p_{2k}^{1}/p_{2k}^{0*}) - \Sigma_{n=1}^{N}\ s_{1nO}^{1}\ \ln(p_{1n}^{1}/p_{1n}^{0})]$$
$$= (1/2)\Sigma_{k=1}^{K}\ s_{2k}^{1}[\ln(p_{2k}^{1}/p_{2k}^{0*}) - \ln P_{JO}^{1}];$$
$$(193)\ \ln\mu \equiv (1/2)\Sigma_{m=1}^{M}\ s_{3m}^{0}[\ln(p_{3m}^{1*}/p_{3m}^{0}) - \Sigma_{n=1}^{N}\ s_{1nO}^{0}\ \ln(p_{1n}^{1}/p_{1n}^{0})]$$
$$= (1/2)\Sigma_{m=1}^{M}\ s_{3m}^{0}[\ln(p_{3m}^{1*}/p_{3m}^{0}) - \ln P_{JO}^{0}]$$

where the (weighted) *Jevons index* using the maximum overlap share weights of period 1 is P_{JO}^{1} and the (weighted) *Jevons index* using the maximum overlap share weights of period 0 is P_{JO}^{0}; i.e., the logarithm of these two indexes are defined as follows:[117]

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[116] These relationships are due to de Haan and Krsinich (2012; 31-32).

[117] These indexes could also be described as Cobb Douglas indexes. The indexes defined by (194) have also been described as geometric Paasche and geometric Laspeyres indexes, respectively.

(194) $\ln P_{JO}^1 \equiv \Sigma_{n=1}^N s_{1nO}^1 \ln(p_{1n}^1/p_{1n}^0)$; $\ln P_{JO}^0 \equiv \Sigma_{n=1}^N s_{1nO}^0 \ln(p_{1n}^1/p_{1n}^0)$.

Exponentiating both sides of (191) leads to the following relationship between the "true" cost of living index P_T and the price index P_{TO} that is defined over products that are available in both periods:[118]

(195) $P_T = P_{TO} \times \kappa \times \mu$.

The term κ defined by (192) can be regarded as a measure of the *reduction* in the true cost of living due to the introduction of new products. The period 0 imputed price for new product k, p_{2k}^{0*}, is likely to be higher than the actual price for new product k in period 1 adjusted for general inflation, p_{2k}^1/P_{JO}^1, and thus κ is likely to be less than 1. The bigger is the share of new products in period 1, $\Sigma_{k=1}^K s_{2k}^1$, the more κ will be less than 1. Note that the logarithmic contribution of each new product to the reduction in the true cost of living can be measured using the additive decomposition that definition (192) provides.

The inflation adjustment term μ defined by (193) can be regarded as a measure of the *increase* in the true cost of living due to the disappearance of existing products. The period 1 imputed price for disappearing product m, p_{3m}^{1*}, is likely to be higher than the actual price for product m in period 0 adjusted for general inflation, $p_{3m}^0 P_{JO}^0$, and thus μ is likely to be greater than 1. The bigger is the share of disappearing products in period 0, $\Sigma_{m=1}^M s_{3m}^0$, the more μ will be greater than 1.

The decomposition defined by (191) is also useful in the context of defining *imputed carry backward or carry forward prices* for products that may be new or unavailable. Recall that the imputed reservation prices in period 0 are the prices p_{2k}^{0*} and the imputed reservation prices in period 1 are the prices p_{3m}^{1*}. Rough estimates or more precise econometric estimates have to be made for these reservation prices. However, it is possible to use available information on prices and quantities for periods 0 and 1 in order to define the following *carry backward prices* p_{2kb}^0 for the missing products in period 0 and the following *carry forward prices* p_{3mf}^1 for the missing products in period 1:

(196) $p_{2kb}^0 \equiv p_{2k}^1/P_{JO}^1$ for k = 1,...,K and $p_{3mf}^1 \equiv p_{3m}^0 P_{JO}^0$ for m = 1,...,M.

Thus the inflation adjusted carry forward price defined by (196) for the missing product m in period 1 takes the observed price for product m in period 0, p_{3m}^0 and adjusts it for general inflation for the group of products that are present in both periods 0 and 1 using the weighted maximum overlap Jevons index P_{JO}^0. Similarly, the inflation adjusted carry backward price defined by (195) for the missing product k in period 0 takes the observed price for product k in period 1, p_{2k}^1 and deflates it by the weighted Jevons maximum overlap price index, P_{JO}^1. The above inflation adjusted imputed prices are more reasonable than the *constant carry forward prices*, p_{3m}^0, or *constant carry backward prices*, p_{2k}^1, which are frequently used to fill in the missing prices. From (190), (191) and (189), it can be seen that if the reservation prices are equal to their inflation adjusted carry forward prices (so that $p_{3m}^{1*} = p_{3mf}^1$ for m = 1,...,M) and inflation adjusted carry backward prices (so that $p_{2k}^{0*} = p_{2kb}^0$ for k = 1,...,K), then the true Törnqvist index P_T will equal its maximum overlap counterpart, P_{TO}.

However, in general, economic theory suggests that the reservation prices will be greater than their inflation adjusted carry forward or backward prices. Thus we define the following *margin terms*, κ_k and μ_m, which express how much higher each reservation price is from its inflation adjusted carry forward or backward price counterpart:

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[118] This formula was first derived by de Haan and Krsinich (2012; 31-32) (2014; 344). They obtained imputed prices for the missing products by using hedonic regressions, which will be studied in some detail in Chapter 8.

(197) $1 + \kappa_k \equiv p_{2k}^{0*}/p_{2kb}^{0}$; $k = 1,...,K$;
(198) $1 + \mu_m \equiv p_{3m}^{1*}/p_{3mf}^{1}$; $m = 1,...,M$.

Now substitute definitions (195)-(198) into (191) and we obtain the following *exact relationship* between the true Törnqvist index P_T and its maximum overlap counterpart P_{TO}:

(199) $\ln(P_T/P_{TO}) = -\Sigma_{k=1}^{K} (1/2)s_{2k}^{1} \ln(1 + \kappa_k) + \Sigma_{m=1}^{M} (1/2)s_{3m}^{0} \ln(1 + \mu_m)$.

Exponentiate both sides of (199) and subtract 1 from both sides of the resulting expression. Define the right hand side of the resulting expression as the function $g(\kappa_1,...,\kappa_K, \mu_1,...,\mu_M)$ and approximate g by taking the first order Taylor series approximation to g evaluated at $0 = \kappa_1 = ... = \kappa_K = \mu_1 =... = \mu_M$. The resulting approximation to $(P_T/P_{TO}) - 1$ is the following one:[119]

(200) $(P_T/P_{TO}) - 1 \approx \Sigma_{m=1}^{M} (1/2)s_{3m}^{0} \mu_m - \Sigma_{k=1}^{K} (1/2)s_{2k}^{1}\kappa_k$.

The period 0 and 1 value aggregates for the goods and services in the group of N + K + M commodities under consideration, V^0 and V^1, are defined as follows:

(201) $V^0 \equiv p_1^0 \cdot q_1^0 + p_3^0 \cdot q_3^0$; $V^1 \equiv p_1^1 \cdot q_1^1 + p_2^1 \cdot q_2^1$.

The "true" *implicit Törnqvist quantity index* Q_T is defined as the value ratio, V^1/V^0, deflated by the "true" Törnqvist price index, P_T ; i.e., we have:

(202) $Q_T \equiv [V^1/V^0]/P_T$.

Statistical agencies can use maximum overlap Törnqvist Theil price indexes to deflate final demand aggregates in order to construct aggregate quantity or volume indexes.[120] Thus in our context, the *maximum overlap Törnqvist Theil quantity index*, Q_{TO}, is defined as follows:

(203) $Q_{TO} \equiv [V^1/V^0]/P_{TO}$.

The *reciprocal* of the bias in Q_{TO} relative to its true counterpart Q_T can be measured by the ratio Q_T/Q_{TO}:

(204) $Q_T/Q_{TO} = P_{TO}/P_T$

where we have used definitions (202) and (203) to derive (204). An exact expression for the logarithm of P_{TO}/P_T can be obtained from (199):

(205) $\ln(P_{TO}/P_T) = \Sigma_{k=1}^{K} (1/2)s_{2k}^{1} \ln(1 + \kappa_k) - \Sigma_{m=1}^{M} (1/2)s_{3m}^{0} \ln(1 + \mu_m)$.

Exponentiate both sides of (205) and subtract 1 from both sides of the resulting expression. Define the right hand side of the resulting expression as the function $h(\kappa_1,...,\kappa_K,\mu_1,...,\mu_M)$ and approximate h by taking the

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[119] This formula is similar in spirit to the highly simplified approximate new goods bias formulae obtained by Diewert (1987; 779) (1998; 51-54).
[120] The US Bureau of Labor Statistics uses the Törnqvist price index as its target index for its chained CPI. Typically, there are no adjustments for new and disappearing products so these Törnqvist price indexes are essentially maximum overlap price indexes.

first order Taylor series approximation to h evaluated at $0 = \kappa_1 = ... = \kappa_K = \mu_1 = ... = \mu_M$. The resulting approximation to $(Q_T/Q_{TO}) - 1$ is the following one:

$$(206)\ (Q_T/Q_{TO}) - 1 \approx \Sigma_{k=1}^K (1/2)s_{2k}^1\kappa_k - \Sigma_{m=1}^M (1/2)s_{3m}^0 \mu_m.$$

Thus if there are no disappearing goods, the right hand side of (206) becomes $\Sigma_{k=1}^K (1/2)s_{2k}^1\kappa_k$ and this number is a measure of the downward bias in the maximum overlap Törnqvist quantity index for the value aggregate in percentage points. That is, (206) gives the downward bias in welfare from ignoring new goods and services.

For analogous bias formulae for price and quantity aggregates that are constructed using maximum overlap Laspeyres, Paasche or Fisher indexes, see Diewert, Fox and Schreyer (2017b).

15. Becker's Theory of the Allocation of Time

Peter Hill (1999), in discussing the classic study by Nordhaus (1997) on the price of light, raised the issue as to how should a cost of living index treat *household production* where consumers combine purchased market goods or "inputs" to produce finally demanded "commodities" that yield utility:[121]

"There is another area in which the definition of a COL requires further clarification and precision. From what is utility derived? Households do not consume many of the goods and services they purchase directly but use them to produce other goods or services from which they derive utility. In a recent stimulating and important paper, Nordhaus has used light as a case study. Households purchase items such as lamps, electric fixtures and fittings, light bulbs and electricity to produce light, which is the product they consume directly. … The light example is striking because Nordhaus provides a plausible case for arguing that the price of light, measured in lumens, has fallen absolutely (at least in US dollars) and dramatically over the last two centuries as a result of major inventions, discoveries and 'tectonic' improvements in the technology of producing light.

The question that arises is whether goods and services that are essentially *inputs* into the production of other goods and services should be treated in a COL as if they provided utility directly. In principle, a COL should include the shadow, or imputed, prices, of the outputs from these processes of production and not the prices of the inputs. … There is a need to clarify exactly how this issue is to be dealt with in a COL index." Peter Hill (1999; 5).

In this section, we address the issues raised by Hill by using the model of household production of final demand commodities that was postulated by Becker (1965) many years ago. Becker's model illustrates not only how household production of the type mentioned by Hill can be integrated into a cost of living framework, but it also indicates the important role that the *allocation of household time* plays in a more realistic model of household behavior. In order to measure *welfare change* more accurately, it is necessary to model how a household manages its allocation of time during the two periods under consideration.

In Becker's model of consumer behavior, a household (consisting of a single individual for simplicity) purchases q_n units of *market commodity n* and combines it with a household input of time, t_n, to produce $Q_n = f^n(q_n,t_n)$ units of a *finally demanded commodity* for n = 1,2,…,N say, where f^n is *the household production function* for the nth finally demanded commodity[122]. Thus using Becker's theory, the purchase of market goods and services alone does not provide utility for the household; these market purchases must be

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121 See also Hill (2009).

122 More complicated household production functions could be introduced but the present assumptions will suffice to show how household production can be modeled in a COLI framework using exact index number formulae. For additional work on Becker's theory of the allocation of time and household production, see Pollak and Wachter (1975) (1977), Diewert (2001), Abraham and Mackie (2005), Hill (2009), Landefeld, Fraumeni and Vojtech (2009), Schreyer and Diewert (2014) and Diewert, Fox and Schreyer (2017a).

combined with household time in order to provide utility. Some examples of Becker's finally demanded commodities (or *basic commodities* to use his terminology) are:

- Making a meal; the inputs are the ingredients used, the use of utensils and possibly a stove and time required to make the meal and the output is the prepared meal.
- Eating a meal; the inputs are the prepared meal and time spent eating and the output is a consumed meal.
- Cleaning a house; the inputs are cleaning utensils, soapy water, polish and time and the output is a clean house.
- Gardening services; the inputs are tools used in the yard, fuel (if power tools are used) and time and the output is a beautiful yard.
- Reading a book; the inputs are computer services or a physical book and time and the output is a book which has been read.

Activities 1, 3 and 4 listed above are examples of basic commodities, which could be *purchased* by the household; i.e., a cook could be hired to prepare a meal, a house cleaning service could be hired to clean the house and a gardening service could be hired to maintain the yard in good condition. These activities could be called examples of household *work activities*. Activities 2 and 5 are examples of *leisure activities* where the utility generated by the activity cannot be outsourced. We will see below why this distinction between the two types of household production can be important.

We follow Becker's example and assume that the household production functions, $f^n(q_n,t_n)$, are linearly homogeneous.[123] If $p_n > 0$ is the price for a unit of q_n and $w > 0$ is the price of household time, then the *unit cost functions* $c^n(p_n,w)$ that correspond to the $f^n(q_n,t_n)$ can be defined as follows:

$$(207)\ c^n(p_n,w) \equiv \min_{q_n,t_n} \{p_nq_n + wt_n : f^n(q_n,t_n) \geq 1;\ q_n \geq 0;\ t_n \geq 0\}\ ; \qquad n = 1,...,N.$$

Assume that the household faces the prices $p^\tau \equiv [p_1^\tau,..., p_N^\tau] >> 0_N$ and $w^\tau > 0$ for periods $\tau = 0,1$. Further assume that the period τ observed purchases of commodity n, q_n^τ, and time allocated to its consumption in period τ, t_n^τ, solve the cost minimization problems, $\min_{q_n,t_n} \{p_n^\tau q_n + w^\tau t_n : f^n(q_n,t_n) \geq f^n(q_n^\tau,t_n^\tau);\ q_n \geq 0;\ t_n \geq 0\}$ for $n = 1,...,N$ and $\tau = 0,1$. In view of the linear homogeneity of the household production functions, f^n, we obtain the following equalities:

$$(208)\ p_n^\tau q_n^\tau + w^\tau t_n^\tau = c^n(p_n^\tau,w^\tau)f^n(q_n^\tau,t_n^\tau) = P_n^\tau Q_n^\tau\ ; \qquad \tau = 0,1;\ n = 1,...,N$$

where the period τ *basic prices and quantities* for the nth household activity are defined as follows:[124]

$$(209)\ P_n^\tau \equiv c^n(p_n^\tau,w^\tau)\ ;\ Q_n^\tau \equiv f^n(q_n^\tau,t_n^\tau)\ ; \qquad \tau = 0,1;\ n = 1,...,N.$$

At this point, the theory of exact index numbers can be used in order to obtain empirical estimates for the unobserved P_n^τ and Q_n^τ. Pick an index number formula that is exact for a certain functional form for either $c^n(p_n,w)$ or $f^n(q_n,t_n)$. For example, pick the Fisher price index, $P_F(p_n^0,w^0;p_n^1,w^1;q_n^0,t_n^0;q_n^1,t_n^1)$, which is exact for certain flexible functional forms[125] for either the nth unit cost function $c^n(p_n,w)$ or the nth household

[123] In addition, following Schreyer and Diewert (2014), we assume that the household production functions are nonnegative, once differentiable, concave and increasing in q_n and t_n. Becker (1965; 496) assumed that the household production functions f^n were of the Leontief, no substitution type.

[124] Becker (1965; 497) called P_n the *full price* for consuming a unit of the nth final commodity; i.e., it is the sum of the prices of the goods and time used to produce a unit of the finally demanded commodity Q_n.

[125] See section 5 above.

production function $f^n(q_n,t_n)$ for n = 1,...,N. The basic prices and quantities for period 0 are defined as follows:[126]

(210) $P_n^0 \equiv 1$; $Q_n^0 \equiv p_n^0 q_n^0 + w^0 t_n^0$; n = 1,...,N.

The basic prices and quantities for period 1 are defined as follows:

(211) $P_n^1 \equiv P_F(p_n^0,w^0;p_n^1,w^1;q_n^0,t_n^0;q_n^1,t_n^1)$; $Q_n^1 \equiv [p_n^1 q_n^1 + w^1 t_n^1]/P_n^1$; n = 1,...,N.

