Consumer Price Index Theory: CHAPTER 2: BASIC INDEX NUMBER THEORY[1]

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1. Introduction

"The answer to the question what is the Mean of a given set of magnitudes cannot in general be found, unless there is given also the object for the sake of which a mean value is required. There are as many kinds of average as there are purposes; and we may almost say in the matter of prices as many purposes as writers. Hence much vain controversy between persons who are literally at cross purposes." Francis Ysidro Edgeworth (1888; 347).

The number of physically distinct goods and unique types of services that consumers can purchase is in the millions. On the business or production side of the economy, there are even more commodities that are actively traded. This is because firms not only produce commodities for final consumption, but they also produce exports and intermediate commodities that are demanded by other producers. Firms collectively also use millions of imported goods and services, thousands of different types of labour services and hundreds of thousands of specific types of capital. If we further distinguish physical commodities by their geographic location or by the season or time of day that they are produced or consumed, then there are *billions* of commodities that are traded within each year in any advanced economy. For many purposes, it is necessary to *summarize* this vast amount of price and quantity information into a much smaller set of numbers. The question that this chapter addresses is: *how exactly should the microeconomic information involving possibly millions of prices and quantities be aggregated into a smaller number of price and quantity variables?* This is the basic *index number problem.*

It is possible to pose the index number problem in the context of microeconomic theory; i.e., given that we wish to implement some economic model based on producer or consumer theory, what is the "best" method for constructing a set of aggregates for the model? However, when constructing aggregate prices or quantities, other points of view (that do not rely on economics) are possible. Some of these alternative points of view will be considered in this chapter and the following two chapters. Economic approaches to index number theory will be pursued in chapters 5 and 8.

The index number problem can be framed as the problem of decomposing the value of a well defined set of transactions in a period of time into an aggregate price term times an aggregate quantity term. This is the price and quantity *levels approach* to index number theory. This approach will be pursued in subsequent chapters but there are some difficulties with the use of this approach and so in section 2 below, the problem of decomposing a *value ratio* pertaining to two periods of time into a component that measures the overall *change in prices* between the two periods (this is the *price index*) times a term that measures the overall *change in quantities* between the two periods (this is the *quantity index*) is considered. Thus instead of attempting to construct aggregate price and quantity levels for each period, a *ratio approach* is adopted. The simplest price index is *a fixed basket type index*; i.e., fixed amounts of the N quantities in the value aggregate are chosen and then this fixed basket of quantities at the prices of period 0 and at the prices of period 1 are calculated and the fixed basket price index is simply the ratio of these two values where the prices vary but the quantities are held fixed. Two natural choices for the fixed basket are the quantities transacted in the base period, period 0, or the quantities transacted in the current period, period 1. These two choices lead to the Laspeyres (1871) and Paasche (1874) price indexes respectively.

Unfortunately, the Paasche and Laspeyres measures of aggregate price change can differ, sometimes substantially. Thus in section 4, taking an average of these two indexes to come up with a single measure of price change is considered. In this section, it is argued that the "best" average to take is the geometric mean, which is Irving Fisher's (1922) ideal price index. In section 5, instead of averaging the Paasche and Laspeyres measures of price change, taking an average of the two baskets is considered. This fixed basket approach to index number theory leads to a price index advocated by Walsh (1901) (1921a). However, other fixed basket approaches are also possible. Instead of choosing the basket of period 0 or 1 (or an average of these two baskets), it is possible to choose a basket that pertains to an entirely different period, say period b. In fact, it is typical statistical agency practice to pick a basket that pertains to an entire year (or even two years) of transactions in a year prior to period 0, which is usually a month. Indexes of this type, where the weight reference period differs from the price reference period, were originally proposed by Joseph Lowe (1823) and in section 6, indexes of this type will be studied.

In section 7, another approach to the use of annual weights with monthly prices will be discussed. This approach is due to Young (1812).

In section 8, the advantages and disadvantages of using a *fixed base* period in the bilateral index number comparison are considered versus always comparing the current period with the previous period, which is called the *chain system*. In the chain system, a *link* is an index number comparison of one period with the previous period. These links are multiplied together in order to make comparisons over many periods. Fixed base or direct indexes will be studied and compared to their chained counterparts in more detail in Chapter 7.[2]

Practical Consumer Price Indexes are usually constructed in two or more stages of aggregation. For example, at the first stage of aggregation, subindexes for various consumption categories, like food, clothing, transportation etc. are constructed, and then in the second stage of aggregation, an overall CPI is constructed. Does a CPI constructed in two stages coincide with a CPI constructed in a single stage? In section 9, this question is addressed for some of the more commonly used index number formulae. In Appendix 5, the consistency in aggregation of various formulae over three (or more) stages of aggregation will be discussed.

The Appendices 1-4 look at the numerical relationships between the Laspeyres, Paasche, Lowe and Young indexes.

2. The Decomposition of Value Aggregates and the Product Test

A *price index* is a measure or function that summarizes the *change* in the prices of many commodities from one situation 0 (a time period or place) to another situation 1. A *price level* can be thought of as an average of the prices pertaining to a single period. More specifically, for most practical purposes, a price index can be regarded as a weighted average of the relative prices of the commodities under consideration in the two situations. To determine a price index, it is necessary to know the following:

- which commodities or items to include in the index;
- how to determine the item prices;
- which transactions that involve these items to include in the index;
- how to determine the weights and from which sources these weights should be drawn;
- what formula or type of mean should be used to average the selected item relative prices.

All of the above price index definition questions except the last can be answered by appealing to the definition of the *value aggregate* to which the price index refers. A *value aggregate* V for a given collection of N items and transactions is computed as:[3]

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[2] In order to compare the advantages and disadvantages of fixed base versus chained indexes, it is useful to be able to draw on other approaches to index number theory, which will be studied in Chapters 3-5.

[3] Notation: The sum of terms, $\Sigma_{n=1}^{N} p_n q_n$, will at times be written as $\Sigma_{i=1}^{N} p_i q_i$ or as $\Sigma_{k=1}^{N} p_k q_k$. In subsequent chapters, $\Sigma_{n=1}^{N} p_n q_n$ will sometimes be written as $p \cdot q$, which is called the inner product of the vectors p and q defined as $p \equiv [p_1,...,p_N]$ and $q \equiv [q_1,...,q_N]$.

(1) $V = \Sigma_{n=1}^{N} p_n q_n$

where p_n represents the price of the nth item in national currency units, q_n represents the corresponding quantity transacted in the time period under consideration and the subscript n identifies the nth elementary item in the group of N items that make up the chosen value aggregate V. Included in this definition of a value aggregate is the specification of the group of included commodities[4] (which items to include) and of the economic agents engaging in transactions involving those commodities (which transactions to include), as well as the valuation and time of recording principles motivating the behavior of the economic agents undertaking the transactions (determination of prices). The included elementary items, their valuation (the p_n), the eligibility of the transactions and the item weights (the q_n) are all within the domain of definition of the value aggregate. The precise determination of the p_n and q_n can be a tricky business.[5]

The value aggregate V defined by (1) above referred to a certain set of transactions pertaining to a single (unspecified) time period. Now the same value aggregate for two places or time periods, periods 0 and 1, is considered. For the sake of definiteness, period 0 is called *the base period* and period 1 is called *the current period* and it is assumed that observations on the base period price and quantity vectors, $p^0 \equiv [p_1^0,\ldots,p_N^0]$ and $q^0 \equiv [q_1^0,\ldots,q_N^0]$ respectively, have been collected.[6] The value aggregates in the two periods are defined in the obvious way as:

(2) $V^0 \equiv \Sigma_{n=1}^{N} p_n^0 q_n^0$; $V^1 \equiv \Sigma_{n=1}^{N} p_n^1 q_n^1$.

In the previous paragraph, a price index was defined as a function or measure that summarizes the *change* in the prices of the N commodities in the value aggregate from situation 0 to situation 1. In this paragraph, a *price index* $P(p^0,p^1,q^0,q^1)$ along with the corresponding *quantity index* (or *volume index*) $Q(p^0,p^1,q^0,q^1)$ is defined to be two functions of the 4N variables p^0,p^1,q^0,q^1 (these variables describe the prices and quantities pertaining to the value aggregate for periods 0 and 1) where these two functions satisfy the following equation:[7]

(3) $V^1/V^0 = P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1)$.

If there is only one item in the value aggregate, then the price index P should collapse down to the single price ratio, p_1^1/p_1^0 and the quantity index Q should collapse down to the single quantity ratio, q_1^1/q_1^0. In the case of many items, the price index P is to be interpreted as some sort of weighted average of the individual price ratios, $p_1^1/p_1^0,\ldots, p_n^1/p_n^0$.

Thus the first approach to index number theory can be regarded as the problem of *decomposing* the change in a value aggregate, V^1/V^0, into the product of a part that is due to *price change*, $P(p^0,p^1,q^0,q^1)$, and a part that is due to *quantity change*, $Q(p^0,p^1,q^0,q^1)$. This

[4] The terms "commodity", "item" and "product" will be used interchangeably in what follows. Different statistical agencies may have more specific definitions for these terms.
[5] Ralph Turvey has noted that some values may be difficult to decompose into unambiguous price and quantity components. Some examples of difficult to decompose values are bank charges, gambling expenditures and life insurance payments. The problems associated with precisely defining p_n^t and q_n^t are discussed in some detail in Eurostat (2018). There is a great deal of valuable information in this Manual.
[6] Note that it is assumed that there are no new or disappearing commodities in the value aggregates. Approaches to the "new goods problem" and the problem of accounting for quality change are discussed in Chapter 8.
[7] The first person to suggest that the price and quantity indexes should be jointly determined in order to satisfy equation (3) was Fisher (1911; 418). Frisch (1930; 399) called (3) the *product test*.

approach to the determination of the price index is the approach that is taken in the national accounts, where a price index is used to *deflate* a value ratio in order to obtain an estimate of quantity change. Thus, in this approach to index number theory, the primary use for the price index is as a *deflator*. Note that once the functional form for the price index $P(p^0,p^1,q^0,q^1)$ is known, then the corresponding quantity or volume index $Q(p^0,p^1,q^0,q^1)$ is completely determined by P; i.e., rearranging (3):

$$(4)\ Q(p^0,p^1,q^0,q^1) = [V^1/V^0]/P(p^0,p^1,q^0,q^1).$$

Conversely, if the functional form for the quantity index $Q(p^0,p^1,q^0,q^1)$ is known, then the corresponding price index function $P(p^0,p^1,q^0,q^1)$ is completely determined by the quantity index function $Q(p^0,p^1,q^0,q^1)$. Thus using this deflation approach to index number theory, separate theories for the determination of the price and quantity indexes are not required: if either P or Q is determined, then the other function is implicitly determined by the product test (3).

In the next subsection, two concrete choices for the price index $P(p^0,p^1,q^0,q^1)$ are considered and the corresponding quantity indexes $Q(p^0,p^1,q^0,q^1)$ that result from using equation (4) are also calculated. These are the two choices used most frequently by national income accountants.

3. The Laspeyres and Paasche Indexes

One of the simplest approaches to the determination of the price index formula was described in great detail by Lowe (1823). His approach to measuring the price change between periods 0 and 1 was to specify an approximate *representative commodity basket*[8], which is a quantity vector $q \equiv [q_1,\ldots,q_N]$ that is representative of purchases made during the two periods under consideration, and then calculate the level of prices in period 1 relative to period 0 as the ratio of the period 1 cost of the basket, $\sum_{n=1}^N p_n^1 q_n$, to the period 0 cost of the basket, $\sum_{n=1}^N p_n^0 q_n$. This *fixed basket approach* to the determination of the price index leaves open the question as to how exactly is the fixed basket vector q to be chosen?

As time passed, economists and price statisticians demanded a bit more precision with respect to the specification of the basket vector q. There are two natural choices for the reference basket: the base period 0 commodity vector q^0 or the current period 1 commodity vector q^1. These two choices lead to the *Laspeyres* (1871) price index[9] P_L defined by (5) and the *Paasche* (1874) price index[10] P_P defined by (6):[11]

$$(5)\ P_L(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^N p_n^1 q_n^0/\Sigma_{i=1}^N p_i^0 q_i^0\ ;$$

[8] Lowe (1823; Appendix page 95) suggested that the commodity basket vector q should be updated every five years. Lowe indexes will be studied in more detail in sections 5 and 6 below.

[9] This index was actually introduced and justified by Drobisch (1871a; 147) slightly earlier than Laspeyres. Laspeyres (1871; 305) in fact explicitly acknowledged that Drobisch showed him the way forward. However, the contributions of Drobisch have been forgotten for the most part by later writers because Drobisch aggressively pushed for the ratio of two unit values as being the "best" index number formula. While this formula has some excellent properties if all of the N commodities being compared have the same unit of measurement, the formula is useless when say, both goods and services are in the index basket. Unit value price indexes will be studied in more detail in subsequent chapters.

[10] Again Drobisch (1871b; 424) appears to have been the first to define explicitly and justify this formula. However, he rejected this formula in favor of his preferred formula, the ratio of unit values, and so again he did not get any credit for his early suggestion of the Paasche formula.

[11] Note that $P_L(p^0,p^1,q^0,q^1)$ does not actually depend on q^1 and $P_P(p^0,p^1,q^0,q^1)$ does not actually depend on q^0. However, it does no harm to include these vectors and the notation indicates that the reader is in the realm of *bilateral index number theory*; i.e., the prices and quantities for a value aggregate pertaining to *two periods* are being compared.

(6) $P_P(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^N p_n^1q_n^1/\Sigma_{i=1}^N p_i^0q_i^1$.

The above formulae can be rewritten in an alternative manner that is more useful for statistical agencies. Define the period t *expenditure share* on commodity n as follows:

(7) $s_n^t \equiv p_n^tq_n^t/\Sigma_{i=1}^N p_i^tq_i^t$ for $n = 1,...,N$ and $t = 0,1$.

Then the Laspeyres index (5) can be rewritten as follows:[12]

$$\begin{aligned}(8)\ P_L(p^0,p^1,q^0,q^1) &\equiv \Sigma_{n=1}^N p_n^1q_n^0/\Sigma_{i=1}^N p_i^0q_i^0 \\ &= \Sigma_{n=1}^N (p_n^1/p_n^0)p_n^0q_n^0/\Sigma_{i=1}^N p_i^0q_i^0 \\ &= \Sigma_{n=1}^N (p_n^1/p_n^0)s_n^0\end{aligned}$$

where the last equality follows using definitions (7).

Thus the Laspeyres price index P_L can be written as a base period expenditure share weighted arithmetic average of the N price ratios, p_n^1/p_n^0. The Laspeyres formula (until the recent past) has been widely used as the target index number concept for Consumer Price Indexes around the world. To implement it, a statistical agency needs only to collect information on expenditure shares s_n^0 for the index domain of definition for the base period 0 and then collect information on item *prices* alone on an ongoing basis. *Thus the Laspeyres CPI can be produced on a timely basis without having to know current period quantity information.*

The Paasche index can also be written in expenditure share and price ratio form as follows:[13]

$$\begin{aligned}(9)\ P_P(p^0,p^1,q^0,q^1) &\equiv \Sigma_{n=1}^N p_n^1q_n^1/\Sigma_{i=1}^N p_i^0q_i^1 \\ &= 1/[\Sigma_{i=1}^N p_i^0q_i^1/\Sigma_{n=1}^N p_n^1q_n^1] \\ &= 1/[\Sigma_{i=1}^N (p_i^0/p_i^1)p_i^1q_i^1/\Sigma_{n=1}^N p_n^1q_n^1] \\ &= 1/[\Sigma_{i=1}^N (p_i^0/p_i^1)s_i^1] \\ &= [\Sigma_{i=1}^N s_i^1(p_i^1/p_i^0)^{-1}]^{-1}\end{aligned}$$

where definitions (7) for $t = 1$ were used to derive the above equality. Thus the Paasche price index P_P can be written as a period 1 (or current period) expenditure share weighted *harmonic* average of the N item price ratios p_i^1/p_i^0.[14] Note that if the statistical agency lacks timely information on quantities, then the Paasche index cannot be produced in a timely manner.

The quantity index that corresponds to the Laspeyres price index using the product test (3) is the *Paasche quantity index*; i.e., if P in (4) is replaced by P_L defined by (5), then the following Paasche quantity index Q_P is obtained:

(10) $Q_P(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^N p_n^1q_n^1/\Sigma_{i=1}^N p_i^1q_i^0$.