The P_n^τ and Q_n^τ defined by (210) and (211) will be consistent with equations (208) provided that the c^n or f^n have the functional forms that are exact for the Fisher index. For future reference, note that the following equations will hold:

(212) $P^\tau \cdot Q^\tau \equiv \Sigma_{n=1}^N P_n^\tau Q_n^\tau = \Sigma_{n=1}^N [p_n^\tau q_n^\tau + w^\tau t_n^\tau] = p^\tau \cdot q^\tau + w^\tau[\Sigma_{n=1}^N t_n^\tau]$; $\tau = 0,1$

where $P^\tau \equiv [P_1^\tau,...,P_N^\tau]$, $Q^\tau \equiv [Q_1^\tau,...,Q_N^\tau]$, $p^\tau \equiv [p_1^\tau,...,p_N^\tau]$ and $q^\tau \equiv [q_1^\tau,...,q_N^\tau]$ for $\tau = 0,1$.

We return to Becker's model of the allocation of time. In addition to spending time on the N household production activities, Becker assumed that the household provides $t_L > 0$ hours of labour market services at the after tax wage rate of $w_L > 0$. Becker also assumed that the household spends the amount of Y of nonlabour income on the purchase of market goods and services.[127] Finally, Becker assumed that the consumer-worker has preferences over different combinations of the finally demanded commodities, $Q_1,\ldots,Q_N$, that are summarized by the (macro) *utility function*, $U(Q_1,\ldots,Q_N) \equiv U[f^1(q_1,t_1),...,f^N(q_N,t_N)]$.[128] In addition to the household budget constraint, $\Sigma_{n=1}^N p_n q_n \leq Y + w_L t_L$, the household has to satisfy the *time constraint*, $\sum_{n=1}^N t_n + t_L = H$, where H is the number of hours available in the period under consideration.

Let $p^\tau \equiv [p_1^\tau,...,p_N^\tau] >> 0_N$ and $w_L^\tau > 0$ be the observed prices for purchases of market goods and services for period τ, let $t^\tau \equiv [t_1^\tau,...,t_N^\tau] >> 0_N$ be the household's period τ vector of time inputs into the household production functions and let $t_L^\tau > 0$ be the observed household labor supply for periods $\tau = 0,1$. We assume that q^τ, t^τ and t_L^τ solve the following period τ household constrained utility maximization problem for $\tau = 0,1$:[129]

(213) $\max_{q_1,...,q_N,t_1,...,t_N,t_L} \{U[f^1(q_1,t_1),...,f^N(q_N,t_N)] : Y^\tau + w_L^\tau t_L - \Sigma_{n=1}^N p_n^\tau q_n \geq 0; H - \Sigma_{n=1}^N t_n - t_L \geq 0\}$.

We assume that the inequality constraints in (213) are satisfied as equalities when evaluated at the q^τ, t^τ and t_L^τ solutions to (213). This means that the following equations hold:

(214) $\Sigma_{n=1}^N p_n^\tau q_n^\tau = Y^\tau + w_L^\tau t_L^\tau$; $\tau = 0,1$;
(215) $w_L^\tau[\Sigma_{n=1}^N t_n^\tau] = w_L^\tau[H - t_L^\tau]$; $\tau = 0,1$.

[126] Definitions (210) and (211) make specific cardinalizations for measuring the unobserved outputs of the N household production functions.
[127] If $w_L t_L$ (equal to after tax labour earnings) is large enough, it could be the case that Y is negative; i.e., some of the household labour earnings are saved. This does not affect Becker's theory.
[128] The utility function U is assumed to be once differentiable, linearly homogeneous, concave and increasing in the $Q_1,...,Q_N$.
[129] We have omitted the nonnegativity constraints $t_n \geq 0$, $t_L \geq 0$ and $q_n \geq 0$ for n = 1,...,N from (212) to save space. Since we have assumed a strictly positive solution to (212) for each time period τ, these nonnegativity constraints will not be binding and hence can be ignored in what follows.

Equations (212) will also hold with $w^\tau = w_L{}^\tau$ for $\tau = 0,1$ as will be seen below. These equations along with (214) and (215) imply that the following equations will hold:

$$(216)\ P^\tau \cdot Q^\tau = \Sigma_{n=1}{}^N [p_n{}^\tau q_n{}^\tau + w_L{}^\tau t_n{}^\tau] = Y^\tau + w_L{}^\tau t_L{}^\tau + w_L{}^\tau [H - t_L{}^\tau] = Y^t + w_L{}^\tau H \equiv F^\tau ; \qquad \tau = 0,1$$

where F^τ is Becker's *full income*.[130] To see why the consumer's regular budget constraint and time constraint can be combined into a single constraint, form the Lagrangian $L^\tau(q,t,t_L,\lambda,\omega)$ for the constrained maximization problem defined by (213) for $\tau = 0$ or 1:

$$(217)\ L^\tau(q,t,t_L,\lambda,\omega) \equiv U[f^1(q_1,t_1),...,f^N(q_N,t_N)] + \lambda[Y^\tau + w_L{}^\tau t_L - \Sigma_{n=1}{}^N p_n{}^\tau q_n] + \omega[H - \Sigma_{n=1}{}^N t_n - t_L] ; \qquad \tau = 0,1.$$

Under our regularity conditions on the functions U and $f^1,...,f^N$, there will exist positive Lagrange multipliers, $\lambda^\tau > 0$ and $\omega^\tau > 0$ such that the observed period τ solution to the period τ constrained maximization problem defined by (213), q^τ, t^τ and $t_L{}^\tau$, will satisfy the following first order conditions:

$$(218)\ [\partial U(Q_1{}^\tau,...,Q_N{}^\tau)/\partial Q_n][\partial f^n(q_n{}^\tau,t_n{}^\tau)/\partial q_n] = \lambda^\tau p_n{}^\tau ; \qquad n = 1,...,N;\ \tau = 0,1;$$
$$(219)\ [\partial U(Q_1{}^\tau,...,Q_N{}^\tau)/\partial Q_n][\partial f^n(q_n{}^\tau,t_n{}^\tau)/\partial t_n] = \omega^\tau ; \qquad n = 1,...,N;\ \tau = 0,1;$$
$$(220)\qquad 0 = \lambda^\tau w_L{}^\tau - \omega^\tau ; \qquad \tau = 0,1.$$

Equations (220) show that $\omega^\tau = \lambda^\tau w_L{}^\tau$ for $\tau = 0,1$. These equations justify Becker's statement that the household budget constraint and the corresponding time constraint can be combined into a single constraint. Using (220), equations (219) become the following equations:

$$(221)\ [\partial U(Q_1{}^\tau,...,Q_N{}^\tau)/\partial Q_n][\partial f^n(q_n{}^\tau,t_n{}^\tau)/\partial t_n] = \lambda^\tau w_L{}^\tau ; \qquad n = 1,...,N;\ \tau = 0,1.$$

For each τ, take equation n in (218), multiply both sides by $q_n{}^\tau$. Take equation n in (221) and multiply both sides by $t_n{}^\tau$. For each τ and n, add these equations. Using the linear homogeneity of $\partial f^n(q_n,t_n)/\partial t_n$ and using definitions (209) with $w^\tau \equiv w_L{}^\tau$ which imply that $Q_n{}^\tau \equiv f^n(q_n{}^\tau,t_n{}^\tau)$ for each n, we obtain the following equations:

$$(222)\ [\partial U(Q_1{}^\tau,...,Q_N{}^\tau)/\partial Q_n]Q_n{}^\tau = \lambda^\tau[p_n{}^\tau q_n{}^\tau + w_L{}^\tau t_n{}^\tau] ; \qquad n = 1,...,N;\ \tau = 0,1;$$
$$= \lambda^\tau[P_n{}^\tau Q_n{}^\tau] \qquad \text{using (208) with } w^\tau \equiv w_L{}^\tau.$$

For each τ, sum the N equations in (222). Using the linear homogeneity of $U(Q_1,...,Q_N)$ and equations (216), we obtain the following equations:

$$(223)\ U(Q_1{}^\tau,...,Q_N{}^\tau) = \lambda^\tau P^\tau \cdot Q^\tau ; \qquad \tau = 0,1$$
$$= \lambda^\tau F^\tau \qquad \text{using definitions (216).}$$

Equations (223) can be solved for the Lagrange multipliers λ^τ. The solutions are $\lambda^\tau = U(Q_1{}^\tau,...,Q_N{}^\tau)/P^\tau \cdot Q^\tau$ for $\tau = 0,1$. Substitute these values for λ^τ back into equations (222). After some rearrangement, we obtain the following equations, which are *Wold's Identity equations* applied to the macro utility function $U(Q_1,...,Q_N)$:

$$(224)\ P^\tau/P^\tau \cdot Q^\tau = \nabla_Q U(Q^\tau)/U(Q^\tau) ; \qquad \tau = 0,1.$$

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[130] "This suggests dropping the approach based on explicitly considering separate goods and time constraints and substituting one in which the total resource constraint necessarily equalled the maximum money income achievable, which will be simply called 'full income'." Gary Becker (1965; 497).

Recall that the P^τ and Q^τ are well defined by equations (210) and (211) with $w^0 \equiv w_L{}^0$ and $w^1 \equiv w_L{}^1$. At this stage, we can assume a functional form for the macro utility function $U(Q_1,...,Q_N) = U(Q)$, which has an exact index number formula associated with it. Thus assume that U(Q) can be approximated by the homogeneous quadratic utility function, $U(Q) \equiv [Q^TAQ]^{1/2}$, where the symmetric matrix A has one positive eigenvalue with a strictly positive eigenvector and the other eigenvalues of A are equal to zero or are negative. Then the Fisher index is exact for this functional form. The nominal growth of full consumption going from period 0 to 1 is equal to the nominal growth of full income, $F^1/F^0 = P^1{\cdot}Q^1/P^0{\cdot}Q^0$, and the real growth of household full consumption is equal to the Fisher ideal quantity index, $Q_F(P^0,P^1,Q^0,Q^1)$.[131] The appropriate consumer price index under these conditions is the Fisher ideal price index, $P_F(P^0,P^1,Q^0,Q^1)$.

In the above model of consumer behavior, the household price of time for period τ turns out to be the *after tax wage rate*, $w_L{}^\tau$. But there are many households that do not offer market labour services; i.e., individuals who are retired or who are simply not in the labour force. How can we value household time in this situation? It is possible to modify Becker's model of the consumer-worker household to deal with non worker households. Make the same assumptions as in the model explained above with one exception: we assume that the Nth household production activity is one where the household time input, t_N, can be replaced by hiring market services, s_N, at the price $w_N > 0$. Thus if the Nth activity is yard maintenance, time spent maintaining the yard can be replaced by hiring a service that will undertake the necessary work. Thus the production function for the Nth activity is $Q_N = f^N(q_N,t_N + s_N)$.[132]

Let $p^\tau \equiv [p_1{}^\tau,...,p_N{}^\tau] >> 0_N$ and $w_S{}^\tau > 0$ be the observed prices for purchases of market goods and services for period τ and let $t^\tau \equiv [t_1{}^\tau,...,t_N{}^\tau] >> 0_N$ be the household's period τ vector of time inputs into the household production functions and for periods $\tau = 0,1$. Let $q_S{}^\tau > 0$ be the household's purchases of market labour services for activity N for $\tau = 0,1$. We assume that q^τ, t^τ and $q_S{}^\tau$ solve the following period τ household constrained utility maximization problem:[133]

$$(225)\ \max_{q_1,...,q_N,t_1,...,t_N,q_S} \{U[f^1(q_1,t_1),...,f^{N-1}(q_{N-1},t_{N-1}),f^N(q_N,t_N+q_S)] : Y^\tau - w_S{}^\tau q_S - \Sigma_{n=1}^N p_n{}^\tau q_n \geq 0;\ H - \Sigma_{n=1}^N t_n \geq 0\}; \qquad \tau = 0,1.$$

We assume that the functions U, $f^1,...,f^N$ satisfy the same regularity conditions as in the Becker model above. Thus the two constraints in (225) will hold as equalities. Hence we will have $Y^\tau = \Sigma_{n=1}^N p_n{}^\tau q_n{}^\tau + w_S{}^\tau q_S{}^\tau$, $w_S{}^\tau H = w_S{}^\tau \Sigma_{n=1}^N t_n{}^\tau$ for $\tau = 0,1$ as well as the following equations:

$$(226)\ \Sigma_{n=1}^N p_n{}^\tau q_n{}^\tau + w_S{}^\tau q_S{}^\tau + \Sigma_{n=1}^N w_S{}^\tau t_n{}^\tau = Y^\tau + w_S{}^\tau H \equiv F^\tau ; \qquad \tau = 0,1$$

where the new *period τ full income* F^τ is equal to period τ nonlabour income Y^τ plus the value of period τ household time H valued at the period τ market service wage for the Nth activity, $w_S{}^\tau$.

Form the Lagrangians $L^\tau(q,q_S,t,\lambda,\omega)$ for the constrained maximization problems defined by (225) for $\tau = 0,1$:

$$(227)\ L^\tau(q,q_S,t,\lambda,\omega) \equiv U[f^1(q_1,t_1),...,f^N(q_N,t_N+q_S)] + \lambda[Y^\tau - \Sigma_{n=1}^N p_n{}^\tau q_n - w_S{}^\tau q_S] + \omega[H - \Sigma_{n=1}^N t_n - t_L] ; \ \tau = 0,1.$$

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131 The period 0 and 1 levels of household real consumption are set equal to $U^0 \equiv P^0{\cdot}Q^0 = p^0{\cdot}q^0 + w_L{}^0[\Sigma_{n=1}^N t_n{}^0]$ and $U^1 \equiv U^0{\times}Q_F(P^0,P^1,Q^0,Q^1) = U^0{\times}[P^0{\cdot}Q^1P^1{\cdot}Q^1/P^0{\cdot}Q^0P^1{\cdot}Q^0]^{1/2}$ respectively.

132 Thus we are assuming that personal yard work and hired yard work are perfect substitutes.

133 We have omitted the nonnegativity constraints $q_S \geq 0$, $t_n \geq 0$ and $q_n \geq 0$ for n = 1,...,N from (225) to save space. Since we have assumed a strictly positive solution to (225) for each time period τ, these nonnegativity constraints will not be binding and hence can be ignored in what follows.

Under our regularity conditions on the functions U and $f^1,...,f^N$, there will exist positive Lagrange multipliers, $\lambda^\tau > 0$ and $\omega^\tau > 0$ such that the observed period τ solution, q^τ, q_S^τ and t^τ, to the period τ constrained maximization problem defined by (225) will satisfy the following first order conditions:

(228) $[\partial U(Q_1^\tau,...,Q_N^\tau)/\partial Q_n][\partial f^n(q_n^\tau,t_n^\tau)/\partial q_n] = \lambda^\tau p_n^\tau$; n = 1,...,N−1; τ = 0,1;
(229) $[\partial U(Q_1^\tau,...,Q_N^\tau)/\partial Q_N][\partial f^N(q_N^\tau,t_N^\tau+q_S^\tau)/\partial q_N] = \lambda^\tau p_N^\tau$; τ = 0,1;
(230) $[\partial U(Q_1^\tau,...,Q_N^\tau)/\partial Q_n][\partial f^n(q_n^\tau,t_n^\tau)/\partial t_n] = \omega^\tau$; n = 1,...,N−1; τ = 0,1;
(231) $[\partial U(Q_1^\tau,...,Q_N^\tau)/\partial Q_N][\partial f^N(q_N^\tau,t_N^\tau+q_S^\tau)/\partial t_N] = \omega^\tau$; τ = 0,1;
(232) $[\partial U(Q_1^\tau,...,Q_N^\tau)/\partial Q_N][\partial f^N(q_N^\tau,t_N^\tau+q_S^\tau)/\partial q_S] = \lambda^\tau w_S^\tau$; τ = 0,1.

For τ = 0 or 1, it can be seen that the derivatives on the left hand sides of (231) and (232) are identical. Hence the right hand sides are equal and we obtain the equations $\omega^\tau = \lambda^\tau w_S^\tau$ for τ = 0,1. Substitute these solutions for the ω^τ into equations (230) and (231) and we obtain the following equations:

(233) $[\partial U(Q^\tau)/\partial Q_n][\partial f^n(q_n^\tau,t_n^\tau)/\partial t_n] = \lambda^\tau w_S^\tau$; n = 1,...,N−1; τ = 0,1;
(234) $[\partial U(Q^\tau)/\partial Q_N][\partial f^N(q_N^\tau,t_N^\tau+q_S^\tau)/\partial t_N] = \lambda^\tau w_S^\tau$; τ = 0,1.

For τ = 0,1 and n = 1,...,N−1, multiply both sides of equation n in (228) by q_n^τ and both sides of equation n in (233) by t_n^τ and add the resulting two equations. Using the linear homogeneity of $f^n(q_n,t_n)$, we have $q_n^\tau[\partial f^n(q_n^\tau,t_n^\tau)/\partial q_n] + t_n^\tau[\partial f^n(q_n^\tau,t_n^\tau)/\partial t_n] = f^n(q_n^\tau,t_n^\tau)$. Thus we obtain the following equations:

(235) $[\partial U(Q^\tau)/\partial Q_n]f^n(q_n^\tau,t_n^\tau) = \lambda^\tau[p_n^\tau q_n^\tau + w_S^\tau t_n^\tau]$; n = 1,...,N−1; τ = 0,1.