Note that Q_P is the value of the period 1 quantity vector valued at the period 1 prices, $\sum_{n=1}^N p_n^1q_n^1$, divided by the (hypothetical) value of the period 0 quantity vector valued at the period

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[12] This method of rewriting the Laspeyres index (or any fixed basket index) as a share weighted arithmetic average of price ratios is due to Fisher (1897; 517) (1911; 397) (1922; 51) and Walsh (1901; 506) (1921a; 92). Note that this alternative formula for the Laspeyres price index requires that all base period prices be positive.
[13] This method of rewriting the Paasche index (or any fixed basket index) as a share weighted harmonic average of the price ratios is due to Walsh (1901; 511) (1921a; 93) and Fisher (1911; 397-398). Note that this alternative formula for the Paasche price index requires all current period prices to be positive.
[14] Note that the derivation in (9) shows how harmonic averages arise in index number theory in a very natural way.

1 prices, $\sum_{n=1}^N p_n^1 q_n^0$. Thus the period 0 and 1 quantity vectors are valued at the same set of prices, the current period prices, p^1.

The quantity index that corresponds to the Paasche price index using the product test (3) is the *Laspeyres quantity index*; i.e., if P in (4) is replaced by P_P defined by (6), then the following quantity index Q_L is obtained:

(11) $Q_L(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^N p_n^0 q_n^1 / \Sigma_{i=1}^N p_i^0 q_i^0$.

Note that Q_L is the (hypothetical) value of the period 1 quantity vector valued at the period 0 prices, $\sum_{n=1}^N p_n^0 q_n^1$, divided by the value of the period 0 quantity vector valued at the period 0 prices, $\sum_{n=1}^N p_n^0 q_n^0$. Thus the period 0 and 1 quantity vectors are valued at the same set of prices, the base period prices, p^0.

The problem with the Laspeyres and Paasche index number formulae is that they are equally plausible but in general, they will give *different* answers. For most purposes, it is not satisfactory for the statistical agency to provide *two* answers to the question:[15] what is the "best" overall summary measure of price change for the value aggregate over the two periods in question? Thus in the following section, it is considered how "best" averages of these two estimates of price change can be constructed. Before doing this, it is asked what is the "normal" relationship between the Paasche and Laspeyres indexes? Under "normal" economic conditions when the price ratios pertaining to the two situations under consideration are negatively correlated with the corresponding quantity ratios, it can be shown that the Laspeyres price index will be larger than the corresponding Paasche index.[16] In Appendix 1 below, a precise statement of this result is presented.[17] This divergence between P_L and P_P suggests that if a *single estimate* for the price change between the two periods is required, then some sort of evenly weighted average of the two indexes should be taken as the final estimate of price change between periods 0 and 1. As mentioned above, this strategy will be pursued in the following section. However, it should be kept in mind that usually statistical agencies will not have information on current expenditure weights and hence averages of Paasche and Laspeyres indexes can only be produced on a delayed basis (perhaps using national accounts information) or not at all.

4. The Fisher Index as an Average of the Paasche and Laspeyres Indexes

[15] In principle, instead of averaging the Paasche and Laspeyres indexes, the statistical agency could think of providing both (the Paasche index on a delayed basis). This suggestion would lead to a *matrix* of price comparisons between every pair of periods instead of a time series of comparisons. Walsh (1901; 425) noted this possibility: "In fact, if we use such direct comparisons at all, we ought to use all possible ones."

[16] Peter Hill (1993; 383) summarized this inequality as follows: "It can be shown that relationship (13) [i.e., that P_L is greater than P_P] holds whenever the price and quantity relatives (weighted by values) are negatively correlated. Such negative correlation is to be expected for price takers who react to changes in relative prices by substituting goods and services that have become relatively less expensive for those that have become relatively more expensive. In the vast majority of situations covered by index numbers, the price and quantity relatives turn out to be negatively correlated so that Laspeyres indexes tend systematically to record greater increases than Paasche with the gap between them tending to widen with time."

[17] There is another way to see why P_P will often be less than P_L. If the period 0 expenditure shares s_i^0 are exactly equal to the corresponding period 1 expenditure shares s_i^1, then by Schlömilch's (1858) Inequality (see Hardy, Littlewood and Pólya (1934; 26)), it can be shown that a weighted harmonic mean of N numbers is equal to or less than the corresponding arithmetic mean of the N numbers and the inequality is strict if the N numbers are not all equal. If expenditure shares are approximately constant across periods, then it follows that P_P will usually be less than P_L under these conditions.

As was mentioned in the previous paragraph, since the Paasche and Laspeyres price indexes are equally plausible but often give different estimates of the amount of aggregate price change between periods 0 and 1, it is useful to consider taking an evenly weighted average of these fixed basket price indices as a single estimator of price change between the two periods. Examples of such *symmetric averages*[18] are the arithmetic mean, which leads to the Drobisch (1871b; 425) Sidgwick (1883; 68) Bowley (1901; 227)[19] index, $P_D \equiv (1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the Fisher[20] (1922) ideal index, P_F defined as follows:

(12) $P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)P_P(p^0,p^1,q^0,q^1)]^{1/2}$.

At this point, the fixed basket approach to index number theory is transformed into the *test approach* to index number theory; i.e., in order to determine which of these fixed basket indexes or which averages of them might be "best", desirable *criteria* or *tests* or *properties* are needed for the price index. This topic will be pursued in more detail in the next chapter but an introduction to the test approach is provided in the present section because a test is used to determine which average of the Paasche and Laspeyres indexes might be "best".

What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical consumer price index? It is very desirable for a price index formula that depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test*[21]. An index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

(13) $P(p^1,p^0,q^1,q^0) = 1/P(p^0,p^1,q^0,q^1)$;

i.e., if the period 0 and period 1 price and quantity data are interchanged and the index number formula is evaluated, then this new index $P(p^1,p^0,q^1,q^0)$ should be equal to the reciprocal of the original index $P(p^0,p^1,q^0,q^1)$. This is a property that is satisfied by a single price ratio and it seems desirable that the measure of aggregate price change should also satisfy this property so that it does not matter which period is chosen as the base period. Put another way, the index number comparison between any two points of time should not depend on the choice of which period we regard as the base period: if the other period is chosen as the base period, then the new index number should simply equal the reciprocal of the original index. It should be noted that the Laspeyres and Paasche price indexes *do not* satisfy this time reversal property.

Having defined what it means for a price index P to satisfy the time reversal test, then it is possible to establish the following result:[22] the Fisher ideal price index defined by (12) above

[18] For a discussion of the properties of symmetric averages, see Diewert (1993b). Formally, an average m(a,b) of two numbers a and b is symmetric if m(a,b) = m(b,a). In other words, the numbers a and b are treated in the same manner in the average. An example of a nonsymmetric average of a and b is (1/4)a + (3/4)b. In general, Walsh (1901; 105) argued for a symmetric treatment if the two periods (or countries) under consideration were to be given equal importance.
[19] Walsh (1901; 99) also suggested this index. See Diewert (1993a; 36) for additional references to the early history of index number theory.
[20] Bowley (1899; 641) appears to have been the first to suggest the use of this index. Walsh (1901; 428-429) also suggested this index while commenting on the big differences between the Laspeyres and Paasche indexes in one of his numerical examples: "The figures in columns (2) [Laspeyres] and (3) [Paasche] are, singly, extravagant and absurd. But there is order in their extravagance; for the nearness of their means to the more truthful results shows that they straddle the true course, the one varying on the one side about as the other does on the other."
[21] See Diewert (1992; 218) for early references to this test. If we want the price index to have the same property as a single price ratio, then it is important to satisfy the time reversal test. However, other points of view are possible. For example, we may want to use our price index for compensation purposes in which case, satisfaction of the time reversal test may not be so important.
[22] See Diewert (1997; 138).

is the *only* index that is a homogeneous[23] symmetric average of the Laspeyres and Paasche price indexes, P_L and P_P, *and* satisfies the time reversal test (13) above. Thus the Fisher ideal price index emerges as perhaps the "best" evenly weighted average of the Paasche and Laspeyres price indexes.

It is interesting to note that this *symmetric basket approach* to index number theory dates back to one of the early pioneers of index number theory, Bowley, as the following quotations indicate:

"If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean … as a first approximation." Arthur. L. Bowley (1901; 227).

"When estimating the factor necessary for the correction of a change found in money wages to obtain the change in real wages, statisticians have not been content to follow Method II only [to calculate a Laspeyres price index], but have worked the problem backwards [to calculate a Paasche price index] as well as forwards. … They have then taken the arithmetic, geometric or harmonic mean of the two numbers so found." Arthur. L. Bowley (1919; 348).[24]

The quantity index that corresponds to the Fisher price index using the product test (3) is the Fisher quantity index; i.e., if P in (4) is replaced by P_F defined by (12), then the following quantity index is obtained:

(14) $Q_F(p^0,p^1,q^0,q^1) = [Q_L(p^0,p^1,q^0,q^1)Q_P(p^0,p^1,q^0,q^1)]^{1/2}$.

Thus the Fisher quantity index is equal to the square root of the product of the Laspeyres and Paasche quantity indexes. It should also be noted that $Q_F(p^0,p^1,q^0,q^1) = P_F(q^0,q^1,p^0,p^1)$; i.e., if the role of prices and quantities is interchanged in the Fisher price index formula, then the Fisher quantity index is obtained.[25]

Rather than take a symmetric average of the two basic fixed basket price indexes pertaining to two situations, P_L and P_P, it is also possible to return to Lowe's basic formulation and choose the basket vector q to be a symmetric average of the base and current period basket vectors, q^0 and q^1. This approach to index number theory is pursued in the following subsection.

5. The Walsh Index and the Theory of the "Pure" Price Index

Price statisticians tend to be very comfortable with a concept of the price index that is based on pricing out a constant "representative" basket of commodities, $q \equiv (q_1,q_2,\ldots,q_N)$, at the prices of period 0 and 1, $p^0 \equiv (p_1{}^0,p_2{}^0,\ldots,p_N{}^0)$ and $p^1 \equiv (p_1{}^1,p_2{}^1,\ldots,p_N{}^1)$ respectively. Price statisticians refer to this type of index as a *fixed basket index* or a *pure price index*[26] and it corresponds to Knibbs' (1924; 43) *unequivocal price index*.[27] Since Lowe (1823) was the first

[23] An average or mean of two numbers a and b, m(a,b), is *homogeneous* if when both numbers a and b are multiplied by a positive number λ, then the mean is also multiplied by λ; i.e., m satisfies the following property: $m(\lambda a,\lambda b) = \lambda m(a,b)$. The importance of linear homogeneity will be explained in Chapter 3 when the test approach to index number theory is studied.

[24] Fisher (1911; 417-418) (1922) also considered the arithmetic, geometric and harmonic averages of the Paasche and Laspeyres indexes.

[25] Fisher (1922; 72) said that P and Q satisfied the *factor reversal test* if $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ and P and Q satisfied the product test (3) as well.

[26] See section 7 in Diewert (2001).

[27] "Suppose however that, for each commodity, $Q' = Q$, then the fraction, $\sum(P'Q) / \sum(PQ)$, viz., the ratio of aggregate value for the second unit-period to the aggregate value for the first unit-period is no longer merely a ratio of totals, it also shows unequivocally the effect of the change in price. Thus it is an unequivocal price index for the quantitatively unchanged complex of commodities, A, B, C, etc.

person to describe systematically this type of index, it is referred to as a *Lowe index*. Thus the general functional form for the *Lowe price index* is

(15) $P_{Lo}(p^0,p^1,q) \equiv \Sigma_{i=1}^N p_i^1 q_i/\Sigma_{n=1}^N p_n^0 q_n = \Sigma_{i=1}^N s_i(p_i^1/p_i^0)$

where the (hypothetical) *hybrid expenditure shares* s_i [28] corresponding to the quantity weights vector q are defined by:

(16) $s_i \equiv p_i^0 q_i/\Sigma_{k=1}^N p_k^0 q_k$ for i = 1,...,N.

The main reason why price statisticians might prefer a member of the family of Lowe or fixed basket price indices defined by (15) is *that the fixed basket concept is easy to explain to the public*. Note that the Laspeyres and Paasche indices are special cases of the pure price concept if we choose $q = q^0$ (which leads to the Laspeyres index) or if we choose $q = q^1$ (which leads to the Paasche index).[29] The practical problem of picking q remains to be resolved and that is the problem that will be addressed in this section.

It should be noted that Walsh (1901; 105) (1921a) also saw the price index number problem in the above framework:

"Commodities are to be weighted according to their importance, or their full values. But the problem of axiometry always involves at least two periods. There is a first period, and there is a second period which is compared with it. Price variations have taken place between the two, and these are to be averaged to get the amount of their variation as a whole. But the weights of the commodities at the second period are apt to be different from their weights at the first period. Which weights, then, are the right ones—those of the first period? Or those of the second? Or should there be a combination of the two sets? There is no reason for preferring either the first or the second. Then the combination of both would seem to be the proper answer. And this combination itself involves an averaging of the weights of the two periods." Correa Moylan Walsh (1921a; 90).

Walsh's suggestion will be followed below and thus the ith quantity weight, q_i, is restricted to be an average or *mean* of the base period quantity q_i^0 and the current period quantity for commodity i q_i^1, say $m(q_i^0,q_i^1)$, for i = 1,2,…,N.[30] Under this assumption, the Lowe price index (15) becomes:

(17) $P_{Lo}(p^0,p^1,q^*) \equiv \Sigma_{n=1}^N p_n^1 q_n^*/\Sigma_{n=1}^N p_n^0 q_n^*$

where $q_n^* \equiv m(q_n^0,q_n^1)$ for n = 1,...,N and m(x,y) is an average or mean of the positive numbers x and y.

It is obvious that if the quantities were different on the two occasions, and if at the same time the prices had been unchanged, the preceding formula would become $\sum(PQ') / \sum(PQ)$. It would still be the ratio of the aggregate value for the second unit-period to the aggregate value for the first unit period. But it would be also more than this. It would show in a generalized way the ratio of the quantities on the two occasions. Thus it is an unequivocal quantity index for the complex of commodities, unchanged as to price and differing only as to quantity.

Let it be noted that the mere algebraic form of these expressions shows at once the logic of the problem of finding these two indices is identical." Sir George H. Knibbs (1924; 43-44).

[28] Fisher (1922; 53) used the terminology "weighted by a hybrid value" while Walsh (1932; 657) used the term "hybrid weights".

[29] Note that the ith share defined by (16) in this case is the hybrid share $s_i \equiv p_i^0 q_i^1/\sum_{j=1}^N p_j^0 q_j^1$, which uses the prices of period 0 and the quantities of period 1.

[30] Note that we have chosen the mean function $m(q_i^0,q_i^1)$ to be the same for each item i. We assume that m(a,b) has the following two properties: m(a,b) is a positive and continuous function, defined for all positive numbers a and b and m(a,a) = a for all a > 0.

In order to determine the functional form for the mean function m, it is necessary to impose some *tests* or *axioms* on the pure price index defined by (17). For the first such test or property, we ask that P_{Lo} satisfy the *time reversal test*, (13) above. Under this hypothesis, it can be shown that the mean function m must be a *symmetric mean*;[31] i.e., m must satisfy the following property: $m(a,b) = m(b,a)$ for all $a > 0$ and $b > 0$. This assumption still does not pin down the functional form for the pure price index defined by (17) above. For example, the function m(a,b) could be the *arithmetic mean*, $(½)a + (½)b$, in which case (17) reduces to the *Marshall* (1887) *Edgeworth* (1925) *price index* P_{ME}, which was the pure price index preferred by Knibbs (1924; 56):

$$(18)\ P_{ME}(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N}\ p_n^{\,1}\,[(½)q_n^{\,0}+(½)q_n^{\,1}]/\Sigma_{n=1}^{N}\ p_n^{\,0}[(½)q_n^{\,0}+(½)q_n^{\,1}].$$

On the other hand, the function m(a,b) could be the *geometric mean*, $(ab)^{1/2}$, in which case (17) reduces to the *Walsh* (1901; 398) (1921a; 97) *price index*, P_W[32]:

$$(19)\ P_W(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^{N}\ p_n^{\,1}\,[q_n^{\,0}q_n^{\,1}]^{1/2}/\Sigma_{n=1}^{N}\ p_n^{\,0}[q_n^{\,0}q_n^{\,1}]^{1/2}.$$

There are many other possibilities for the mean function m, including the mean of order r, $[(½)a^r + (½)b^r\,]^{1/r}$ for $r \neq 0$. Obviously, in order to completely determine the functional form for the pure price index defined by (17), it is necessary to impose at least one additional test or axiom on $P_{Lo}(p^0,p^1,q^*)$.