For each τ = 0,1 and n = 1,...,N−1, equation n in equations (235) can be solved for λ^τ and this value for λ^τ can be substituted back into equations n in (228) and (223). After a bit of rearrangement, the following equations are obtained:

(236) $[\partial f^n(q_n^\tau,t_n^\tau)/\partial q_n]/f^n(q_n^\tau,t_n^\tau) = p_n^\tau/[p_n^\tau q_n^\tau + w_S^\tau t_n^\tau]$; n = 1,...,N−1; τ = 0,1;
(237) $[\partial f^n(q_n^\tau,t_n^\tau)/\partial t_n]/f^n(q_n^\tau,t_n^\tau) = w_S^\tau/[p_n^\tau q_n^\tau + w_S^\tau t_n^\tau]$; n = 1,...,N−1; τ = 0,1.

For each n = 1,...,N−1 and for τ =0,1, equations (236) and (237) are the Wold Identity equations (14) for the household production function $f^n(q_n,t_n)$. Thus we can approximate f^n by a homogeneous quadratic utility function and use the Fisher price and quantity indexes to estimate $Q_n^0 \equiv f^n(q_n^0,t_n^0)$ and $Q_n^1 \equiv f^n(q_n^1,t_n^1)$ for n = 1,...,N−1; i.e., define Q_n^τ and the companion prices $P_n^\tau \equiv c^h(p_n^\tau,w_S^\tau)$ as follows:

(238) $P_n^0 \equiv 1 \equiv c^n(p_n^0,w_S^0)$; $Q_n^0 \equiv p_n^0q_n^0 + w_S^0t_n^0 \equiv f^n(q_n^0,t_n^0)$; n = 1,...,N−1;
(239) $P_n^1 \equiv P_F(p_n^0,w^0;p_n^1,w^1;q_n^0,t_n^0;q_n^1,t_n^1) \equiv c^n(p_n^1,w_S^1)$; $Q_n^1 \equiv [p_n^1q_n^1 + w^1t_n^1]/P_n^1 \equiv f^n(q_n^1,t_n^1)$; n = 1,...,N−1.

Now use equations (229), (231) and (232) and repeat the above operations for $f^N(q_N,t_N+q_S)$ and obtain the following counterparts to (236)-(239):

(240) $[\partial f^N(q_N^\tau,t_N^\tau + q_S^\tau)/\partial q_N]/f^N(q_N^\tau,t_N^\tau + q_S^\tau) = p_N^\tau/[p_N^\tau q_N^\tau + w_S^\tau(t_N^\tau + q_S^\tau)]$; τ = 0,1;
(241) $[\partial f^N(q_N^\tau,t_N^\tau)/\partial t_N]/f^N(q_N^\tau,t_N^\tau) = w_S^\tau(t_N^\tau + q_S^\tau)/[p_N^\tau q_N^\tau + w_S^\tau(t_N^\tau + q_S^\tau)]$; τ = 0,1.
(242) $P_N^0 \equiv 1 \equiv c^N(p_N^0,w_S^0)$; $Q_N^0 \equiv p_N^0q_N^0 + w_S^0(t_N^0 + q_S^0) \equiv f^N(q_N^0,t_N^0 + q_S^0)$;
(243) $P_N^1 \equiv P_F(p_N^0,w_S^0;p_N^1,w_S^1;q_N^0,t_N^0 + q_S^0;q_N^1,t_N^1 + q_S^1) \equiv c^N(p_N^1,w_S^1)$;
$Q_N^1 \equiv p_N^1q_N^1+w_S^1(t_N^1+q_S^{1)}]/P_N^1 \equiv f^N(q_N^1,t_N^1 + q_S^1)$.

Definitions (240)-(243) can be substituted back into equations (228)-(235) in order to derive the following equations:

(244) $[\partial U(Q_1^\tau,...,Q_N^\tau)/\partial Q_n]Q_n^\tau = \lambda^\tau[p_n^\tau q_n^\tau + w_S^\tau t_n^\tau] = \lambda^\tau[P_n^\tau Q_n^\tau]$; n = 1,...,N−1; τ = 0,1;
(245) $[\partial U(Q_1^\tau,...,Q_N^\tau)/\partial Q_N]Q_N^\tau = \lambda^\tau[p_N^\tau q_N^\tau + w_S^\tau(t_n^\tau+q_S^\tau)] = \lambda^\tau[P_N^\tau Q_N^\tau]$ τ = 0,1.

For each τ, sum the N equations in (244) and (245). Using the linear homogeneity of $U(Q_1,...,Q_N)$ and the definition (226) for period τ full income F^τ, we obtain the following equations:

(246) $U(Q_1^\tau,...,Q_N^\tau) = \lambda^\tau P^\tau \cdot Q^\tau = \lambda^\tau F^\tau$; τ = 0,1.

Equations (246) can be solved for the Lagrange multipliers, λ^τ for τ = 0,1. We obtain $\lambda^\tau = U(Q_1^\tau,...,Q_N^\tau)/P^\tau \cdot Q^\tau$ for τ = 0,1. Substitute these values for λ^τ back into equations (244) and (245). After some rearrangement, we obtain the following equations which are *Wold's Identity equations* applied to the macro utility function $U(Q_1,...,Q_N)$:

(247) $P^\tau/P^\tau \cdot Q^\tau = \nabla_Q U(Q^\tau)/U(Q^\tau)$; τ = 0,1.

Recall that the P^τ and Q^τ are well defined by equations (238), (239), (242) and (243). Now assume a functional form for the macro utility function $U(Q_1,...,Q_N) = U(Q)$ which has an exact index number formula associated with it. Thus assume that U(Q) can be approximated by the homogeneous quadratic utility function, $U(Q) \equiv [Q^TAQ]^{1/2}$, where the symmetric matrix A has one positive eigenvalue with a strictly positive eigenvector and the other eigenvalues of A are either equal to zero or are negative. Then the Fisher price and quantity indexes are exact for this functional form. The nominal growth of full consumption going from period 0 to 1 is equal to the nominal growth of full income, $F^1/F^0 = P^1 \cdot Q^1/P^0 \cdot Q^0$ where the F^τ are defined by (226) and the real growth of household full consumption is equal to the Fisher ideal quantity index, $Q_F(P^0,P^1,Q^0,Q^1)$.[134] The appropriate consumer price index under these conditions is the Fisher ideal price index, $P_F(P^0,P^1,Q^0,Q^1)$.

Here are the important points that emerge from our analysis of the above two models for the household's allocation of time:[135]

- Depending on the type of household, the valuation of household time is either the *after tax wage rate* for the household or the *price of market services* that can substitute for household work.
- It is possible to use "normal" index number theory to provide price and volume indexes for utility maximizing households that face both a budget constraint and a time constraint.

However, there are many problems with the two models of household behavior that were considered above:

- The first model did not take into account the possible *disutility of providing market labour supply* while neither model did not take into account the possible *disutility of providing household work*.[136] Taking possible disutility into account greatly complicates the analysis. In particular, the scaling of the utility functions, F and $f^1,...,f^N$ is no longer straightforward.

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[134] The period 0 and 1 levels of household real full consumption are set equal to $U^0 \equiv F^0 = P^0 \cdot Q^0 = p^0 \cdot q^0 + w_S^0 q_S^0 + w_S^0[\Sigma_{n=1}^N t_n^0]$ and $U^1 \equiv U^0 \times Q_F(P^0,P^1,Q^0,Q^1) = U^0 \times [P^0 \cdot Q^1 P^1 \cdot Q^1/P^0 \cdot Q^0 P^1 \cdot Q^0]^{1/2}$ respectively.

[135] These two models are considered in more detail by Schreyer and Diewert (2014).

[136] The utility function $U[f^1(q_1,t_1),...,f^N(q_N,t_N)]$ should be replaced by $U[f^1(q_1,t_1),...,f^N(q_N,t_N),t_L]$ for the Becker model where $U[f^1(q_1,t_1),...,f^N(q_N,t_N),t_L]$ is decreasing as labour supply t_L increases. For the second model, the utility function $U[f^1(q_1,t_1),...,f^{N-1}(q_{N-1},t_{N-1}),f^N(q_N,t_N+q_S)]$ should be replaced by $U[f^1(q_1,t_1),...,f^{N-1}(q_{N-1},t_{N-1}),f^N(q_N,t_N+q_S),t_N]$ where this function could be decreasing in the household's supply of time spent t_N on final demand activity N.

- In more realistic models of household behavior, *corner solutions* to the household utility maximization problems emerge as realistic possibilities.[137]
- In more realistic models of household behavior, it is possible to identify the "correct" prices of time to value household labour supply, household time in leisure activities and household time in work activities but econometric estimation is required.[138] This means that it will be difficult for statistical agencies to deal with these difficulties in practical settings.
- There are also problems in forming household utility functions when there are multiple persons in the household.[139]
- Finally, the household production functions for work and leisure type activities could be subject to *technological change*. In this case, it will be necessary to measure the constant quality outputs produced by the household production functions directly instead of using the indirect methods that rely on inputs that were used in the above models.

In spite of the above difficulties, there is no doubt that the allocation of time plays an important role in determining household welfare. Hopefully, future research will address some of the above problems.

16. Aggregate Cost of Living Indexes

In previous sections, we have considered the theory of the cost of living index for only a single consumer or household. In this section, we consider some of the problems involved in the construction of a cost of living index when there are many households or regions in the economy and the goal is the production of a national index. Below, we allow for an arbitrary number of households, H, so in principle, each household in the economy under consideration could have its own consumer price index. However, in practice, it will be necessary to group households into various classes and within each class, it will be necessary to assume that the group of households in the class behaves as if it were a single household in order to apply the economic approach to index number theory.[140] Our partition of the economy into H household classes can also be given a regional interpretation: each household class can be interpreted as a group of households within a region of the country under consideration.

In this section, we will consider an economic approach to the CPI that was initiated by Pollak (1980) (1981), who called his index concept a *social cost of living index*. It is a straightforward extension of the *Konüs Cost of Living Index* (COLI) for an individual household to a group of households.

Suppose that there are H households (or regions) in the economy and suppose further that there are N commodities in the economy in periods 0 and 1 that households consume in the two periods. Denote an N dimensional vector of commodity consumption in a given period by $q \equiv (q_1,q_2,\ldots, q_N)$ as usual. Denote the vector of period t market prices faced by household h by $p_h^t \equiv (p_{h1}^t,p_{h2}^t,\ldots,p_{hN}^t)$ for $t = 0,1$. Denote the corresponding observed consumption vector for household h in period t by $q_h^t \equiv (q_{h1}^t,q_{h2}^t,\ldots,q_{hN}^t)$ for $t = 0,1$. Note that we are *not* assuming that each household faces the same vector of commodity prices. The preferences of household h over different combinations of market commodities q is represented by the

[137] A corner solution to a household utility maximization problem is one where the nonnegativity constraints in the consumer's constrained utility maximization problem is binding (i.e, some q_n or t_n are equal to 0) and hence cannot be ignored. See Diewert, Fox and Schreyer (2017a) for the analysis of corner solutions.
[138] See Diewert, Fox and Schreyer (2017a) for approaches to the econometric estimation problems. The econometrics of consumer demand models where there are two constraints instead of a single budget constraint is not a well developed area.
[139] There are also complications due to changes in the composition of households over time resulting from demographic changes.
[140] The problems associated with grouping households will be discussed in section 18 below.

continuous utility function $f^h(q)$ for h = 1,2,…,H.[141] For periods t = 0,1 and for households h = 1,2,…,H, it is assumed that the observed household h consumption vector $q_h^t \equiv (q_{h1}^t,\ldots,q_{hN}^t)$ is a solution to the following household h expenditure minimization problem:

$$(248)\quad \min_q \{p_h^t\cdot q : f^h(q) \geq u_h^t\} \equiv C^h(u_h^t,p_h^t) = p_h^t\cdot q_h^t\,; \qquad t = 0,1;\ h = 1,2,\ldots H$$

where $u_h^t \equiv f^h(q_h^t)$ is the utility level achieved by household h during period t and C^h is the cost or expenditure function that is dual to the utility function f^h. Basically, these assumptions mean that each household has *stable preferences* over the same list of commodities during the two periods under consideration, the same households appear in each period and each household chooses its consumption bundle in the most cost efficient way during each period. Let p^t be defined as the period t price vector of dimension HN that combines all of the household specific period t observed price vectors $p_1^t,\ldots,p_H^t$ into one big price vector and let q^t be the companion economy wide quantity vector that combines all of the observed period t quantity vectors $q_1^t,\ldots,q_H^t$ into a single vector of dimension HN. Let q be a reference quantity vector of dimension HN; i.e., $q \equiv [q_{11},\ldots,q_{1N};q_{21},\ldots,q_{2N};\ldots;q_{H1},\ldots,q_{HN}]$.

With the above definitions in mind, the family of *social cost of living indexes* or *aggregate Konüs cost of living indexes* for the group of households under consideration is defined as follows:[142]

$$(249)\quad P_K(p^0,p^1,q) \equiv \Sigma_{h=1}^H C^h(f^h(q_h),p_h^1)/\Sigma_{h=1}^H C^h(f^h(q_h),p_h^0).$$

The numerator on the right hand side of (249) is the sum over households of the minimum cost, $C^h(u_h,p_h^1)$, for household h to achieve the reference utility level $u_h \equiv f^h(q_h)$ given that the household h faces the period 1 vector of prices p_h^1. The denominator on the right hand side of (249) is the sum over households of the minimum cost, $C^h(u_h,p_h^0)$, for household h to achieve the *same* reference utility level u_h, given that the household faces the period 0 vector of prices p_h^0. Thus in the numerator and denominator of (249), only the price variables are different, which is precisely what we want in a theoretical definition of a consumer price index.

We now specialize the general definition (249) by replacing the general utility vector u by either the period 0 vector of household utilities $u^0 \equiv (u_1^0,u_2^0,\ldots u_H^0)$ or the period 1 vector of household utilities $u^1 \equiv (u_1^1,u_2^1,\ldots u_H^1)$. The choice of the base period vector of utility levels leads to the *Laspeyres Konüs cost of living index*, $P_K(p^0,p^1,q^0)$, while the choice of the period 1 vector of utility levels leads to the *Paasche Konüs cost of living index*, $P_K(p^0,p^1,q^1)$. It turns out that these two indexes satisfy some inequalities, which are counterparts to the inequalities (3) and (4) in section 2 above.

$$\begin{aligned}(250)\quad P_K(p^0,p^1,q^0) &\equiv \Sigma_{h=1}^H C^h(u_h^0,p_h^1)/\Sigma_{h=1}^H C^h(u_h^0,p_h^0) && \text{where } u_h^0 \equiv f^h(q_h^0) \text{ for } h = 1,\ldots,H\\ &= \Sigma_{h=1}^H C^h(u_h^0,p_h^1)/\Sigma_{h=1}^H p_h^0\cdot q_h^0 && \text{using (248) for } t = 0\ ^{143}\\ &\leq \Sigma_{h=1}^H p_h^1\cdot q_h^0/\Sigma_{h=1}^H p_h^0\cdot q_h^0\\ &\qquad \text{since } q_h^0 \text{ is feasible for the cost minimization problem } C^h(u_h^0,p_h^1) \text{ for } h = 1,2,\ldots,H\\ &\equiv P_L(p^0,p^1,q^0,q^1)\end{aligned}$$

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[141] As usual, we assume that each $f^h(q)$ is continuous, concave and increasing in the components of q.

[142] See Pollak (1980; 276) (1981; 328) (1989; 182) and Diewert (1983; 190-192) (2001; 170) for additional materials on social cost of living indexes.

[143] It can be seen that $P_K(p^0,p^1,q^0)$ is also equal to a weighted average of the individual Laspeyres Konüs cost of living indexes; i.e., $P_K(p^0,p^1,q^0) = \Sigma_{h=1}^H S_h^0C^h(u_h^0,p_h^1)/C^h(u_h^0,p_h^0)$ where $S_h^0 \equiv p_h^0\cdot q_h^0/\Sigma_{i=1}^H p_i^0\cdot q_i^0$ for h = 1,...,H. Since the weights for the individual household cost of living indexes are equal to the household's share of total nominal consumption in period 0, $P_K(p^0,p^1,q^0)$ is a *plutocratic aggregate cost of living index* to use the terminology of Prais (1959). Prais (1959) defined a *democratic COLI* as $\Sigma_{h=1}^H (1/H)C^h(u_h^0,p_h^1)/C^h(u_h^0,p_h^0)$.

where $P_L(p^0,p^1,q^0,q^1)$ is defined to be the economy wide observable (in principle) *Laspeyres price index*, $\sum_{h=1}^H p_h^1 \cdot q_h^0 / \sum_{h=1}^H p_h^0 \cdot q_h^0 = p^1 \cdot q^0 / p^0 \cdot q^0$ which treats each household consumption vector as a separate commodity so that p^0, p^1 and q^0 are HN dimensional vectors.