There is a potential problem with the use of the Edgeworth Marshall price index (18) that has been noticed in the context of using the formula to make international comparisons of prices. If the price levels of a very large country are compared to the price levels of a small country using formula (18), then the quantity vector of the large country may totally overwhelm the influence of the quantity vector corresponding to the small country. In technical terms, the Edgeworth Marshall formula is not homogeneous of degree 0 in the components of both q^0 and q^1.[33] To prevent this problem from occurring in the use of the pure price index defined by (17), it is asked that $P_{Lo}(p^0,p^1,q^*)$ satisfy the following *invariance to proportional changes in current quantities test*.[34]

$$(20)\ P_{Lo}(p^0,p^1,m(q_1^{\,0},\lambda q_1^{\,1}),...,m(q_N^{\,0},\lambda q_N^{\,1})) = P_{Lo}(p^0,p^1,m(q_1^{\,0},q_1^{\,1}),...,m(q_N^{\,0},q_N^{\,1})) \text{ for all } \lambda > 0.$$

The two tests, the time reversal test (13) and the linear homogeneity invariance test (20), enable one to determine the precise functional form for the pure price index $P_{Lo}(p^0,p^1,q^*)$ defined by (17) above: the pure price index $P_{Lo}(p^0,p^1,q^*)$ must be the Walsh index P_W defined by (19).[35]

[31] See section 7 of Diewert (2001) for a proof. For more on symmetric means, see Diewert (1993b; 361).

[32] Walsh endorsed P_W as being the best index number formula: "We have seen reason to believe formula 6 better than formula 7. Perhaps formula 9 is the best of the rest, but between it and Nos. 6 and 8 it would be difficult to decide with assurance." C.M. Walsh (1921a; 103). His formula 6 is P_W defined by (19) and his 9 is the Fisher ideal defined by (12) above. The *Walsh quantity index*, $Q_W(p^0,p^1,q^0,q^1)$ is defined as $P_W(q^0,q^1,p^0,p^1)$; i.e., the role of prices and quantities in definition (19) is interchanged. If the Walsh quantity index is used to deflate the value ratio, an implicit price index is obtained, which is Walsh's formula 8.

[33] Thus using (4), the companion quantity index defined by (4) will not be homogeneous of degree 1 in the components of the vector q^1 and homogeneous of degree -1 in the components of q^0.

[34] This is the terminology used by Diewert (1992; 216). Vogt (1980) was the first to propose this test. If this test holds, then the corresponding implicit quantity index defined by (4) will be linearly homogeneous in the components of q^1, which is a desirable property for a quantity index.

[35] See section 7 in Diewert (2001).

In order to be of practical use by statistical agencies, an index number formula must be able to be expressed as a function of the base period expenditure shares, s_i^0, the current period expenditure shares, s_i^1, and the N price ratios, p_i^1/p_i^0. The Walsh price index defined by (19) above can be rewritten in this format:

$$
\begin{aligned}
(21)\ P_W(p^0,p^1,q^0,q^1) &\equiv \sum_{n=1}^N p_n^1(q_n^0q_n^1)^{1/2} / \sum_{j=1}^N p_j^0(q_j^0q_j^1)^{1/2} \\
&= \sum_{n=1}^N [p_n^1/(p_n^0p_n^1)^{1/2}]\,(s_n^0s_n^1)^{1/2} / \sum_{j=1}^N [p_j^0/(p_j^0p_j^1)^{1/2}]\ (s_j^0s_j^1)^{1/2} \\
&= \sum_{n=1}^N (s_n^0s_n^1)^{1/2}\,[p_n^1/p_n^0]^{1/2}\ / \sum_{j=1}^N (s_j^0s_j^1)^{1/2}\,[p_j^0/p_j^1]^{1/2}\ .
\end{aligned}
$$

The approach taken to index number theory in this section was to consider averages of various fixed basket type price indexes. The first approach was to take an even handed average of the two primary fixed basket indexes: the Laspeyres and Paasche price indexes. These two primary indices are based on pricing out the baskets that pertain to the two periods (or locations) under consideration. Taking an average of them led to the Fisher ideal price index P_F defined by (12) above. The second approach was to average the basket quantity weights and then price out this average basket at the prices pertaining to the two situations under consideration. This approach led to the Walsh price index P_W defined by (19) above. Both of these indexes can be written as a function of the base period expenditure shares, s_i^0, the current period expenditure shares, s_i^1, and the N price ratios, p_i^1/p_i^0. Assuming that the statistical agency has information on these three sets of variables, which index should be used? Experience with normal time series data at higher levels of aggregation has shown that these two indexes will not differ substantially and thus it is a matter of indifference which of these indexes is used in practice.[36] Both of these indexes are examples of *superlative indexes*, which will be defined in Chapter 5. However, note that both of these indexes treat the data pertaining to the two situations in a *symmetric* manner. Hill[37] commented on superlative price indexes and the importance of a symmetric treatment of the data as follows:

"Thus economic theory suggests that, in general, a symmetric index that assigns equal weight to the two situations being compared is to be preferred to either the Laspeyres or Paasche indices on their own. The precise choice of superlative index—whether Fisher, Törnqvist or other superlative index—may be of only secondary importance as all the symmetric indices are likely to approximate each other, and the underlying theoretic index fairly closely, at least when the index number spread between the Laspeyres and Paasche is not very great." Peter Hill (1993; 384).

6. The Lowe Index with Monthly Prices and Annual Base Year Quantities

It is now necessary to discuss a major practical problem with the above theory of basket type indexes. Up to now, it has been assumed that the quantity vector $q \equiv (q_1,q_2,\ldots,q_N)$ that appeared in the definition of the Lowe index, $P_{Lo}(p^0,p^1,q)$ defined by (15), is either the base period quantity vector q^0 or the current period quantity vector q^1 or an average of these two quantity vectors. In fact, in terms of actual statistical agency practice, the quantity vector q is frequently taken to be an *annual quantity vector* that refers to a *base year*, b say, that is prior to the base period for the prices, period 0. Typically, a statistical agency will produce a Consumer Price Index at a monthly or quarterly frequency but for the sake of definiteness, a monthly frequency will be assumed in what follows. Thus a typical price index will have the

[36] Diewert (1978; 887-889) showed that these two indexes will approximate each other to the second order around an equal price and quantity point. Thus for normal time series data where prices and quantities do not change much going from the base period to the current period, the indexes will approximate each other quite closely. However, if scanner data from retail outlets or from individual households is used at the first stage of aggregation, and the price and quantity data are very volatile, then second order approximations may not be very accurate and the Walsh and Fisher indexes may differ substantially. As will be seen in Chapter 3, the Fisher index may be preferred over the Walsh index because of its better axiomatic properties.

[37] See also Hill (1988).

form $P_{Lo}(p^0,p^t,q^b)$, where p^0 is the price vector pertaining to the base period month for prices, month 0, p^t is the price vector pertaining to the current period month for prices, month t say, and q^b is a reference basket quantity vector that refers to the base year b, which is equal to or prior to month 0.[38] Note that this Lowe index $P_{Lo}(p^0,p^t,q^b)$ is *not* a true Laspeyres index (because the annual quantity vector q^b is not equal to the monthly quantity vector q^0 in general).[39]

The question is: why do statistical agencies *not* pick the reference quantity vector q in the Lowe formula to be the monthly quantity vector q^0 that pertains to transactions in month 0 (so that the index would reduce to an ordinary Laspeyres price index)? There are two main reasons why this is not done:

- Most economies are subject to seasonal fluctuations and so picking the quantity vector of month 0 as the reference quantity vector for all months of the year would not be representative of transactions made throughout the year.
- Monthly household quantity or expenditure weights are usually collected by the statistical agency using a household expenditure survey with a relatively small sample. Hence the resulting weights are usually subject to very large sampling errors and so standard practice is to average these monthly expenditure or quantity weights over an entire year (or in some cases, over several years) in an attempt to reduce these sampling errors.

The index number problems that are caused by seasonal monthly weights will be studied in more detail in chapter 9. For now, it can be argued that the use of annual weights in a monthly index number formula is simply a method for dealing with poor estimates of monthly quantities or for dealing with the seasonality problem.[40] However, it should be noted that the use of annual weights in a monthly consumer price index is not consistent with the economic approach to index number theory.[41]

One problem with using annual weights corresponding to a perhaps distant year in the context of a monthly Consumer Price Index must be noted at this point: if there are systematic (but divergent) trends in commodity prices and households increase their purchases of commodities that decline (relatively) in price and decrease their purchases of commodities that increase (relatively) in price, then the use of distant quantity weights will tend to lead to an upward bias in this Lowe index compared to one that used more current weights. This observation suggests that statistical agencies should strive to get up to date weights on an ongoing basis.