The inequality (250) says that the theoretical Laspeyres Konüs cost of living index, $P_K(p^0,p^1,q^0)$, is bounded from above by the observable Laspeyres price index P_L. In a similar manner, specializing definition (249), *the Paasche Konüs cost of living index*, $P_K(p^0,p^1,q^1)$, satisfies the following inequality:

$$
\begin{aligned}
(251)\ P_K(p^0,p^1,q^1) &\equiv \textstyle\sum_{h=1}^H C^h(u_h^1,p_h^1)/\sum_{h=1}^H C^h(u_h^1,p_h^0) && \text{where } u_h^1 \equiv f^h(q_h^1,e_h^1) \text{ for } h = 1,\ldots,H \\
&= \textstyle\sum_{h=1}^H p_h^1 \cdot q_h^1 / \sum_{h=1}^H C^h(u_h^1,p_h^0) && \text{using (248) for } t = 1\ ^{144} \\
&\geq \textstyle\sum_{h=1}^H p_h^1 \cdot q_h^1 / \sum_{h=1}^H p_h^0 \cdot q_h^1 && \text{using feasibility arguments} \\
&\equiv P_P(p^0,p^1,q^0,q^1)
\end{aligned}
$$

where $P_P(p^0,p^1,q^0,q^1)$ is defined to be the observable (in principle) *Paasche price index*, $\sum_{h=1}^H p_h^1 \cdot q_h^1 / \sum_{h=1}^H p_h^0 \cdot q_h^1 = p^1 \cdot q^1 / p^0 \cdot q^1$. The inequality (251) says that the theoretical Paasche *Konüs* cost of living index, $P_K(p^0,p^1,q^1)$, is bounded from below by the observable Paasche price index P_P.

It is possible to find two sided bounds for a Konüs cost of living index; i.e., we have the following proposition:

Proposition 15: Under suitable continuity assumptions on preferences, there exists a number λ^* between 0 and 1 such that

$$(252)\quad P_L \leq P_K(p^0,p^1,\lambda^* q^0+(1-\lambda^*)q^1) \leq P_P \ \text{ or } \ P_P \leq P_K(p^0,p^1,\lambda^* q^0+(1-\lambda^*)q^1) \leq P_L$$

where $P_L \equiv p^1 \cdot q^0 / p^0 \cdot q^0$ and $P_P \equiv p^1 \cdot q^1 / p^0 \cdot q^1$. The proof of Proposition 15 is similar to the proof of Proposition 1; see Diewert (2001; 173) for the details.

The above result tells us that *the theoretical aggregate Konüs cost of living index consumer price index* $P_K(p^0,p^1,q^*)$ lies between the observable Laspeyres index P_L and the Paasche index P_P, where $q^* \equiv \lambda^* q^0+(1-\lambda^*)q^1$ is an intermediate quantity vector that lies between q^0 and q^1. Hence if P_L and P_P are not too different, a good approximation to a theoretical aggregate cost of living index will be the *Fisher index* $P_F(p^0,p^1,q^0,q^1)$ defined as $P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)P_P(p^0,p^1,q^0,q^1)]^{1/2}$. This Fisher price index is computed just like the usual Fisher price index, except that each commodity in each region (or for each household) is regarded as a separate commodity.

It is possible to obtain an alternative estimator for an aggregate cost of living index if stronger assumptions on household preferences are made. Thus assume that the preferences of household h are represented by the linearly homogeneous utility function $f^h(q_h) \equiv [q_h^T A^h q_h]^{1/2}$ where A^h is a symmetric matrix which satisfies the regularity conditions discussed in section 5 above for h = 1,...,H. Under these assumptions, the Fisher price and quantity indexes will be exact for these preferences; see section 5 above. Let $c^h(p_h) = c^h(p_{h1},\ldots,p_{hN})$ be the unit cost function that corresponds to $f^h(q_h)$ for h = 1,...,H. Assuming utility maximizing behavior on the part of each household, the following equations will be satisfied:

$$(253)\ p_h^t \cdot q_h^t = f^h(q_h^t)c^h(p_h^t)\ ; \qquad h = 1,\ldots,H;\ t = 0,1.$$

[144] It can be verified that $P_K(p^0,p^1,q^1)$ is equal to the following weighted harmonic average of the individual Paasche Konüs cost of living indexes: $P_K(p^0,p^1,q^1) = \{\Sigma_{h=1}^H S_h^1[C^h(u_h^1,p_h^1)/C^h(u_h^1,p_h^0)]^{-1}\}^{-1}$ where $S_h^1 \equiv p_h^1 \cdot q_h^1 / \Sigma_{i=1}^H p_i^1 \cdot q_i^1$ for h = 1,...,H.

Now use Fisher price and quantity indexes to estimate household quantity and price levels, $Q_h^t \equiv f^n(q_h^t)$ and $P_h^t \equiv c^h(p_h^t)$, for t = 0,1 and h = 1,...,H as follows:

$$(254)\ P_h^0 \equiv 1 \equiv c^h(p_h^0)\ ;\ Q_h^0 \equiv p_h^0\cdot q_h^0 \equiv f^n(q_h^0)\ ; \qquad h = 1,\ldots,H;$$
$$(255)\ P_h^1 \equiv P_F(p_h^0,p_h^1,q_h^0,q_h^1) \equiv c^h(p_h^1)\ ;\ Q_h^1 \equiv [p_h^1\cdot q_h^1]/P_h^1\ ; \qquad h = 1,\ldots,H.$$

Under our new assumption of homothetic preferences for each household, definition (250) for the *Laspeyres Konüs cost of living index* $P_K(p^0,p^1,q^0)$ simplifies into the following expression:

$$\begin{aligned}(256)\ P_K(p^0,p^1,q^0) &\equiv \textstyle\sum_{h=1}^H C^h(u_h^0,p_h^1)/\sum_{h=1}^H C^h(u_h^0,p_h^0) && \text{where } u_h^0 \equiv f^h(q_h^0) \text{ for } h = 1,\ldots,H\\ &= \textstyle\sum_{h=1}^H u_h^0c^h(p_h^1)/\sum_{h=1}^H u_h^0c^h(p_h^0) && \text{since } C^h(u_h,p_h) = u_hc(p_h) \text{ for each h}\\ &= \textstyle\sum_{h=1}^H P_h^1Q_h^0/\sum_{h=1}^H P_h^0Q_h^0 && \text{using (254) and (255)}\\ &= P_L(P^0,P^1,Q^0,Q^1)\end{aligned}$$

where $P^t \equiv [P_1^t,\ldots,P_H^t]$ and $Q^t \equiv [Q_1^t,\ldots,Q_H^t]$ for t = 0,1 and $P_L(P^0,P^1,Q^0,Q^1)$ is the ordinary Laspeyres price index using the aggregate household prices and quantities for the two periods under consideration as the price and quantity variables.

Similarly, definition (251) for the *Paasche Konüs cost of living index* $P_K(p^0,p^1,q^1)$ simplifies into the following expression:

$$\begin{aligned}(257)\ P_K(p^0,p^1,q^1) &\equiv \textstyle\sum_{h=1}^H C^h(u_h^1,p_h^1)/\sum_{h=1}^H C^h(u_h^1,p_h^0) && \text{where } u_h^1 \equiv f^h(q_h^1) \text{ for } h = 1,\ldots,H\\ &= \textstyle\sum_{h=1}^H u_h^1c^h(p_h^1)/\sum_{h=1}^H u_h^1c^h(p_h^0) && \text{since } C^h(u_h,p_h) = u_hc(p_h) \text{ for each h}\\ &= \textstyle\sum_{h=1}^H P_h^1Q_h^1/\sum_{h=1}^H P_h^0Q_h^1 && \text{using (254) and (255)}\\ &\equiv P_P(P^0,P^1,Q^0,Q^1)\end{aligned}$$

where $P_P(P^0,P^1,Q^0,Q^1)$ is the ordinary Paasche price index using the aggregate household prices and quantities for the two periods under consideration as the price and quantity variables.

The aggregate price indexes $P_K(p^0,p^1,q^0)$ and $P_K(p^0,p^1,q^1)$ defined by (256) and (257) are equally plausible measures of overall consumer price inflation between periods 0 and 1 and so it is reasonable to take an average of these two indexes to obtain a "final" estimate of inflation between the two periods. As usual, the geometric average leads to an index that will satisfy a time reversal test. Thus we have:

$$(258)\ [P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2} = [P_L(P^0,P^1,Q^0,Q^1)P_P(P^0,P^1,Q^0,Q^1)]^{1/2} \equiv P_F(P^0,P^1,Q^0,Q^1)$$

where $P_F(P^0,P^1,Q^0,Q^1)$ is the *Fisher index* defined over the aggregate household prices and quantities for the two periods under consideration. It is actually a two stage Fisher index where the first stage of aggregation uses the price and quantity data for each household to construct household specific Fisher price and quantity levels for each household. The two stage Fisher price index $P_F(P^0,P^1,Q^0,Q^1)$ defined by (258) can be compared to the single stage Fisher price index $P_F(p^0,p^1,q^0,q^1)$ defined earlier as the geometric mean of $P_L(p^0,p^1,q^0,q^1)$ and $P_P(p^0,p^1,q^0,q^1)$ defined by (250) and (251). Using the results listed in section 8 above, we know that the single stage Fisher index will approximate its two stage counterpart to the second order around an equal price and quantity point. Thus normally, we would not expect much difference between these alternative measures of overall consumer price inflation.

In the following section, we turn our attention to the definition of aggregate quantity indexes.

17. Aggregate Allen Quantity Indexes

Recall the definition of the Allen quantity index for a single household defined above in section 11. In this section, we will generalize this index concept to cover the case of many households.

Make the same assumptions on households and their preference functions that were made at the beginning of the previous section. Again assume that the observed household h consumption vector $q_h^t \equiv (q_{h1}^t,\ldots,q_{hN}^t)$ is a solution to the following household h expenditure minimization problem defined by (248) for t = 0,1 and h = 1,...,H. Using the same notation that was used at the beginning of the previous section, the family of *aggregate Allen quantity indexes* for the group of households under consideration is defined as follows:

$$(259)\ Q_A(q^0,q^1,p) \equiv \Sigma_{h=1}^H C^h(f^h(q_h^1),p_h)/\Sigma_{h=1}^H C^h(f^h(q_h^0),p_h) = \Sigma_{h=1}^H C^h(u_h^1,p_h)/\Sigma_{h=1}^H C^h(u_h^0,p_h)$$

where $u_h^t \equiv f^h(q_h^t)$ for t = 0,1 and h = 1,...,H and $p \equiv [p_1,\ldots p_H]$ is an NH dimensional vector of reference prices.

Note that in the numerator and denominator of the last equation in (259), only the household utility variables are different, which is appropriate for an overall measure of household utility which in turn is an overall quantity or volume measure. Note also that if H = 1, definition (259) reduces to the definition of an Allen (1949) quantity index.

We now specialize the general definition (259) by replacing the reference price vector p by either the period 0 economy wide price vector p^0 or the period 1 economy wide price vector p^1. Thus define the *Laspeyres aggregate Allen quantity index* by $Q_A(q^0,q^1,p^0)$ and the *Paasche aggregate Allen quantity index* by $Q_A(q^0,q^1,p^1)$. It turns out that these two indexes satisfy some inequalities, which are counterparts to the inequalities (3) and (4) in section 2 above. Thus choosing $p = p^0$ leads to the following index:

$$\begin{aligned}(260)\ Q_A(q^0,q^1,p^0) &\equiv \textstyle\sum_{h=1}^H C^h(f^h(q_h^1),p_h^0)/\sum_{h=1}^H C^h(f^h(q_h^0),p_h^0) \\ &= \textstyle\sum_{h=1}^H C^h(f^h(q_h^1),p_h^0)/\sum_{h=1}^H p_h^0\cdot q_h^0 \qquad \text{using (248) for } t = 0\ ^{145} \\ &\leq \textstyle\sum_{h=1}^H p_h^0\cdot q_h^1/\sum_{h=1}^H p_h^0\cdot q_h^0 \\ &\qquad \text{since } q_h^1 \text{ is feasible for the cost minimization problem } C^h(f^h(q_h^1),p_h^0) \text{ for } h = 1,2,\ldots,H \\ &\equiv Q_L(p^0,p^1,q^0,q^1)\end{aligned}$$

where $Q_L(p^0,p^1,q^0,q^1)$ is defined to be the observable (in principle) *Laspeyres quantity index*, $\sum_{h=1}^H p_h^0\cdot q_h^1/\sum_{h=1}^H p_h^0\cdot q_h^0 = p^0\cdot q^1/p^0\cdot q^0$, which treats each household consumption vector as a separate commodity so that p^0, q^0 and q^1 are HN dimensional vectors.

The inequality (260) says that the theoretical Laspeyres Allen aggregate quantity index, $Q_A(q^0,q^1,p^0)$, is bounded from above by the observable Laspeyres quantity index Q_L. In a similar manner, specializing definition (259) by setting $p = p^1$, *the Paasche Allen aggregate quantity index*, $Q_A(q^0,q^1,p^1)$, satisfies the following inequality:

$$\begin{aligned}(261)\ Q_A(q^0,q^1,p^1) &\equiv \textstyle\sum_{h=1}^H C^h(f^h(q_h^1),p_h^1)/\sum_{h=1}^H C^h(f^h(q_h^0),p_h^1) \\ &= \textstyle\sum_{h=1}^H p_h^1\cdot q_h^1/\sum_{h=1}^H C^h(f^h(q_h^0),p_h^1) \qquad \text{using (248) for } t = 1\ ^{146}\end{aligned}$$

[145] It can be seen that $Q_A(q^0,q^1,p^0)$ is equal to a weighted average of the individual household Laspeyres Allen quantity indexes; i.e., $Q_A(q^0,q^1,p^0) = \Sigma_{h=1}^H S_h^0 C^h(u_h^1,p_h^0)/C^h(u_h^0,p_h^0)$ where $S_h^0 \equiv p_h^0\cdot q_h^0/\Sigma_{i=1}^H p_i^0\cdot q_i^0$ for h = 1,...,H. Since the weights for the individual household quantity indexes are equal to the household's share of total nominal consumption in period 0, $Q_A(q^0,q^1,p^0)$ can be interpreted as a *plutocratic aggregate quantity index*. A *democratic aggregate quantity index* can be defined as $\Sigma_{h=1}^H (1/H)[C^h(u_h^1,p_h^0)/C^h(u_h^0,p_h^0)]$.

[146] It can be seen that $Q_A(q^0,q^1,p^1)$ is equal to a weighted harmonic average of the individual household Paasche Allen quantity indexes; i.e., $Q_A(q^0,q^1,p^1) = \{\Sigma_{h=1}^H S_h^1 [C^h(u_h^1,p_h^0)/C^h(u_h^0,p_h^0)]^{-1}\}^{-1}$.

$\geq \sum_{h=1}^{H} p_h^1 \cdot q_h^1 / \sum_{h=1}^{H} p_h^1 \cdot q_h^0$
since q_h^0 is feasible for the cost minimization problem $C^h(f^h(q_h^0),p_h^1)$ for h = 1,2,…,H
$\equiv Q_P(p^0,p^1,q^0,q^1)$

where $Q_P(p^0,p^1,q^0,q^1)$ is defined to be the observable (in principle) *Paasche quantity index*, $\sum_{h=1}^{H} p_h^1 \cdot q_h^1/\sum_{h=1}^{H} p_h^1 \cdot q_h^0 = p^1 \cdot q^1/p^1 \cdot q^0$. The inequality (261) says that the theoretical Paasche Allen aggregate quantity index, $Q_A(q^0,q^1,p^1)$, is bounded from below by the observable Paasche quantity index $Q_P \equiv p^1 \cdot q^1/p^1 \cdot q^0$.

As usual, it is possible to find two sided bounds for a relevant Allen aggregate quantity index; i.e., we have the following proposition:

Proposition 16: Under our regularity conditions, there exists a number λ^* between 0 and 1 such that

(262) $Q_L \leq Q_A(q^0,q^1,\lambda^* p^0+(1-\lambda^*)p^1) \leq Q_P$ or $Q_P \leq Q_A(q^0,q^1,\lambda^* p^0+(1-\lambda^*)p^1) \leq Q_L$

where $Q_L \equiv p^0 \cdot q^1/p^0 \cdot q^0$ and $Q_P \equiv p^1 \cdot q^1/p^1 \cdot q^0$. The proof of Proposition 16 is similar to the proof of Proposition 1.

The above result tells us that *the theoretical aggregate Allen quantity index*, $Q_A(q^0,q^1,\lambda^* p^0+(1-\lambda^*)p^1)$, lies between the observable Laspeyres and Paasche quantity indexes, Q_L and Q_P, where the reference price vector is the intermediate price vector ,$\lambda^* p^0+(1-\lambda^*)p^1$. Hence if Q_L and Q_P are not too different, a good approximation to a theoretical aggregate quantity index will be the *single stage Fisher quantity index* $Q_F(p^0,p^1,q^0,q^1)$ defined as $[p^0 \cdot q^1 p^1 \cdot q^1/p^0 \cdot q^0 p^1 \cdot q^0]^{1/2}$. This single stage Fisher quantity index is computed just like the usual Fisher quantity index, except that each commodity in each region (or for each household) is regarded as a separate commodity.

The two special cases of the family of aggregate Allen quantity indexes defined by (260) and (261) are connected to the two special cases of family of Konüs cost of living indexes defined by (250) and (251) in the previous section. Using these definitions, it is straightforward to show that the following two relationships hold:

(263) $P_K(p^0,p^1,q^0)Q_A(q^0,q^1,p^1) = \sum_{h=1}^{H} C^h(f^h(q_h^1),p_h^1)/\sum_{h=1}^{H} C^h(f^h(q_h^0),p_h^0) = \sum_{h=1}^{H} p_h^1 \cdot q_h^1/\sum_{h=1}^{H} p_h^0 \cdot q_h^0$;
(264) $P_K(p^0,p^1,q^1)Q_A(q^0,q^1,p^0) = \sum_{h=1}^{H} C^h(f^h(q_h^1),p_h^1)/\sum_{h=1}^{H} C^h(f^h(q_h^0),p_h^0) = \sum_{h=1}^{H} p_h^1 \cdot q_h^1/\sum_{h=1}^{H} p_h^0 \cdot q_h^0$.

Thus the aggregate Laspeyres Konüs price index $P_K(p^0,p^1,q^0)$ times the aggregate Paasche Allen quantity index $Q_A(q^0,q^1,p^1)$ equals the aggregate value ratio for the group of households, $p^1 \cdot q^1/p^0 \cdot q^0$, and the aggregate Paasche Konüs price index $P_K(p^0,p^1,q^1)$ times the aggregate Laspeyres Allen quantity index $Q_A(q^0,q^1,p^0)$ also equals the aggregate value ratio, $p^1 \cdot q^1/p^0 \cdot q^0$.

As was the case in the previous section, it is possible to obtain an alternative estimator for an aggregate quantity index if stronger assumptions on household preferences are made. Thus as in the previous section, assume that the preferences of household h are represented by the linearly homogeneous utility function $f^h(q_h) \equiv [q_h{}^T A^h q_h]^{1/2}$ where A^h is a symmetric matrix, which satisfies the regularity conditions discussed in section 5 above for h = 1,...,H. Under these assumptions, the individual household Fisher price and quantity indexes, $P_F(p_h^0,p_h^1,q_h^0,q_h^1)$ and $Q_F(p_h^0,p_h^1,q_h^0,q_h^1)$, will be exact for these preferences. As in the previous section, let $c^h(p_h) = c^h(p_{h1},...,p_{hN})$ be the unit cost function that corresponds to $f^h(q_h)$ for h = 1,...,H. Assuming utility maximizing behavior on the part of each household, equations (253)-(255) will be satisfied.

Under the above homothetic utility function assumptions on household preferences, definition (260) for the Laspeyres Allen aggregate quantity index, $Q_A(q^0,q^1,p^0)$, simplifies into the following expression:

$$(265)\ Q_A(q^0,q^1,p^0) \equiv \sum_{h=1}^H C^h(u_h^1,p_h^0)/\sum_{h=1}^H C^h(u_h^0,p_h^0) \qquad \text{where } u_h^t \equiv f^h(q_h^t) \text{ for } h = 1,\ldots,H \text{ and } t = 0,1$$
$$= \sum_{h=1}^H u_h^1 c^h(p_h^0)/\sum_{h=1}^H u_h^0 c^h(p_h^0) \qquad \text{since } C^h(u_h,p_h) = u_h c(p_h) \text{ for each } h$$
$$= \sum_{h=1}^H P_h^0 Q_h^1/\sum_{h=1}^H P_h^0 Q_h^0 \qquad \text{using (254) and (255)}$$
$$= Q_L(P^0,P^1,Q^0,Q^1)$$

where $P^t \equiv [P_1^t,\ldots,P_H^t]$ and $Q^t \equiv [Q_1^t,\ldots,Q_H^t]$ for $t = 0,1$ and $Q_L(P^0,P^1,Q^0,Q^1)$ is the ordinary Laspeyres quantity index using the aggregate household prices and quantities, P^t and Q^t, for the two periods under consideration as the household aggregate price and quantity variables.

Similarly, definition (261) for the Paasche Allen aggregate quantity index $Q_A(q^0,q^1,p^1)$ simplifies into the following expression:

$$(266)\ Q_A(q^0,q^1,p^1) \equiv \sum_{h=1}^H C^h(u_h^1,p_h^1)/\sum_{h=1}^H C^h(u_h^0,p_h^1)$$
$$= \sum_{h=1}^H u_h^1 c^h(p_h^1)/\sum_{h=1}^H u_h^0 c^h(p_h^1) \qquad \text{since } C^h(u_h,p_h) = u_h c(p_h) \text{ for each } h$$
$$= \sum_{h=1}^H P_h^1 Q_h^1/\sum_{h=1}^H P_h^1 Q_h^0 \qquad \text{using (254) and (255)}$$
$$\equiv Q_P(P^0,P^1,Q^0,Q^1)$$

where $Q_P(P^0,P^1,Q^0,Q^1)$ is the ordinary Paasche quantity index using the aggregate household prices and quantities for the two periods under consideration as the price and quantity variables.

The aggregate quantity indexes $Q_A(q^0,q^1,p^0)$ and $Q_A(q^0,q^1,p^1)$ defined by (265) and (266) are equally plausible measures of overall consumer quantity or volume growth between periods 0 and 1 and so it is reasonable to take an average of these two indexes to obtain a "final" estimate of aggregate quantity growth between the two periods. As usual, the geometric average leads to an index that will satisfy a time reversal test. Thus we have:

$$(267)\ [Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2} = [Q_L(P^0,P^1,Q^0,Q^1)Q_P(P^0,P^1,Q^0,Q^1)]^{1/2} \equiv Q_F(P^0,P^1,Q^0,Q^1)$$

where $Q_F(P^0,P^1,Q^0,Q^1)$ is the *Fisher quantity index* defined over the aggregate household prices and quantities for the two periods under consideration. It is a two stage Fisher index where the first stage of aggregation uses the price and quantity data for each household to construct household specific Fisher price and quantity levels for each household. The two stage Fisher quantity index $Q_F(P^0,P^1,Q^0,Q^1)$ defined by (267) can be compared to the single stage Fisher quantity index $Q_F(p^0,p^1,q^0,q^1)$ defined as the geometric mean of $Q_L(p^0,p^1,q^0,q^1) \equiv p^0\cdot q^1/p^0\cdot q^0$ and $Q_P(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/p^1\cdot q^0$. Using the results listed in section 8 above, we know that the single stage Fisher quantity index will approximate its two stage counterpart to the second order around an equal price and quantity point. Thus normally, we would not expect much difference between these alternative measures of overall real aggregate consumption growth.

18. Social Welfare Functions and Inequality Indexes

Equations (265) and (266) have some interesting implications. These equations give the following decompositions for an aggregate quantity index: $Q_A(q^0,q^1,p^0) = \sum_{h=1}^H u_h^1 c^h(p_h^0)/\sum_{h=1}^H u_h^0 c^h(p_h^0)$ and $Q_A(q^0,q^1,p^1) = \sum_{h=1}^H u_h^1 c^h(p_h^1)/\sum_{h=1}^H u_h^0 c^h(p_h^1)$. The numerators in these equations can be interpreted as aggregate period 1 quantity levels and the denominators as aggregate period 0 quantity levels. These quantity levels have the same general form; i.e., the period t aggregate quantity level Q^t is equal to a weighted sum of the period t household utility levels so that $Q^t \equiv \Sigma_{h=1}^H \omega_h u_h^t$ for $t = 0,1$ where the weights ω_h are fixed nonnegative numbers. Functions like $\Sigma_{h=1}^H \omega_h u_h^t$ are called *social welfare functions* in the economics literature. Thus the two aggregate Allen indexes can be regarded as specific examples where the indexes are equal to ratios of social welfare functions.

Choosing the appropriate weights for a social welfare function is a nontrivial problem, which has not been complety resolved in the economics literature but there is a demand for statistical agencies to produce measures of social welfare that take into account possible inequality in the distribution of income between households.[147] We will not go into great detail on the complex issues surrounding the measurement of social welfare but we will indicate some of the problems that are associated with the construction of indexes of social welfare.

The first problem that needs to be addressed is that the individual household utility measures have to be made cardinally comparable in some way. Recall the assumptions made on household preferences made above equations (253). In order to construct meaningful measures for the levels of social welfare, it is necessary to make stronger assumptions; i.e., we now assume that the preferences of household h are represented by the linearly homogeneous utility function $f^h(q_h) \equiv [q_h{}^T A q_h]^{1/2}$ for each h where A is a symmetric matrix which satisfies the regularity conditions discussed in section 5 above. Thus under these stronger assumptions, we are now assuming that the household preference functions are *identical* across households for h = 1,...,H. Under these assumptions, the Fisher price and quantity indexes will be exact across households within a time period as well as across time periods. Let $c(p_h) = c(p_{h1},...,p_{hN})$ be the unit cost function that corresponds to $f(q_h)$ for h = 1,...,H. Assuming utility maximizing behavior on the part of each household, the following equations should be satisfied:

(268) $p_h{}^t \cdot q_h{}^t = f(q_h{}^t)c(p_h{}^t)$; h = 1,...,H; t = 0,1.

Now use Fisher price and quantity indexes to estimate household quantity and price levels, $Q_h{}^t \equiv f(q_h{}^t)$ and $P_h{}^t \equiv c(p_h{}^t)$, for t = 0,1 and h = 1,...,H as follows:

(269) $P_1{}^0 \equiv 1 \equiv c(p_1{}^0)$; $Q_1{}^0 \equiv p_1{}^0 \cdot q_1{}^0 \equiv f(q_1{}^0) \equiv u_1{}^0$;
(270) $P_h{}^0 \equiv P_F(p_1{}^0,p_h{}^0,q_1{}^0,q_h{}^0) \equiv c(p_h{}^0)$; $Q_h{}^0 \equiv p_h{}^0 \cdot q_h{}^0/P_h{}^0 \equiv f(q_h{}^0) \equiv u_h{}^0$; h = 2,...,H;
(271) $P_h{}^1 \equiv P_F(p_1{}^0,p_h{}^1,q_1{}^0,q_h{}^1) \equiv c(p_h{}^1)$; $Q_h{}^1 \equiv [p_h{}^1 \cdot q_h{}^1]/P_h{}^1 \equiv f(q_h{}^1) \equiv u_h{}^1$; h = 1,...,H.

Thus household 1 in period 0 acts as a *numeraire household*; the Fisher price and quantity indexes for the other households in periods 0 and 1 are computed relative to household 1 in period 0.[148] Once the cardinally comparable utility levels $u_h{}^t$ have been computed using definitions (269)-(271), they can be used to determine the level of *social welfare* in each period t. For example, the period t level of social welfare could be defined as $Q^t \equiv \Sigma_{h=1}{}^H \omega_h u_h{}^t$ for t = 0,1 where the weights ω_h are somehow chosen by the statistical office.

However, it has proven to be difficult to come up with consensus social welfare weights for the ω_h. A simple solution is to set $\omega_h = 1$ for h = 1,...,H. The resulting function is the *utilitarian social welfare function*. However, this function shows no concern of the distribution of utility across all households. An allocation of the economy's real expenditures on consumer goods and services that gave most of the total group expenditure to one household would generate the same level of social welfare using the utilitarian function as the distribution that divided the total real expenditures equally across households. In order to address distributional issues, it is necessary to introduce nonlinear social welfare functions.

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[147] See Hays, Martin and Mkandawire (2019).

[148] This is known as a "star" approach to the construction of multilateral indexes and the resulting indexes will depend on the choice of the numeraire household. We will introduce more symmetric methods for making multilateral comparisons in Chapter 7.

Atkinson (1970; 257) introduced the following *mean of order r social welfare function*:[149]

(272) $W^r(u_1,...,u_H) \equiv [\Sigma_{h=1}^H (1/H)(u_h)^r]^{1/r}$

where $r \leq 1$ and $r \neq 0$.[150] Note that $W^r(u_1,...,u_H)$ is a measure of per capita utility rather than a measure of total utility for the period under consideration. Using the earlier materials on CES utility functions, we know that $W^r(u_1,...,u_H) \equiv W^r(u)$ is a linearly homogeneous, concave increasing function of the household utility levels, $u \equiv [u_1,...,u_H]$. When $r = 1$, $W^1(u) = \Sigma_{h=1}^H (1/H)u_h$ which is *per capita utility*. As r approaches minus infinity, $W^r(u_1,...,u_H)$ approaches $\min_h \{u_h : h = 1,...,H\}$, which is the social welfare function advocated by Rawls (1971).[151]

It proves to be useful to compare an Atkinson measure of social welfare $W^r(u_1,...,u_H)$ with per capita utility for each period. *Period t per capita utility* is defined as follows:

(273) $u_A^t \equiv \Sigma_{h=1}^H (1/H)u_h^t \equiv W^1(u_1^t,...,u_H^t)$; $t = 0,1.$

Thus per capita utility is a special case of the Atkinson family of social welfare measures with $r = 1$. For a general $r < 1$, Atkinson's (1970; 250) period t *equally distributed equivalent real income per head*, u_E^t, is defined (implicitly) by the following equation:

(274) $W^r(u_1^t,...,u_H^t) = W^r(u_E^t 1_H)$; $t = 0,1$
$= u_E^t W^r(1_H)$ using the linear homogeneity property of $W^r(u_1,...,u_H)$
$= u_E^t$ using definition (272) which implies $W^r(1_H) = 1$.

Thus actual social welfare in period t, $W^r(u_1^t,...,u_H^t)$, is set equal to a level of social welfare where each household gets the same level of utility, u_E^t. Hardy, Littlewood and Polya (1934; 26) show that the mean of order r function, $W^r(u_1,...,u_H)$, is increasing in r provided that not all u_h are the same and nondecreasing in r in general. Since $r < 1$, $W^r(u_1^t,...,u_H^t) \leq W^1(u_1^t,...,u_H^t)$ for $t = 0,1$. Using these inequalities and definitions (273) and (274), we have the following inequalities:

(275) $u_E^t/u_A^t \leq 1$; $t = 0,1.$

Kolm's (1969; 186) period t *index of relative injustice* or Atkinson's (1970; 257) and Sen's (1973; 42) period t *relative inequality index*, I^t, is defined as follows:

(276) $I^t \equiv 1 - (u_E^t/u_A^t) \geq 0$; $t = 0,1.$

Thus if household utility levels in period t are identical, u_E^t will equal u_A^t and period t inequality I^t will equal 0. If r is a very large negative number, and one or more households in period t has a very low utility level, then u_E^t will be close to 0 and I^t will be close to 1, the maximum amount of inequality that can occur.

Define the *period t equality index* as

(277) $E^t \equiv u_E^t/u_A^t$; $t = 0,1.$

[149] Atkinson worked with continuous distributions of nominal incomes whereas we work with discrete distributions of real incomes. Fleurbaey (2009; 1032) has a discrete version of Atkinson's approach, which is similar to the approach presented here except that nominal incomes are used in place of our real incomes. Finally, Jorgenson and Schreyer (2017; S466) use a version of the approach presented here except they assume all households face the same prices.
[150] As usual, if $r = 0$, define the logarithm of $W^0(u_1,...,u_H)$ as $\Sigma_{h=1}^H (1/H)\ln u_h$.
[151] See also Blackorby and Donaldson (1978).

Thus the closer E^t is to its maximum value 1, the more equal is the distribution of real consumption in the group of households under consideration. Since period t Atkinson welfare is equal to $W^r(u_1^t,...,u_H^t) = u_E^t$, we can write period t welfare as the product of per capita real consumption, u_A^t, times E^t:[152]

$$(278)\ W^r(u_1^t,...,u_H^t) = u_A^t E^t \qquad t = 0,1.$$

A practical problem with the above approach for measuring social welfare is that it is necessary to pick a specific value for r in order to implement it.[153] Since the results will depend on which r is chosen and since there is no general consensus on which r to choose, statistical agencies have largely not produced practical measures of social welfare. Thus we will conclude this section by considering one more approach to the production of social welfare indexes: an approach that, at first glance, does not require chosing parameters for the social welfare function.

Our final approach to the measurement of social welfare relies on a discrete version of the Gini (1921) coefficient. We first convert the household utility levels u_h^t defined by (269)-(271) into *household shares of total utility* σ_h^t for each time period:

$$(279)\ \sigma_h^t \equiv u_h^t/\Sigma_{i=1}^H u_i^t\ ; \qquad h = 1,...,H;\ t = 0,1.$$

Now order the households so that household 1 has the lowest utility in period t, household 2 has the next lowest utility and so on. Thus for each period t, the shares σ_h^t will satisfy the following inequalities:

$$(280)\ \sigma_1^t \le \sigma_2^t \le ... \le \sigma_H^t\ ; \qquad t = 0,1.$$

The *area under the cumulative distribution function of the share variables* σ_h^t is proportional to A^t defined as follows:

$$(281)\ A^t \equiv \sigma_1^t + (\sigma_1^t+\sigma_2^t) + (\sigma_1^t+\sigma_2^t+\sigma_3^t) + ... + (\Sigma_{h=1}^{H-1}\ \sigma_h^t) + (\Sigma_{h=1}^H\ \sigma_h^t)\ ; \qquad t = 0,1$$
$$= H\sigma_1^t + (H-1)\sigma_2^t + (H-2)\sigma_3^t + ... + 2\sigma_{H-1}^t + \sigma_H^t.$$

Consider the following linear programming problem:

$$(282)\ \max_{\sigma_1,...,\sigma_H} \{H\sigma_1+(H-1)\sigma_2+(H-2)\sigma_3+ ... +2\sigma_{H-1}+\sigma_H : 0 \le \sigma_1^t \le \sigma_2^t \le ... \le \sigma_H^t;\ \Sigma_{h=1}^H\ \sigma_h = 1\}.$$

The solution to this problem is $\sigma_h = 1/H$ for h = 1,...,H. Substitute this solution into the objective function in (282) and this will determine the maximum value A^* for the objective function in (282):

$$(283)\ A^* \equiv [H + (H-1) + (H-2) + ... + 2 + 1][1/H] = [H(H+1)/2][1/H] = (H+1)/2.$$

Define the period t *Gini index of equality* for the distribution of household utilities, E^{t*}, as:

$$(284)\ E^{t*} \equiv A^t/A^* \le 1\ ; \qquad t = 0,1$$

[152] See Atkinson (1970; 250) and Fleurbaey (2009; 1032) for this type of decomposition applied to nominal incomes and see Jorgenson and Schreyer (2017; S470) for this type of decomposition applied to real incomes.