It is useful to explain how the annual quantity vector q^b could be obtained from monthly expenditures on each commodity during the chosen base year b. Let the month m expenditure of the reference population in the base year b for commodity i be $v_i^{b,m}$ and let the

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[38] Month 0 is called the price reference period and year b is called the weight reference period.

[39] Triplett (1981; 12) defined the Lowe index, calling it a Laspeyres index, and calling the index that has the weight reference period equal to the price reference period, a pure Laspeyres index. However, Balk (1980; 69) asserted that although the Lowe index is of the fixed base type, it is not a Laspeyres price index. Triplett also noted the hybrid share representation for the Lowe index defined by (15) and (16) above. Triplett noted that the ratio of two Lowe indexes using the same quantity weights was also a Lowe index. Baldwin (1990; 255) called the Lowe index an *annual basket index*.

[40] In fact, the use of the Lowe index $P_{Lo}(p^0,p^t,q^b)$ in the context of seasonal commodities corresponds to Bean and Stine's (1924; 31) Type A index number formula. Bean and Stine made 3 additional suggestions for price indexes in the context of seasonal commodities. Their contributions will be evaluated in Chapter 9.

[41] Thus if one takes the economic approach to index number theory, then the use of annual weights will lead to a certain amount of substitution bias; see Chapter 7.

corresponding price and quantity be $p_i^{b,m}$ and $q_i^{b,m}$ respectively. Of course, value, price and quantity for each commodity are related by the following equations:

$$(22)\ v_i^{b,m} = p_i^{b,m}q_i^{b,m}\ ; \qquad i = 1,...,N;\ m = 1,...,12.$$

For each commodity i, an estimate for the annual total quantity, q_i^b can be obtained by price deflating monthly values and summing over months in the base year b as follows:

$$(23)\ q_i^b \equiv \Sigma_{m=1}^{12}\ v_i^{b,m}/p_i^{b,m} = \Sigma_{m=1}^{12}\ q_i^{b,m}\ ; \qquad i = 1,...,N.$$

where (22) was used to derive the second equation in (23). In practice, the above equations will be evaluated using aggregate expenditures over closely related commodities and the price $p_i^{b,m}$ will be the month m price index for this elementary commodity group i in year b relative to the first month of year b.

For some purposes, it is also useful to have annual prices by commodity to match up with the annual quantities defined by (23). Following national income accounting conventions, a reasonable[42] price p_i^b to match up with the annual quantity q_i^b is the value of total consumption of commodity i in year b divided by q_i^b. Thus we have:

$$\begin{aligned}(24)\ p_i^b &\equiv \Sigma_{m=1}^{12}\ v_i^{b,m}/\Sigma_{m=1}^{12}\ q_i^{b,m} && i = 1,...,N \\ &= \Sigma_{m=1}^{12}\ v_i^{b,m}/\Sigma_{m=1}^{12}\ [v_i^{b,m}/p_i^{b,m}] && \text{using (22)} \\ &= \Sigma_{m=1}^{12}\ [s_i^{b,m}(p_i^{b,m})^{-1}]^{-1} &&\end{aligned}$$

where the share of annual expenditure on commodity i in month m of the base year b is

$$(25)\ s_i^{b,m} \equiv\ v_i^{b,m}/\Sigma_{k=1}^{12}\ v_i^{b,k}\ ; \qquad i = 1,...,N;\ m = 1,...,12.$$

Thus the annual base year price for commodity i, p_i^b, turns out to be a monthly expenditure weighted *harmonic mean* of the monthly prices for commodity i in the base year, $p_i^{b,1}$, $p_i^{b,2}$,…, $p_i^{b,12}$.

Using the annual commodity prices for the base year defined by (24), a vector of these prices can be defined as $p^b \equiv [p_1^b,\ldots,p_N^b]$. Using this definition, the Lowe index $P_{Lo}(p^0,p^t,q^b)$ can be expressed as a ratio of two Laspeyres indexes where the price vector p^b plays the role of base period prices in each of the two Laspeyres indexes:

$$\begin{aligned}(26)\ P_{Lo}(p^0,p^t,q^b) &\equiv \Sigma_{i=1}^N\ p_i^tq_i^b/\Sigma_{i=1}^N\ p_i^0q_i^b \\ &= [\Sigma_{i=1}^N\ p_i^tq_i^b/\Sigma_{i=1}^N\ p_i^bq_i^b]/[\Sigma_{i=1}^N\ p_i^0q_i^b/\Sigma_{i=1}^N\ p_i^bq_i^b] \\ &= \Sigma_{i=1}^N\ s_i^b\ (p_i^t/p_i^b)/\Sigma_{i=1}^N\ s_i^b\ (p_i^0/p_i^b) \\ &= P_L(p^b,p^t,q^b)/P_L(p^b,p^0,q^b)\end{aligned}$$

where the *base year expenditure shares* are defined as $s_i^b \equiv p_i^bq_i^b/\Sigma_{n=1}^N\ p_n^bq_n^b$ and the Laspeyres formula P_L was defined by (5) above. Thus the above equation shows that the Lowe monthly price index comparing the prices of month 0 to those of month t using the quantities of base year b as weights, $P_{Lo}(p^0,p^t,q^b)$, is equal to the Laspeyres index that compares the prices of month t to those of year b, $P_L(p^b,p^t,q^b)$, divided by the Laspeyres index that compares the prices of month 0 to those of year b, $P_L(p^b,p^0,q^b)$. Note that the Laspeyres

[42] Hence these annual commodity prices are essentially unit value prices. Under conditions of high inflation, the annual prices defined by (24) may no longer be "reasonable" or representative of prices during the entire base year because the expenditures in the final months of the high inflation year will be somewhat artificially blown up by general inflation. Under these conditions, the annual prices and annual commodity expenditure shares should be interpreted with caution. For more on dealing with situations where there is high inflation within a year, see Hill (1996).

index in the numerator can be calculated if the base year commodity expenditure shares, s_i^b, are known along with the price ratios that compare the prices of commodity i in month t, p_i^t, with the corresponding annual average prices in the base year b, p_i^b. The Laspeyres index in the denominator can be calculated if the base year commodity expenditure shares, s_i^b, are known along with the price ratios that compare the prices of commodity i in month 0, p_i^0, with the corresponding annual average prices in the base year b, p_i^b.

There is another convenient formula for evaluating the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, and that is to use the hybrid weights formula, (15). In the present context (assuming that all prices in the base period are positive), the formula becomes:

$$(27)\ P_{Lo}(p^0,p^t,q^b) \equiv \Sigma_{i=1}^N p_i^t q_i^b/\Sigma_{n=1}^N p_n^0 q_n^b = \Sigma_{i=1}^N (p_i^t/p_i^0)p_i^0 q_i^b /\Sigma_{n=1}^N p_n^0 q_n^b = \Sigma_{i=1}^N (p_i^t/p_i^0)s_i^{0b}$$

where the hybrid weights s_i^{0b} using the prices of month 0 and the quantities of year b are defined by

$$(28)\ s_i^{0b} \equiv p_i^0 q_i^b/\Sigma_{n=1}^N p_n^0 q_n^b = (p_i^0/p_i^b)p_i^b q_i^b/\Sigma_{n=1}^N (p_n^0/p_n^b)p_n^b q_n^b\ ; \qquad i = 1,\ldots,N.$$

The second equation in (28) shows how the base year expenditures, $p_i^b q_i^b$, can be multiplied by the commodity price indexes, p_i^0/p_i^b, in order to calculate the hybrid shares.

There is one additional formula for the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, that will be exhibited. Note that the Laspeyres decomposition of the Lowe index defined by the third line in (26) involves the very long term price relatives, p_i^t/p_i^b, which compare the prices in month t, p_i^t, with the possibly distant base year prices, p_i^b, and that the hybrid share decomposition of the Lowe index defined by the last equality in (27) involves the long term monthly price relatives, p_i^t/p_i^0, which compare the prices in month t, p_i^t, with the base month prices, p_i^0. Both of these formulae are not satisfactory in practice because of the problem of sample attrition: each month, a substantial fraction of commodities disappears from the marketplace and thus it is useful to have a formula for updating the previous month's price index using just month over month price relatives. In other words, long term price relatives disappear at a rate that is too large in practice to base an index number formula on their use. The Lowe index for month t+1, $P_{Lo}(p^0,p^{t+1},q^b)$, can be written in terms of the Lowe index for the prior month t, $P_{Lo}(p^0,p^t,q^b)$, and an *updating factor* as follows:

$$\begin{aligned}(29)\ P_{Lo}(p^0,p^{t+1},q^b) &\equiv \Sigma_{i=1}^N p_i^{t+1} q_i^b/\Sigma_{n=1}^N p_n^0 q_n^b \\ &= [\Sigma_{i=1}^N p_i^t q_i^b/\Sigma_{n=1}^N p_n^0 q_n^b][\Sigma_{i=1}^N p_i^{t+1} q_i^b/\Sigma_{n=1}^N p_n^t q_n^b] \\ &= P_{Lo}(p^0,p^t,q^b)\ [\Sigma_{i=1}^N p_i^{t+1} q_i^b/\Sigma_{n=1}^N p_n^t q_n^b] \\ &= P_{Lo}(p^0,p^t,q^b)[\Sigma_{i=1}^N (p_i^{t+1}/p_i^t)p_i^t q_i^b/\Sigma_{n=1}^N p_n^t q_n^b] \qquad \text{if all } p_i^t > 0 \\ &= P_{Lo}(p^0,p^t,q^b)\Sigma_{i=1}^N (p_i^{t+1}/p_i^t)s_i^{tb}\end{aligned}$$

where the hybrid weights s_i^{tb} are defined by

$$(30)\ s_i^{tb} \equiv p_i^t q_i^b/\Sigma_{n=1}^N p_n^t q_n^b\ ; \qquad i = 1,\ldots,N.$$

Thus the required updating factor, going from month t to month t+1, is the chain link index $\sum_{i=1}^N s_i^{tb} (p_i^{t+1}/p_i^t)$, which uses the hybrid share weights s_i^{tb} corresponding to month t and base year b.[43]

The Lowe index $P_{Lo}(p^0,p^t,q^b)$ can be regarded as an approximation to the ordinary Laspeyres index, $P_L(p^0,p^t,q^0)$, that compares the prices of the base month 0, p^0, to those of month t, p^t, using the quantity vector of month 0, q^0, as weights. It turns out that there is a relatively

[43] If one or more of the p_i^{tb} are equal to 0, then define the link factor by $\Sigma_{i=1}^N p_i^{t+1} q_i^b/\Sigma_{n=1}^N p_n^t q_n^b$.

simple formula that relates these two indexes.[44] In order to explain this formula, it is first necessary to make a few definitions. Define the *nth price relative* between month 0 and month t as

$$(31)\ r_n \equiv p_n^t/p_n^0\ ; \qquad n = 1,\ldots,N.$$

The ordinary Laspeyres price index, going from month 0 to t, can be defined in terms of these price relatives as follows:

$$(32)\ P_L(p^0,p^t,q^0) \equiv \Sigma_{n=1}^N\, p_n^t q_n^0/\Sigma_{i=1}^N\, p_i^0 q_i^0 = \Sigma_{n=1}^N\, (p_n^t/p_n^0)p_n^0 q_n^0/\Sigma_{i=1}^N\, p_i^0 q_i^0 = \Sigma_{n=1}^N\, s_n^0 r_n \equiv r^*$$

using definitions (7) and (31) in order to derive the penultimate equality.

Define the nth quantity relative t_n as the ratio of the quantity of commodity n used in the base year b, q_n^b, to the quantity used in month 0, q_n^0, as follows:

$$(33)\ t_n \equiv q_n^b/q_n^0\ ; \qquad n = 1,\ldots,N.$$

The Laspeyres quantity index, $Q_L(q^0,q^b,p^0)$, that compares quantities in year b, q^b, to the corresponding quantities in month 0, q^0, using the prices of month 0, p^0, as weights can be defined as a weighted average t^* of the quantity ratios t_n as follows:

$$\begin{aligned}(34)\ Q_L(q^0,q^b,p^0) &\equiv \Sigma_{n=1}^N\, p_n^0 q_n^b/\Sigma_{i=1}^N\, p_i^0 q_i^0 \\ &= \Sigma_{n=1}^N\, p_n^0 q_n^0 (q_n^b/q_n^0)/\Sigma_{i=1}^N\, p_i^0 q_i^0 \\ &= \Sigma_{n=1}^N\, s_n^0 t_n \qquad \text{using (7) and (33)} \\ &\equiv t^*.\end{aligned}$$

The relationship between the Lowe index $P_{Lo}(p^0,p^t,q^b)$ that uses the quantities of year b as weights to compare the prices of month t to month 0 and the corresponding ordinary Laspeyres index $P_L(p^0,p^t,q^0)$ that uses the quantities of month 0 as weights is the following one:[45]

$$\begin{aligned}(35)\ P_{Lo}(p^0,p^t,q^b) &\equiv \Sigma_{n=1}^N\, p_n^t q_n^b/\Sigma_{n=1}^N\, p_n^0 q_n^b \\ &= P_L(p^0,p^t,q^0) + \Sigma_{n=1}^N\, (r_n - r^*)(t_n - t^*)s_n^0/Q_L(q^0,q^b,p^0).\end{aligned}$$

Thus the Lowe price index using the quantities of year b as weights, $P_{Lo}(p^0,p^t,q^b)$, is equal to the usual Laspeyres index using the quantities of month 0 as weights, $P_L(p^0,p^t,q^0)$, plus a covariance term $\sum_{n=1}^N\, (r_n - r^*)(t_n - t^*)s_n^0$ between the price relatives $r_n \equiv p_n^t/p_n^0$ and the quantity relatives $t_n \equiv q_n^b/q_n^0$, divided by the Laspeyres quantity index $Q_L(q^0,q^b,p^0)$ between month 0 and base year b.

Formula (35) shows that the Lowe price index will coincide with the Laspeyres price index if the covariance or correlation between the month 0 to t price relatives $r_n \equiv p_n^t/p_n^0$ and the month 0 to year b quantity relatives $t_n \equiv q_n^b/q_n^0$ is zero. Note that this covariance will be zero under three different sets of conditions:

- If the month t prices are proportional to the month 0 prices so that all $r_n = r^*$;
- If the base year b quantities are proportional to the month 0 quantities so that all $t_n = t^*$;
- If the distribution of the relative prices r_n is independent of the distribution of the relative quantities t_n.

[44] In what follows, it is assumed that all prices and quantities in month 0 are positive.
[45] See Appendix 2 for a derivation of this formula.

The first two conditions are unlikely to hold empirically but the third is possible, at least approximately, if consumers do not systematically change their purchasing habits in response to changes in relative prices.

If this covariance in (35) is negative, then the Lowe index will be less than the Laspeyres and finally, if the covariance is positive, then the Lowe index will be greater than the Laspeyres index. Although the sign and magnitude of the covariance term, $\Sigma_{n=1}^{N} (r_n - r^*)(t_n - t^*)$, is ultimately an empirical matter, it is possible to make some reasonable conjectures about its likely sign. If the base year b precedes the price reference month 0 *and there are long term trends in prices*, then it is likely that this covariance is *positive* and hence this implies that the Lowe index will exceed the corresponding Laspeyres price index[46]; i.e.,

(36) $P_{Lo}(p^0,p^t,q^b) > P_L(p^0,p^t,q^0)$.

To see why this covariance is likely to be positive, suppose that there is a long term upward trend in the price of commodity n so that $r_n - r^* \equiv (p_n^t/p_n^0) - r^*$ is positive. With normal consumer substitution responses[47], q_n^t/q_n^0 less an average quantity change of this type is likely to be negative, or, upon taking reciprocals, q_n^0/q_n^t less an average quantity change of this (reciprocal) type is likely to be positive. But if the long term upward trend in prices has persisted back to the base year b, then $t_n - t^* \equiv (q_n^b/q_n^0) - t^*$ is also likely to be positive. Hence, the covariance will be positive under these circumstances. Moreover, the more distant is the base year b from the base month 0, the bigger the residuals $t_n - t^*$ will likely be and the bigger will be the positive covariance. Similarly, the more distant is the current period month t from the base period month 0, the bigger the residuals $r_n - r^*$ will likely be and the bigger will be the positive covariance. *Thus under the assumptions that there are long term trends in prices and normal consumer substitution responses, the Lowe index will normally be greater than the corresponding Laspeyres index.*[48]

The Paasche index between months 0 and t is defined as follows:

(37) $P_P(p^0,p^t,q^t) \equiv \Sigma_{n=1}^{N} p_n^t q_n^t / \Sigma_{i=1}^{N} p_i^0 q_i^t$.