[153] For alternative social welfare functions that require exogenous parameterization, see Diewert (1985; 77-82), Fleurbaey (2009; 1032-1036) and Jorgenson and Schreyer (2017).

where A^t is defined by (281) and A^* is defined by (283). The inequalities $A^t/A^* \leq 1$ follow since A^t is necessarily less than the maximum possible value for A^t, which is A^*. The period t *Gini coefficient* or *Gini index of inequality* for the discrete income distribution, G^t, is defined as:

$$(285)\ G^t \equiv 1 - E^{t*}\ ; \qquad t = 0,1.$$

The Gini coefficient as a measure of inequality in *nominal* income distributions is well understood and well accepted in economic measurement circles. The above algebra simply adapts it as a measure of inequality for *real* income distributions. There are no additional parameters that have to be determined by the official statistician. [154]

The final step is to use E^{t*} to adjust per capita real consumption $u_A{}^t$ defined above by definitions (273) for inequality in the real income distribution; i.e., define period t welfare, W^t, as:

$$(286)\ W^t \equiv u_A{}^t E^{t*} = u_A{}^t(1 - G^t)\ ; \qquad t = 0,1.$$

Thus for each period t, per capita real consumption for the group under consideration, $u_A{}^t$, is multiplied by the Gini equality index E^{t*} to give an estimate of social welfare for the group that takes into account the distribution of real incomes within the group. Since the Gini coefficient is a generally accepted measure of inequality, the social welfare estimates defined by (286) are likely to be acceptable to the public.[155]

However, there are a number of practical measurement problems that are not addressed in the above material:

- Real income distributions (or more accurately, distributions of real consumption over households in a country) do not exist. Thus the real "income" distribution described above may have to be approximated by a corresponding nominal distribution of household consumption expenditures for a period. This approximation may be satisfactory if all households in the group under consideration face approximately the same prices.
- Some households have more members than other households but the theory outlined above implicitly assumed that all households had the same size. This problem can be addressed by the use of *household equivalence scales* but some measurement error will be introduced by their use.[156] For

[154] However, the fact that the economic statistician using the Gini equality index to adjust per capita real income for inequality does not have to pick a particular value of r as is the case if an Atkinson social welfare function is used to measure inequality does not imply that the use of the Gini coefficient methodology is free of value judgements. The social welfare function defined by (286) does imply specific judgements about the relative welfare of the individuals in the welfare comparison; see Atkinson (1970; 257).

[155] For related work on the use of the Gini coefficient in measures of welfare, see Sen (1976; 30-31) and Fleurbaey (2009; 1034-1035).

[156] The simplest way to deal with households that differ in the number of members is to divide their utility, say $u_h{}^t$ for household h in period t, by $n_h{}^t$, which is the number of household members. Then when constructing the distribution of utilities for period t, replace $u_h{}^t$ by $n_h{}^t$ copies of per person utility, $u_h{}^t/n_h{}^t \equiv u_h{}^{t*}$. This crude adjustment of utility for household composition neglects the fact that multiple person households can share the services of the durable goods owned by the household. A *household equivalence scale* for household h in period t is a *household efficiency factor* $a_h{}^t$ which is equal to 1 if $n_h{}^t = 1$ and if $n_h{}^t > 1$, $a_h{}^t > 1$. The new adjusted per person utility for the household $u_h{}^{t*}$ is set equal to unadjusted per person utility, $u_h{}^t/n_h{}^t$, times the household efficiency factor $a_h{}^t$. Thus the new adjusted for composition per person household utility is $u_h{}^{t*} \equiv u_h{}^t a_h{}^t/n_h{}^t \geq u_h{}^t/n_h{}^t$. Thus when constructing the distribution of utilities for period t, replace $u_h{}^t$ by $n_h{}^t$ copies of the *composition adjusted per person utility*, $u_h{}^{t*} = u_h{}^t a_h{}^t/n_h{}^t$. Our suggested approach to adjusting social welfare measures for household composition is more or less the same as the procedure suggested by Jorgenson and Schreyer (2017; S466).

references to the literature on alternative household equivalence scales, see Fleurbaey (2009; 1051-1052), Jorgenson and Slesnick (1987) and Jorgenson and Schreyer (2017; S462-S465).

- The services of consumer durables should be included in household consumption.[157] Most nominal income (or consumption) distributions for countries ignore the services provided by household durable goods. In particular, the services provided by Owner Occupied Housing are typically missing in published income distributions.[158] This is a serious omission.
- Finally, adjustments to household nominal expenditures should be made for households that receive goods and services provided by governments and charitable organizations at no cost or at highly subsidized prices. These subsidized goods and services should be valued at comparable market prices.[159]

19. The Matching of Prices Problem

The economic approach to index number theory starts out by developing a theory of individual household behavior. With the exception of the material in section 14, our analysis of the economic approach has assumed that prices faced by households were all positive in the two periods being compared and the quantities purchased by each household during the two periods were also positive. However, individual households rarely purchase positive amounts of the same commodities in two consecutive periods. The shorter is the time period, the greater will be this *lack of matching problem*. Part of the problem is due to the existence of seasonal commodities and part is due to the fact that consumers can store goods purchased in one period and consume them over multiple periods and the economic approach to index number theory does not take the storage problem into consideration. In recent years, an increasing number of firms have used *dynamic pricing*; i.e., they vary the prices of their products by introducing deeply discounted prices at random intervals. Thus individuals can purchase these discounted products in one period and gradually consume them over multiple periods.

There are a number of ways to address this lack of matching problem:

- Make the reference time period longer; i.e., move from a weekly index to a monthly index or move from a monthly index to a quarterly index.
- Instead of defining products narrowly (i.e., by a product code and by a particular point of purchase), group similar products together and use *broadly defined unit value prices* instead of narrowly defined unit value prices. This reduces the number of products in scope for the index from N to a number considerably less than N and this will increase the number of "matched" products.
- Aggregate households that are "similar" into a group of households and apply the economic approach to the group.
- Acknowledge that the economic approach is difficult to implement at the level of individual households and apply the fixed basket approach to index number theory that was developed in Chapter 2 to groups of households.

We will address each of the above points in turn.

[157] Christensen and Jorgenson (1969) advocated this inclusion many years ago and provided estimates for the US.

[158] Various approaches to the measurement of the services provided by consumer durables will be considered in Chapter 10.

[159] Thus there is a difference between *household expenditures* (final consumption expenditures in the System of National Accounts) and *actual individual consumption*, which includes social transfers in kind such as free or subsidized services such as health, education and housing services provided by governments at free or below market prices by government agencies. The latter concept is the correct concept to use in welfare measures.

There are a few countries that construct quarterly CPIs but most countries find that a monthly CPI seems to satisfy most user needs. Thus moving from a monthly CPI to a quarterly CPI is not feasible for most countries. Moving to weekly or daily CPIs is likely to encounter severe lack of matching problems if they are constructed at the individual level.

The problem with moving from narrowly defined products to more broadly defined products is that *unit value bias* or *quality adjustment bias* is likely to result. It is difficult to quantify the tradeoff between obtaining more product matches versus increased unit value bias.

The economic approach to index number theory can be applied to a group of households under some restrictive assumptions. Suppose we have a group of similar households which have the same homothetic preferences. In particular, suppose we have H households and N commodities and the unit cost function for each household is $c(p) \equiv (p^TBp)^{1/2}$ where B is an N by N symmetric matrix with one positive eigenvalue with a strictly positive eigenvector and the remaining eigenvalues are nonpositive. We know that the Fisher price and quantity indexes for each household are exact for this functional form. Let the utility function that corresponds to this unit cost function be f(q). Let $p_h^t >> 0_N$ and $q_h^t > 0_N$ be the "observed" price and quantity vectors for household h in period t for h = 1,...,H and t = 0,1.[160] Assuming cost minimizing behavior for each household in each period and using Shephard's Lemma, the following equations will hold, where $u_h^t \equiv f(q_h^t)$ for h = 1,...,H and t = 0,1:

(287) $q_h^t \equiv \nabla_p c(p_h^t)u_h^t = Bp_h^t u_h^t/c(p_h^t)$;   h = 1,...,H and t = 0,1.

Define the period t *aggregate quantity vector* q^t and *aggregate utility level* u^t as follows:

(288) $q^t \equiv \Sigma_{h=1}^H q_h^t$; $u^t \equiv \Sigma_{h=1}^H u_h^t$;   t = 0,1.

Our final assumption is that all households in each period t face the same vector of prices p^t:

(289) $p_h^t = p^t$;   h = 1,...,H and t = 0,1.

Using (287)-(289), we have the following equations:

(290) $q^t \equiv \Sigma_{h=1}^H q_h^t = \Sigma_{h=1}^H Bp^t u_h^t/c(p^t) = Bp^t[\Sigma_{h=1}^H u_h^t]/c(p^t) = Bp^t u^t/c(p^t)$;   t = 0,1.

Thus q^t, p^t and u^t satisfy the Shephard's Lemma equations (287) where q^t, p^t and u^t have replaced q_h^t, p_h^t and u_h^t. Thus the period t aggregate price and quantity vectors, p^t and q^t, along with the aggregate utility level u^t for t = 0 and 1 will be exact for the following Fisher aggregate quantity index:

(291) $u^1/u^0 = [p^0\cdot q^1 p^1\cdot q^1/p^0\cdot q^0 p^1\cdot q^0]^{1/2}$.

Thus under the above hypotheses, the aggregate data will satisfy the same equations as the micro data. The above assumptions justify treating the data for the group as if it were generated by a single utility maximizing household. This result is better than having no result at all but it does rest on two restrictive assumptions: (i) identical homothetic preferences and (ii) all members of the group face the same vector of prices in each period. Thus if we apply this theory, we should try to group households so that they are demographically similar (so that their preferences can be better represented by the same preference function)

[160] We have assumed that all prices are positive but some quantities are allowed to equal 0. We assume that positive reservation prices are used for products that are not consumed by a household in some period.

and so that they face similar prices (so grouping households by location is also a useful thing to do).[161] Jorgenson and Schreyer summarized the need to group households in the following quotation:

"Another, related measurement issue is the level of detail at which distributional measures are put in place. Ideally, the equivalence scales are directly applied to household-level information. In practice, another simplifying assumption is often used in empirical measurements. Rather than applying equivalence scales (and, as will be discussed below, price indices) at the level of individual households, groups of households are the object of measurement in the simplified case. Each group is treated like a single, homogenous household." Dale Jorgenson and Paul Schreyer (2017; S464).

Finally, it is possible to fall back on our very first approach to index number theory that was explained in Chapter 2. This theory works as follows: a group of households collectively purchase the vector of goods and services q^t in periods $t = 0,1$. The corresponding unit value price vector for period t is p^t for $t = 0,1$. Two equally reasonable measures of price inflation for this group of purchasers are the Laspeyres and Paasche price indexes, $P_L \equiv p^1 \cdot q^0/p^0 \cdot q^0$ and $P_P \equiv p^1 \cdot q^1/p^0 \cdot q^1$. Since both indexes are equally plausible, it makes sense to take an average of the two to obtain a point estimate of the price inflation facing this group of purchasers. The Fisher index is perhaps the "best" average because it ends up satisfying the time reversal test. A similar theory works well for measuring the growth of consumption at constant prices. If we use the base period prices as weights, the Laspeyres quantity index, $Q_L \equiv p^0 \cdot q^1/p^0 \cdot q^0$ is a reasonable measure and if we use the current period prices as weights, the Paasche quantity index, $Q_P \equiv p^1 \cdot q^1/p^1 \cdot q^0$ is another reasonable measure for the growth of consumption at constant prices. Again, it is reasonable to take a symmetric average of these two measures to end up with a point estimate for real consumption growth. The Fisher quantity index is again "best" because it satisfies the time reversal test.

Appendix: Proofs of Propositions

Proof of Proposition 1: Define $g(\lambda)$ for $0 \leq \lambda \leq 1$ by $g(\lambda) \equiv P_K(p^0,p^1,(1-\lambda)q^0 + \lambda q^1)$. Note that $g(0) = P_K(p^0,p^1,q^0)$ and $g(1) = P_K(p^0,p^1,q^1)$. There are 24 = (4)(3)(2)(1) possible a priori inequality relations that are possible between the four numbers $g(0)$, $g(1)$, P_L and P_P. However, the inequalities (3) and (4) above imply that $g(0) \leq P_L$ and $P_P \leq g(1)$. This means that there are only six possible inequalities between the four numbers:

(A1) $g(0) \leq P_L \leq P_P \leq g(1)$;
(A2) $g(0) \leq P_P \leq P_L \leq g(1)$;
(A3) $g(0) \leq P_P \leq g(1) \leq P_L$;
(A4) $P_P \leq g(0) \leq P_L \leq g(1)$;
(A5) $P_P \leq g(1) \leq g(0) \leq P_L$;
(A6) $P_P \leq g(0) \leq g(1) \leq P_L$.

Using the assumptions that: (a) the consumer's utility function f is continuous over its domain of definition; (b) the utility function is increasing in the components of q and hence is subject to local nonsatiation and (c) the price vectors p^t have strictly positive components, it is possible to use Debreu's (1959; 19) Maximum Theorem (see also Diewert (1993a; 112-113) for a statement of the Theorem) to show that the consumer's cost function $C(f(q),p^t)$ will be continuous in the components of q. Thus using definition (2), it can be seen that $P_K(p^0,p^1,q)$ will also be continuous in the components of the vector q. Hence $g(\lambda)$ is a continuous function of λ and assumes all intermediate values between $g(0)$ and $g(1)$. By inspecting the inequalities (A1)-(A6) above, it can be seen that we can choose λ between 0 and 1, λ^* say, such that $P_L \leq g(\lambda^*) \leq P_P$ for case (A1) or such that $P_P \leq g(\lambda^*) \leq P_L$ for cases (A2) to (A6). Thus at least one of the two inequalities in (5) holds.

[161] This last point helps to justify applying the above methodology to the customers of a particular retail outlet.

Proof of Proposition 2: Using assumptions (ii) and (iv), $q^t >> 0_N$ solves the concave programming problem $\max_q \{f(q) : p^t \cdot q \leq e^t ; q \geq 0_N\}$ for t = 0,1. Since q^t is strictly positive, the nonnegativity constraints $q \geq 0_N$ are not binding and hence, using the differentiability assumptions (iii), the following Lagrangian conditions are necessary and sufficient for q^t to solve the period t constrained maximization problem in (13):

(A7) $\nabla f(q^t) = \lambda_t p^t$; t = 0,1;
(A8) $p^t \cdot q^t = e^t$.

Take the inner product of both sides of (A7) with q^t and solve the resulting equation for λ_t. The solution for t = 0,1 is $\lambda_t = q^t \cdot \nabla f(q^t)/p^t \cdot q^t > 0$.[162] Substitute this solution for λ_t back into equation t in (A7). After a bit of rearrangement, we obtain the equations $p^t/p^t \cdot q^t = \nabla f(q^t)/q^t \cdot \nabla f(q^t)$ for t = 0,1.

Proof of Proposition 3: Let $u^t = f(q^t)$ for t = 0,1. By assumption (iii), q^t solves the cost minimization problem defined by $C(u^t,p^t)$ for t = 0,1. Thus q^t is a feasible solution for the following cost minimization problem where the general price vector $p >> 0_N$ has replaced the specific period t price vector p^t:

(A9) $C(u^t,p) \equiv \min_q \{p \cdot q : f(q) \geq u^t; q \geq 0_N\}$; t = 0,1
$\leq p \cdot q^t$

where the inequality follows, since q^t is a feasible (but not necessarily an optimal) solution for the cost minimization problem defined by $C(u^t,p)$. Since by assumption (iii), q^t is a solution to the cost minimization problem defined by $C(u^t,p^t)$, we must have the following equalities:

(A10) $C(u^t,p^t) = p^t \cdot q^t$; t = 0,1.

Define the function $g^t(p) \equiv C(u^t,p) - p \cdot q^t$ for t = 0,1. Since $C(u^t,p)$ is a concave function in p and since the linear function $- p \cdot q^t$ is also concave in p, it can be seen that $g^t(p)$ is also a concave function of p for t = 0,1. The inequalities (A9) and equalities (A10) show that $g^t(p)$ achieves a global maximum at $p = p^t$ for t = 0,1. Since $C(u^t,p)$ is differentiable with respect to the components of p at $p = p^t$, the following first order necessary conditions for maximizing $C(u^t,p)$ with respect to the components of p must hold:

(A11) $\nabla_p g(p^t) = \nabla_p C(u^t,p^t) - q^t = 0_N$; t = 0,1.

Equations (A11) can be rearranged to give us the following equations:

(A12) $q^t = \nabla_p C(u^t,p^t)$; t = 0,1.

To establish the uniqueness of q^t, let q^{t*} be any other solution to the cost minimization problem defined by $C(u^t,p^t)$ for t = 0,1. Repeat the above proof to show that $q^{t*} = \nabla_p C(u^t,p^t)$ for t = 0,1. Thus $q^t = q^{t*}$ for t = 0,1 and the solution to the cost minimization problem defined by $C(u^t,p^t)$ is unique for t = 0,1.[163]

[162] We assume that at least one component of $\nabla f(q^t)$ is positive for t = 0,1.
[163] This method of proof is due to McKenzie (1956). Shephard (1953) (1970) was the first to derive this result starting with a differentiable cost function. However, Hotelling (1932; 594) stated a version of the result in the context of profit functions and Hicks (1946; 331) and Samuelson (1953; 15-16) established the result starting with a differentiable utility or production function. For a more complete exposition of the technical details and references to the literature, see Diewert (1993a; 107-117).

Proof of Proposition 4: Let $f^*(q)$ be a given increasing linearly homogeneous function which is twice continuously differentiable along the ray λq^* where $\lambda > 0$ and $q^* >> 0_N$. We assume that $f^*(q^*) > 0$. Since $f^*(q)$ is linearly homogeneous, we have:

(A13) $f^*(\lambda q^*) = \lambda f^*(q^*)$ for all $\lambda > 0$.

Differentiate both sides of (A13) with respect to λ and evaluate the resulting derivatives at $\lambda = 1$. We obtain the following equation:

(A14) $f^*(q^*) = \nabla f^*(q^*)^T q^* = \Sigma_{n=1}^N q_n^* \partial f^*(q^*)/\partial q_n$.

Thus if the first order partial derivatives of $f^*(q^*)$ are known numbers, then the number $f^*(q^*)$ is also known and is equal to $q^{*T}\nabla f^*(q^*) = \Sigma_{n=1}^N q_n^* \partial f^*(q^*)/\partial q_n$.

Now partially differentiate both sides of (A13) with respect to q_n for $n = 1,...,N$. The following equations are obtained for all $\lambda > 0$:

(A15) $[\partial f^*(\lambda q^*)/\partial(\lambda q_n)][\partial(\lambda q_n)/\partial\lambda] = \lambda \partial f^*(\lambda q^*)/\partial(\lambda q_n) = \lambda \partial f^*(q^*)/\partial q_n$ $\quad n = 1,...,N$.

Let $f_n^*(q) \equiv \partial f^*(q)/\partial q_n$ denote the function that is the partial derivative of $f^*(q)$ with respect to q_n for $n = 1,...,N$. Using this notation, equations (A15) simplify to the following equations:

(A16) $f_n^*(\lambda q^*) = f_n^*(q^*)$ for all $\lambda > 0$; $\quad n = 1,...,N$.

Thus the first order partial derivative functions $f_n^*(q)$ of a linearly homogeneous function $f^*(q)$ are homogeneous of degree 0. Now differentiate both sides of equations (A16) with respect to λ, evaluate the resulting second order partial derivatives $f_{nk}^*(\lambda q^*)$ at $\lambda = 1$ and we obtain the following system of equations:

(A17) $\Sigma_{k=1}^N f_{nk}^*(q^*)q_n^* = 0$; $\quad n = 1,...,N$

where $f_{nk}^*(q^*) \equiv \partial^2 f^*(q)/\partial q_n \partial q_k$ for $n,k = 1,...,N$. The N equations (A17) can be rewritten more succinctly using matrix notation as the following matrix equation:

(A18) $\nabla^2 f^*(q^*)q^* = 0_N$.

Since $f^*(q)$ is assumed to be twice continuously differentiable at $q = q^*$, Young's Theorem in advanced calculus implies that the matrix of second order derivatives, $\nabla^2 f^*(q^*)$, is a symmetric matrix so that $\partial^2 f^*(q)/\partial q_n \partial q_k = \partial^2 f^*(q)/\partial q_k \partial q_n$ for all $n, k = 1,...,N$. Using matrix notation once again, this means that:

(A19) $[\nabla^2 f^*(q^*)]^T = \nabla^2 f^*(q^*)$.

The $1 + N + N^2$ numbers $f^*(q^*)$, $\nabla f^*(q^*)$ and $\nabla^2 f^*(q^*)$ are regarded as given numbers or parameters in what follows. From the above derivations, we see that that these numbers are not independent: equation (A14), $f^*(q^*) = \nabla f^*(q^*)^T q^*$, implies that if the N components in the vector of first order partial derivatives $\nabla f^*(q^*)$ are given numbers, then the level of the function $f^*(q)$ evaluated at the point q^* is determined by these numbers. Similarly, the symmetry conditions (A19) imply that if the N^2 second order partial derivatives of $f^*(q^*)$ are calculated, then these numbers are not independent of each other either. If the $N(N-1)/2$ components of $\nabla^2 f^*(q^*)$ in the upper triangle of this matrix are given (so that $\partial^2 f^*(q)/\partial q_n \partial q_k$ for $1 \leq n < k \leq N$ are given numbers), then the $N(N-1)/2$ numbers in the lower triangle of this matrix are also determined. Furthermore, the N restrictions given by equations (A18) mean that if the upper triangle second order partial derivatives are given (which means that the lower triangle second order partial derivatives are also given), then the main diagonal second order partial derivatives (the N derivatives $\partial^2 f^*(q)/\partial q_n \partial q_n$ for n = 1,...,N) are also determined (provided that the components of the q^* vector are all positive). Thus the assumption of linear homogeneity of $f^*(q)$ (along with the assumption that second order partial derivatives of $f^*(q)$ exist and are continuous at $q = q^*$) implies that there are only $N(N-1)/2$ independent parameters instead of N^2 parameters in the matrix $\nabla^2 f^*(q^*)$.

Define the utility function f(q) over the set $S \equiv (q : q \geq 0_N;\ Aq \geq 0_N;\ q^TAq > 0)$ as:

(A20) $f(q) \equiv (q^TAq)^{1/2}$ where $A = A^T$.

To show that the f(q) is a flexible functional form at $q = q^* >> 0_N$, we need to solve the following equations for the components of the N by N matrix $A \equiv [a_{nk}]$ where $a_{nk} = a_{kn}$ for $1 \leq n < k \leq N$:

(A21) $f(q^*) = f^*(q^*)$;
(A22) $\nabla f(q^*) = \nabla f^*(q^*)$;
(A23) $\nabla^2 f(q^*) = \nabla^2 f^*(q^*)$.

Define the matrix A as follows:

(A24) $A \equiv f^*(q^*)\nabla^2 f^*(q^*) + \nabla f^*(q^*)\nabla f^*(q^*)^T$.

Note that this A matrix is symmetric; i.e., $A = A^T$. Use this A matrix to define $f(q) \equiv (q^TAq)^{1/2}$ and compute $f(q^*)^2$:

(A25) $f(q^*)^2 = q^{*T}Aq^*$
$= q^{*T}[f^*(q^*)\nabla^2 f^*(q^*) + \nabla f^*(q^*)\nabla f^*(q^*)^T]q^*$ using definition (A24)
$= q^{*T}\nabla f^*(q^*)\nabla f^*(q^*)^Tq^*$ using (A18)
$= f^*(q^*)^2$ using (A14).

Take positive square roots of both sides of (A25) and the resulting equation is (A21). Now calculate the vector of first order partial derivatives of f(q) defined by (A20) and (A24) and evaluate these derivatives at $q = q^*$:

(A26) $\nabla f(q^*) = Aq^*/(q^{*T}Aq^*)^{1/2}$
$= [f^*(q^*)\nabla^2 f^*(q^*) + \nabla f^*(q^*)\nabla f^*(q^*)^T]q^*/f^*(q^*)$ using (A24) and (A25)
$= 0_N + \nabla f^*(q^*)[\nabla f^*(q^*)^Tq^*]/f^*(q^*)$ using (A18)
$= \nabla f^*(q^*)$ using (A14).

Thus equations (A22) are satisfied. Finally, calculate the matrix of second order partial derivatives of f(q) defined by (A20) and (A24) and evaluate these derivatives at $q = q^*$. Differentiating the first line in (A26) leads to the following matrix equation:

(A27) $\nabla^2 f(q^*) = \{A/(q^{*T}Aq^*)^{1/2}\} - \{Aq^*q^{*T}A/(q^{*T}Aq^*)^{3/2}\}$
$= [f^*(q^*)]^{-1}\{f^*(q^*)\nabla^2 f^*(q^*) + \nabla f^*(q^*)\nabla f^*(q^*)^T\} - \{Aq^*q^{*T}A/(q^{*T}Aq^*)^{3/2}\}$ using (A24) and (A25)
$= \nabla^2 f^*(q^*) + [f^*(q^*)]^{-1}[\nabla f^*(q^*)\nabla f^*(q^*)^T] - [f^*(q^*)]^{-1}[\nabla f^*(q^*)\nabla f^*(q^*)^T]$ using (A25) and (A26)
$= \nabla^2 f^*(q^*)$.

Thus equations (A23) are satisfied and $f(q) \equiv (q^TAq)^{1/2}$ is a flexible functional form.[164] Note that this functional form has the minimum number of free parameters (which is N(N+1)/2) that is required to satisfy the $1 + N + N^2$ equations (A21)-(A23). In the literature on flexible functional forms, such a function is called a *parsimonious flexible functional form*.

Proof of Proposition 5: Let $c(p) = (p^TBp)^{1/2}$ where $B = B^T$ and B has one positive eigenvalue with a strictly positive eigenvector and the remaining N − 1 eigenvalues of B are negative. The function c(p) is well defined over the set $S^* \equiv \{p: p \geq 0_N;\ Bp \geq 0_N;\ p^TBp > 0\}$. Under our eigenvalue assumptions, a result in Diewert and Hill (2010) will imply that c(p) is a concave function over the set S^*. It will also be increasing, linearly homogeneous and positive over S^*. Let $q^* >> 0_N$ and suppose also that $B^{-1}q^* >> 0_N$. Let f(q) be the utility function that is dual to c(p). Then $f(q^*)$ can be defined by the following modification of definition (50) in the main text:[165]

(A28) $f(q^*) = 1/\max_p \{c(p) : p\cdot q^* = 1;\ p \in S^*\}$.

Consider the maximization problem on the right hand side of (A28). If we temporarily drop the constraints $p \in S^*$, then the resulting problem is:

(A29) $\max_p \{(p^TBp)^{1/2} : p\cdot q^* = 1\}$.

The first order necessary conditions for an interior maximum for the constrained maximization problem (A29) are equivalent to the following conditions:

(A30) $Bp^* = \lambda^* q^*$;
(A31) $p^*\cdot q^* = 1$.

Since B^{-1} exists under our assumptions, $p^* = \lambda^* B^{-1}q^*$. Substitute this equation into (A31) and solve the resulting equation, $\lambda^* q^{*T}B^{-1}q^* = 0$ for $\lambda^* = 1/q^{*T}B^{-1}q^*$, which is positive since q^* and $B^{-1}q^*$ are strictly positive vectors by our assumptions. Thus $p^* = \lambda^* B^{-1}q^* = B^{-1}q^*/q^{*T}B^{-1}q^*$. It can be seen that this p^* is the global maximizer for the problem defined by (A29) under our regularity conditions on B. Thus we have

(A32) $\max_p \{(p^TBp)^{1/2} : p\cdot q^* = 1\} = (p^{*T}Bp^*)^{1/2} = (q^{*T}B^{-1}q^*)^{-1/2}$.

Since $B^{-1}q^* >> 0_N$ and $\lambda^* > 0$, $p^* = \lambda^* B^{-1}q^* >> 0_N$. From (A30), $Bp^* = \lambda^* q^* >> 0_N$. Thus p^* also solves the maximization problem on the right hand side of (A28) since p^* belongs to S^*. Thus we have[166]

$$\begin{aligned}\text{(A33) } f(q^*) &= 1/\max_p \{c(p) : p\cdot q^* = 1;\ p \in S^*\} \\ &= 1/(q^{*T}B^{-1}q^*)^{-1/2} \\ &= (q^{*T}B^{-1}q^*)^{1/2}.\end{aligned}$$

Proof of Proposition 6: Let $A \equiv [a_{ik}]$ be an N by N symmetric matrix with element a_{ik} in row i and column k so that $A = A^T$. Suppose $r \neq 0$, $q >> 0_N$ and define f(q) as follows:[167]

164 The above proof of flexibility is an adaptation of the proof of flexibility for this functional form in Diewert (1974b; 125). See also Diewert (1976; 140-142) for an alternative proof.

165 See Blackorby and Diewert (1979) for additional material on local duality theorems.

166 This seems to be the model considered by Konüs and Byushgens (1926; 171).

167 In order to ensure that f(q) is well defined for any $r \neq 0$, we require that $\sum_{i=1}^N \sum_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2} > 0$. If each $a_{ik} \geq 0$ and at least one $a_{ik} > 0$, then for $q >> 0_N$, $\sum_{i=1}^N \sum_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}$ will be greater than 0. However, as will be seen later in the proof, $\sum_{i=1}^N \sum_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}$ can be positive without assuming that each $a_{ik} \geq 0$.

(A34) $f(q) = f(q_1,\ldots,q_N) \equiv [\sum_{i=1}^N \sum_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}]^{1/r}$.

Denote the nth first order partial derivative of f(q) as $f_n(q) \equiv \partial f(q)/\partial q_n$ for n = 1,...,N. Assuming that $\sum_{i=1}^N\sum_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}$ is positive, $f_n(q)$ is equal to the following expression:

$$\text{(A35)}\quad f_n(q) = (1/r)[\sum_{i=1}^N \sum_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}]^{(1/r)-1} r[\sum_{k=1}^N a_{nk}q_n^{(r/2)-1}q_k^{r/2}] \qquad n = 1,\ldots,N$$
$$= [f(q)]^{1-r} [\sum_{k=1}^N a_{nk}q_n^{(r/2)-1}q_k^{r/2}].$$

Denote the second order partial derivative of f(q) with respect to q_n and q_m as $f_{nm}(q) \equiv \partial^2 f(q)/\partial q_n\partial q_m$ for n = 1,...,N and m = 1,...,N. For n < m, $f_{nm}(q)$ is equal to the following expression:

$$\text{(A36)}\quad f_{nm}(q) = [(1/r)-1][\sum_{i=1}^N \sum_{k=1}^N a_{ik}q_i^{r/2}q_k^{r/2}]^{(1/r)-2}r[\sum_{k=1}^N a_{nk}q_n^{(r/2)-1}q_k^{r/2}][\sum_{k=1}^N a_{mk}q_m^{(r/2)-1}q_k^{r/2}]$$
$$+ [f(q)]^{1-r} [r/2][a_{nm}q_n^{(r/2)-1}q_m^{(r/2)-1}] \qquad 1 \le n < m \le N$$
$$= (1-r)[f(q)]^{-1}f_n(q)f_m(q) + (r/2)a_{nm}q_n^{(r/2)-1}q_m^{(r/2)-1}.$$

As was seen in the proof of Proposition 4, because the f(q) defined by (A34) is linearly homogeneous, we need only to choose the a_{nm} to satisfy equations (A22) and the upper triangle of equations (A23) in order to prove that f(q) is a flexible functional form; i.e., for $q^* >> 0_N$. we need the a_{nm} to satisfy the following equations:[168]

$$\text{(A37)}\quad f_n(q^*) = f^*_n(q^*)\ ; \qquad n = 1,\ldots,N;$$
$$\text{(A38)}\quad f_{nm}(q^*) = f^*_{nm}(q^*)\ ; \qquad 1 \le n < m \le N.$$

Temporarily assume that we have found a set of a_{nm} so that equations (A37) and the following equation are satisfied:

$$\text{(A39)}\quad f(q^*) = f^*(q^*).$$

Evaluate the second order partial derivatives of f(q) at q^* using equations (A36) and set the nmth partial derivative of f(q) equal to the corresponding nmth partial derivative of $f^*(q^*)$. Using equations (A37) and (A39), these equations become the following equations:

$$\text{(A40)}\quad f^*_{nm}(q^*) = (1-r)[f^*(q^*)]^{-1}f^*_n(q^*)f^*_m(q^*) + (r/2)a_{nm}(q_n^*)^{(r/2)-1}(q_m^*)^{(r/2)-1}\ ; \qquad 1 \le n < m \le N.$$

The N(N−1)/2 equations (A40) determine a_{nm} for $1 \le n < m \le N$. Define $a_{mn} = a_{nm}$ for $1 \le n < m \le N$. Thus all of the a_{nm} are determined except for the a_{nn} for n = 1,...,N. Again, assume that $f(q^*) = f^*(q^*)$ and evaluate equations (A35) at $q = q^*$ and set the resulting first order partial derivatives of $f(q^*)$ equal to the corresponding given first order partial derivatives of $f^*(q^*)$. We obtain the following N equations:

$$\text{(A41)}\quad f^*_n(q^*) = [f^*(q^*)]^{1-r} [\sum_{k=1}^N a_{nk} (q_n^*)^{(r/2)-1}(q_k^*)^{r/2}]; \qquad n = 1,\ldots,N.$$

The N equations (A41) determine the a_{nn} for n = 1,...,N. It turns out that this solution for the a_{nm} enables f(q) defined by (A34) to satisfy all of the equations (A21)-(A23). Thus f(q) is a flexible functional form.[169] Note that the resulting f(q) will be positive and the first order derivatives of f(q) will be positive in a

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[168] We assume that the exogenous $f^*(q^*)$ and $\nabla f^*(q^*)$ satisfy the positivity restrictions $\nabla f^*(q^*) >> 0_N$ and hence $f^*(q^*) = q^{*T}\nabla f^*(q^*) > 0$.
[169] This method of proof is due to Diewert (1976; 140-141).

neighbourhood around q^* due to the continuity of the function f(q) defined by (A34). Finally, note that if r = 2, then $f(q) = (q^TAq)^{1/2}$ and so the proof of Proposition 6 provides an alternative proof for Proposition 4.

Proof of Proposition 7: Let $r \neq 0$ and define $f^r(q)$ by (53). The assumption that $q^t >> 0_N$ solves the constrained utility maximization problem $\max_q \{f^r(q) : p^t \cdot q \leq e^t ; q \in S\}$ where S is an open convex set means that q^t is not on the boundary of S and hence q^t will satisfy the first order conditions for the problem $\max_q \{f^r(q) : p^t \cdot q \leq e^t\}$ for t = 0,1. The first order necessary conditions for these problems (which are equivalent to the Wold's Identity conditions (16)) are the following conditions:

(A42) $p_n^t/e^t = p_n^t/p^t \cdot q^t = f^r{}_n(q^t)/f^r(q^t) = [f^r(q^t)]^{-r} [\sum_{k=1}^N a_{nk} (q_n^t)^{(r/2)-1}(q_k^t)^{r/2}]$; n = 1,...,N; t = 0,1

where we have used equations (A35) to establish the last equation in (A42). Using equations (A42), we obtain the following expressions for the shares s_n^t:

(A43) $s_n^t = p_n^t q_n^t/e^t = [f^r(q^t)]^{-r} [\sum_{k=1}^N a_{nk} (q_n^t)^{(r/2)}(q_k^t)^{r/2}]$; n = 1,...,N; t = 0,1.

Now substitute the s_n^t defined by (A43) into (54), the definition of $Q^r(p^0,p^1,q^0,q^1)$:

(A44) $Q^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{n=1}^N s_n^0(q_n^1/q_n^0)^{r/2}\}^{1/r}\{\sum_{n=1}^N s_n^1(q_i^1/q_n^0)^{-r/2}\}^{-1/r}$
$= [f^r(q^0)]^{-1}\{\sum_{n=1}^N \sum_{k=1}^N a_{nk} (q_n^0)^{(r/2)}(q_k^1)^{r/2}\}^{1/r} [f^r(q^1)]\{\sum_{n=1}^N \sum_{k=1}^N a_{nk} (q_n^0)^{(r/2)}(q_k^1)^{r/2}\}^{-1/r}$
$= f^r(q^1)/f^r(q^0)$.

Proof of Proposition 9: Consider the following constrained maximization problem:

(A45) $\max_p \{c^r(p); e^t = p \cdot q^t; p \in S^*\}$.

Since S^* is an open set, the first order necessary conditions for $p^* \in S^*$ to solve (A45) is that there exist λ^* such that the following equations are satisfied:

(A46) $\nabla c^r(p^*) = \lambda^* q^t$;
(A47) $p^* \cdot q^t = e^t$.

Premultiply both sides of (A46) by p^{*T} and we obtain the equation $\lambda^* p^{*T} q^t = p^{*T}\nabla c^r(p^*) = c^r(p^*)$ where the last equality follows from the linear homogeneity of $c^r(p)$. Thus $\lambda^* = c^r(p^*)/p^* \cdot q^t = c^r(p^*)/e^t$ where the last equation follows using (A47). Substituting $\lambda^* = c^r(p^*)/e^t$ into (A47) gives us the equation $\nabla c^r(p^*) = [c^r(p^*)/e^t]q^t$, which in turn can be written as follows:

(A48) $q^t \equiv e^t \nabla c^r(p^*)/c^r(p^*)$.

But from (64), we have $q^t \equiv e^t \nabla c^r(p^t)/c^r(p^t)$. Thus if we set p^* to p^t, equation (A48) will be satisfied. We also have $p^t \cdot q^t = e^t p^t \cdot \nabla c^r(p^t)/c^r(p^t) = e^t c^r(p^t)/c^r(p^t) = e^t$, so equation (A47) is satisfied if $p^* = p^t$. If we define $\lambda^* = c^r(p^t)/e^t$, then (A46) with $p^* = p^t$ becomes $\nabla c^r(p^*) = [c^r(p^t)/e^t]q^t$ which is (A48) and so $p^* \equiv p^t$ and $\lambda^* = c^r(p^t)/e^t$ satisfy equations (A46) and (A47). Thus p^t is a candidate to solve (A45) since it satisfies the first order necessary conditions for an interior solution for (A45).

Next, we show that p^t actually solves the constrained maximization problem defined by (A45). Define $\lambda^* \equiv c^r(p^t)/e^t$ and define the function g(p) as follows:

(A49) $g(p) \equiv c^r(p) + \lambda^*[e^t - p \cdot q^t]$.

Since $c^r(p)$ is concave over S^* by assumption and the function $\lambda^*[e^t - p\cdot q^t]$ is linear in p (and hence concave everywhere), g(p) is a differentiable concave function over S^*. Hence the first order Taylor series approximation to g(p) around the point p^t will be coincident with or lie above the function; i.e., we have the following inequality:

(A50) $g(p) \leq g(p^t) + \nabla g(p^t)(p - p^t)$ for all $p\in S^*$.

Substituting definition (A49) into (A50) and noting that $\nabla g(p^t) = \nabla c^r(p^t) - \lambda^* q^t = 0_N$ (using (A46) with $p^* = p^t$ and $\lambda^* = c^r(p^t)/e^t$), we find that (A50) becomes:

(A51) $c^r(p) + \lambda^*[e^t - p\cdot q^t] \leq c^r(p^t) + \lambda^*[e^t - p^t\cdot q^t]$; $\qquad p\in S^*$.

But the above inequality does not take into account the constraint $e^t = p\cdot q^t$. If we impose this additional constraint on p, the inequality (A51) becomes the following inequality:

(A52) $c^r(p) \leq c^r(p^t)$; $\qquad p\in S^*$ and $p\cdot q^t = e^t$.

Thus p^t solves the constrained maximization problem (A45) and we have:

(A53) $c^r(p^t) = \max_p \{c^r(p); e^t = p\cdot q^t; p\in S^*\}$.

Now use definition (63) with $e = e^t$ to define $f^{r*}(q^t)$ and we obtain the following result using (A53):

(A54) $f^{r*}(q^t) = e^t/\max_p \{c^r(p); e^t = p\cdot q^t; p\in S^*\}$
$= e^t/c^r(p^t)$.

(A54) establishes (65). Now consider the following local utility maximization problem:

(A55) $\max_q \{f^{r*}(q) : p^t\cdot q = e^t; q\in S\}$

where $f^{r*}(q)$ is defined as

(A56) $f^{r*}(q) = e^t/\max_p \{c^r(p); e^t = p\cdot q; p\in S^*\}$.

Let $q\in S$ and we suppose that q also satisfies the consumer's period t budget constraint, $p^t\cdot q = e^t$. Let p^* be a solution to $\max_p \{c^r(p); e^t = p\cdot q; p\in S^*\}$. Thus we have:

(A57) $c^r(p^*) = \max_p \{c^r(p); e^t = p\cdot q; p\in S^*\}$
$\geq c^r(p^t)$

since $p^t\cdot q = e^t$ and hence p^t is a feasible solution for the constrained maximization problem. Using (A54), (A56) and (A57), we have $f^{r*}(q^t) \geq f^{r*}(q)$ for all q belonging to S such that $p^t\cdot q = e^t$. Thus q^t solves the local utility maximization problem (A55).

Proof of Proposition 10: The proof of the previous Proposition showed that q^t solves the local utility maximization problem, $\max_q \{f^{r*}(q) ; p^t\cdot q = e^t; q\in S\}$, for t = 0,1.

Conditions (68) (Shephard's Lemma) and definition (59) imply that the following equations will hold:

(A58) $q_n^t/p^t \cdot q^t = c^r{}_n(p^t)/c^r(p^t) = [c^r(p^t)]^{-r} [\sum_{k=1}^N b_{nk} (p_n^t)^{(r/2)-1}(p_k^t)^{r/2}]$; n = 1,...,N; t = 0,1.

Using equations (A58), we obtain the following expressions for the shares s_n^t:

(A59) $s_n^t = p_n^t q_n^t/p^t \cdot q^t = [c^r(p^t)]^{-r} [\sum_{k=1}^N b_{nk} (p_n^t)^{(r/2)}(p_k^t)^{r/2}]$; n = 1,...,N; t = 0,1.

Now substitute the s_n^t defined by (A59) into (69), the definition of $P^r(p^0,p^1,q^0,q^1)$:

(A60) $P^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{n=1}^N s_n^0(p_n^1/p_n^0)^{r/2}\}^{1/r}\{\sum_{n=1}^N s_n^1(p_n^1/p_n^0)^{-r/2}\}^{-1/r}$
$= [c^r(p^0)]^{-1}\{\sum_{n=1}^N \sum_{k=1}^N b_{nk} (p_n^0)^{(r/2)}(p_k^1)^{r/2}\}^{1/r} [c^r(p^1)]\{\sum_{n=1}^N \sum_{k=1}^N b_{nk} (p_n^0)^{(r/2)}(p_k^1)^{r/2}\}^{-1/r}$
$= c^r(p^1)/c^r(p^0)$.

Proof of Proposition 11: Let $p \equiv [p_1,...,p_N] >> 0_N$. Ignoring the constraints $q \geq 0_N$, the first order necessary (and sufficient) conditions for $q^* >> 0_N$ and $\lambda^* > 0$ to solve the unit cost minimization problem defined by (96) are:

(A61) $p_n = \lambda^*\partial f(q^*)/\partial q_n = \lambda^*\alpha_n f(q^*)/q_n^*$; n = 1,...,N;
(A62) $1 = f(q^*)$.

Substituting (A62) into (A61), we get the N equations $p_n = \lambda^*\alpha_n/q_n^*$ for n = 1,...,N which can be rearranged to give us the following equations:

(A63) $q_n^* = \lambda^*\alpha_n/p_n$; n = 1,...,N.

Now substitute equations (A63) into equation (A62) and using definition (94) for f, we get the following single equation involving λ^*:

(A64) $1 = \alpha_0\Pi_{n=1}^N [\lambda^*\alpha_n/p_n]^{\alpha_n}$
$= \lambda^*\alpha_0\Pi_{n=1}^N [\alpha_n]^{\alpha_n} \Pi_{n=1}^N [1/p_n]^{\alpha_n}$.

Therefore, we have the following expression for λ^*:

(A65) $\lambda^* = [\alpha_0\Pi_{n=1}^N [\alpha_n]^{\alpha_n}]^{-1} \Pi_{n=1}^N [p_n]^{\alpha_n} = \kappa \Pi_{n=1}^N p_n^{\alpha_n} > 0$

where the constant κ is defined as $\kappa \equiv [\alpha_0\Pi_{n=1}^N [\alpha_n]^{\alpha_n}]^{-1}$. Substitute λ^* defined by (A65) back into equations (A63) and we obtain the q^* solution to the cost minimization problem defined by (96):

(A66) $q_n^* = \kappa [\Pi_{n=1}^N p_n^{\alpha_n}]\alpha_n/p_n$; n = 1,...,N.

Thus the optimized objective function for (96) is equal to the following expression:

(A67) $c(p) = \Sigma_{n=1}^N p_n q_n^*$
$= \Sigma_{n=1}^N p_n\kappa[\Pi_{n=1}^N p_n^{\alpha_n}]\alpha_n/p_n$ using (A66)
$= \kappa[\Pi_{n=1}^N p_n^{\alpha_n}][\Sigma_{n=1}^N \alpha_n]$
$= \kappa \Pi_{n=1}^N p_n^{\alpha_n}$ using (95).

Thus c(p) is defined by (97).

Proof of Proposition 12: If r = 0, then the CES preferences collapse to Cobb Douglas preferences, which will imply that $s^0 = s^1$, and thus the Sato Vartia index collapses to the Konüs Byushgens index which was studied in section 9. Hence we assume $r \neq 0$ and define the consumer's unit cost function by (108). Let $p^0 >> 0_N$, $p^1 >> 0_N$ and define q^0 and q^1 using Shephard's Lemma, equations (109). We assume that $q^0 >> 0_N$ and $q^1 >> 0_N$ and hence the share vectors s^0 and s^1 defined by equations (110) also satisfy $s^0 >> 0_N$ and $s^1 >> 0_N$. Given these positivity conditions, equations (110) can be rewritten as follows:

$$(A68)\ \sum_{n=1}^{N} \alpha_n(p_n^t)^r = \alpha_i(p_i^t)^r/s_i^t\ ; \qquad t = 0,1;\ i = 1,\ldots,N.$$

Take the logarithm of both sides of (A68) and obtain the following equations:

$$(A69)\ \ln[\sum_{n=1}^{N} \alpha_n(p_n^t)^r] = \ln\alpha_i + r\ln p_i^t - \ln s_i^t\ ; \qquad t = 0,1;\ i = 1,\ldots,N.$$

The consumer's true cost of living index is $c(p^1)/c(p^0) = \alpha_0 [\sum_{n=1}^{N} \alpha_n (p_n^1)^r]^{1/r}/\alpha_0 [\sum_{n=1}^{N} \alpha_n (p_n^0)^r]^{1/r}$, which equals $[\sum_{n=1}^{N} \alpha_n (p_n^1)^r]^{1/r}/[\sum_{n=1}^{N} \alpha_n (p_n^0)^r]^{1/r}$. Raising both sides of this equation to the power r and taking the logarithm of the resulting equation leads to the following equation:

$$(A70)\ \ln\{[c(p^1)/c(p^0)]^r\} = \ln[\sum_{n=1}^{N} \alpha_n (p_n^1)^r] - \ln[\sum_{n=1}^{N} \alpha_n (p_n^0)^r].$$

From (118), the logarithm of $P_{SV}(p^0,p^1,q^0,q^1)^r$ is defined as follows:

$$(A71)\ \ln\{P_{SV}(p^0,p^1,q^0,q^1)^r\} = r\Sigma_{n=1}^{N} w_i^*[\ln p_i^1 - \ln p_i^0]/\Sigma_{n=1}^{N} w_i^*$$

where $w_i^* \equiv [s_i^1 - s_i^0]/[\ln s_i^1 - \ln s_i^0]$ if $s_i^1 \neq s_i^0$ and $w_i^* \equiv s_i^0$ if $s_i^1 = s_i^0$. Now equate (A71) to (A70) and after a bit of rearrangement, we obtain the following equation:

$$\begin{aligned}(A72)\ r\Sigma_{n=1}^{N} w_i^*[\ln p_i^1 - \ln p_i^0] &= \Sigma_{n=1}^{N} w_i^*\ln[\Sigma_{n=1}^{N} \alpha_n (p_n^1)^r] - \Sigma_{n=1}^{N} w_i^*\ln[\Sigma_{n=1}^{N} \alpha_n (p_n^0)^r] \\ &= \Sigma_{n=1}^{N} w_i^*[\ln\alpha_i + r\ln p_i^1 - \ln s_i^1] - \Sigma_{n=1}^{N} w_i^*[\ln\alpha_i + r\ln p_i^0 - \ln s_i^0] \qquad \text{using (A69)} \\ &= r\Sigma_{n=1}^{N} w_i^*[\ln p_i^1 - \ln p_i^0] - \Sigma_{n=1}^{N} w_i^*[\ln s_i^1 - \ln s_i^0] \\ &= r\Sigma_{n=1}^{N} w_i^*[\ln p_i^1 - \ln p_i^0] - \Sigma_{n=1}^{N} [s_i^1 - s_i^0] \\ &= r\Sigma_{n=1}^{N} w_i^*[\ln p_i^1 - \ln p_i^0] \qquad \text{since } \Sigma_{n=1}^{N} s_i^1 = \Sigma_{n=1}^{N} s_i^0 = 1.\end{aligned}$$

The last equality follows because if $s_i^0 \neq s_i^1$, then $w_i^*[\ln s_i^1 - \ln s_i^0] = \{[s_i^1 - s_i^0]/[\ln s_i^1 - \ln s_i^0]\}[\ln s_i^1 - \ln s_i^0] = s_i^1 - s_i^0$. If $s_i^1 = s_i^0$, then $w_i^* = s_i^0$ but $\ln s_i^1 - \ln s_i^0 = 0$ so $w_i^*[\ln s_i^1 - \ln s_i^0] = 0 = s_i^1 - s_i^0$. Thus we have shown that $\ln\{[c(p^1)/c(p^0)]^r\} = \ln\{P_{SV}(p^0,p^1,q^0,q^1)^r\}$ and thus that $c(p^1)/c(p^0) = P_{SV}(p^0,p^1,q^0,q^1)$.

We note that the Sato Vartia quantity index $Q_{SV}(p^0,p^1,q^0,q^1)$ can be defined by interchanging prices and quantities in the definition of the Sato Vartia price index; i.e., define $Q_{SV}(p^0,p^1,q^0,q^1) \equiv P_{SV}(q^0,q^1,p^0,p^1)$. The above proof can be adapted to show that $f(q^1)/f(q^0) = Q_{SV}(p^0,p^1,q^0,q^1)$ where f(q) is defined by (134). In order to prove this result, we require that $s < 1$ and $s \neq 0$.

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