As was discussed in section 4 above, a reasonable target index to measure the price change going from month 0 to t is some sort of symmetric average of the Paasche index $P_P(p^0,p^t,q^t)$ defined by (37) and the corresponding Laspeyres index, $P_L(p^0,p^t,q^0)$ defined by (32). Using the results in Appendix 1, the relationship between the Paasche and Laspeyres indexes can be written as follows:

[46] It is also necessary to assume that households have normal substitution effects in response to these long term trends in prices; i.e., if a commodity increases (relatively) in price, its consumption will decline (relatively) and if a commodity decreases relatively in price, its consumption will increase relatively.

[47] Walsh (1901; 281-282) was well aware of consumer substitution effects as can be seen in the following comment which noted the basic problem with a fixed basket index that uses the quantity weights of a single period: "The argument made by the arithmetic averagist supposes that we buy the same quantities of every class at both periods in spite of the variation in their prices, which we rarely, if ever, do. As a rough proposition, we –a community –generally spend more on articles that have risen in price and get less of them, and spend less on articles that have fallen in price and get more of them."

[48] If expression (26) is substituted into the left hand side of (36), the resulting inequality becomes $P_L(p^b,p^t,q^b) > P_L(p^b,p^0,q^b)P_L(p^0,p^t,q^0)$. Thus the Laspeyres index from b to t is bigger than a Laspeyres from b to 0 multiplied by a Laspeyres from 0 to t. This is not surprising since the Laspeyres index from b to t continues using period b weights until period t. On the right side of the inequality, period b weights are only used until period 0 when period 0 weights are introduced, which will reflect any substitution households may have made from b to 0. This point is due to Carsten Boldsen.

(38) $P_P(p^0,p^t,q^t) \equiv \Sigma_{n=1}^N p_n^t q_n^t / \Sigma_{n=1}^N p_n^0 q_n^t = P_L(p^0,p^t,q^0) + \Sigma_{n=1}^N (r_n - r^*)(u_n - u^*)s_n^0 / Q_L(q^0,q^t,p^0)$

where the price relatives $r_n \equiv p_n^t/p_n^0$ are defined by (31) and their share weighted average r* by (32) and the u_n, u^* and Q_L are defined as follows:

(39) $u_n \equiv q_n^t/q_n^0$; n = 1,,,.N;
(40) $u^* \equiv \Sigma_{n=1}^N s_n^0 u_n \equiv Q_L(q^0,q^t,p^0)$

and the month 0 expenditure shares s_i^0 are defined by (7). Thus u^* is equal to the Laspeyres quantity index between months 0 and t. This means that the Paasche price index that uses the quantities of month t as weights, $P_P(p^0,p^t,q^t)$, is equal to the usual Laspeyres index using the quantities of month 0 as weights, $P_L(p^0,p^t,q^0)$, plus a weighted covariance term $\Sigma_{n=1}^N (r_n - r^*)(u_n - u^*)s_n^0$ between the price relatives $r_n \equiv p_n^t/p_n^0$ and the corresponding quantity relatives $u_n \equiv q_n^t/q_n^0$, divided by the Laspeyres quantity index $Q_L(q^0,q^t,p^0)$ between month 0 and month t.

Although the sign and magnitude of the covariance term, $\Sigma_{n=1}^N (r_n - r^*)(u_n - u^*)s_n^0$, is again an empirical matter, it is possible to make a reasonable conjecture about its likely sign. If *there are long term trends in prices and consumers respond normally to price changes in their purchases*, then it is likely that that this covariance is *negative* and hence the Paasche index will be less than the corresponding Laspeyres price index; i.e.,

(41) $P_P(p^0,p^t,q^t) < P_L(p^0,p^t,q^0)$.

To see why this covariance is likely to be negative, suppose that there is a long term upward trend in the price of commodity n[49] so that $r_n - r^* \equiv (p_n^t/p_n^0) - r^*$ is positive. With normal consumer substitution responses, q_n^t/q_n^0 less an average quantity change of this type is likely to be negative. Hence $u_n - u^* \equiv (q_n^t/q_n^0) - u^*$ is likely to be negative. Thus, the covariance will be negative under these circumstances. Moreover, the more distant is the base month 0 from the current month t, the bigger in magnitude the residuals $u_n - u^*$ will likely be and the bigger in magnitude will be the negative covariance.[50] Similarly, the more distant is the current period month t from the base period month 0, the bigger the residuals $r_n - r^*$ will likely be and the bigger in magnitude will be the covariance. *Thus under the assumptions that there are long term trends in prices and normal consumer substitution responses, the Laspeyres index will be greater than the corresponding Paasche index*, with the divergence likely growing as month t becomes more distant from month 0.

Putting the arguments in the previous paragraphs together, it can be seen that under the assumptions that there are long term trends in prices and normal consumer substitution responses, the Lowe price index between months 0 and t will exceed the corresponding Laspeyres price index which in turn will exceed the corresponding Paasche price index; i.e., under these hypotheses,

(42) $P_{Lo}(p^0,p^t,q^b) > P_L(p^0,p^t,q^0) > P_P(p^0,p^t,q^t)$.

Thus if the long run target price index is an average of the Laspeyres and Paasche indexes, it can be seen that the Laspeyres index will have an *upward bias* relative to this target index and

[49] The reader can carry through the argument if there is a long term relative decline in the price of the ith commodity. The argument required to obtain a negative covariance requires that there be some differences in the long term trends in prices; i.e., if all prices grow (or fall) at the same rate, we have price proportionality and the covariance will be zero.
[50] However, $Q_L = u^*$ may also be growing in magnitude so the net effect on the divergence between P_L and P_P is ambiguous.

the Paasche index will have a *downward bias*. In addition, *if the base year b is prior to the price reference month, month 0, then the Lowe index will also have an upward bias relative to the Laspeyres index and hence also to the target index*.

7. The Young Index

Recall the definitions for the base year quantities, q_n^b, and the base year prices, p_n^b, given by (23) and (24) above. The *base year expenditure shares* can be defined in the usual way as follows:

$$(43)\ s_n^b \equiv p_n^b q_n^b / \Sigma_{i=1}^N p_i^b q_i^b\ ; \qquad n = 1,...,N.$$

Define the vector of base year expenditure shares in the usual way as $s^b \equiv [s_1^b,\ldots,s_N^b]$. These base year expenditure shares were used to provide an alternative formula for the base year b Lowe price index going from month 0 to t defined in (26) as $P_{Lo}(p^0,p^t,q^b) = [\sum_{i=1}^N s_i^b(p_i^t/p_i^b)]/[\sum_{i=1}^N s_i^b(p_i^0/p_i^b)]$. Rather than using this index as their target index, some statistical agencies use the following related index, which also uses base year expenditure shares as weights:[51]

$$(44)\ P_Y(p^0,p^t,s^b) \equiv \Sigma_{i=1}^N s_i^b(p_i^t/p_i^0).$$

This type of index was first defined by the English economist Arthur Young (1812).[52] Note that there is a change in focus when the Young index is used compared to the other indexes proposed earlier in this chapter. Up to this point, the indexes proposed have been of the fixed basket type (or averages of such indexes) where a *commodity basket* that is somehow representative of the two periods being compared is chosen and then "purchased" at the prices of the two periods and the index is taken to be the ratio of these two costs. On the other hand, for the Young index, one instead chooses *representative expenditure shares* that pertain to the two periods under consideration and then uses these shares to calculate the overall index as a share weighted average of the individual price ratios, p_i^t/p_i^0. Note that this share weighted average of price ratios view of index number theory is a bit different from the view taken at the beginning of this chapter, which viewed the index number problem as the problem of decomposing a value ratio into the product of two terms, one of which expresses the amount of price change between the two periods and the other which expresses the amount of quantity change.[53] However, the two approaches are not necessarily inconsistent; the weighted average of price ratios approach to index number theory generates a price index and the companion quantity index can always be generated using the product test; see equation (4) above.

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[51] We require all prices in the base period to be positive in order for the Young index to be well defined.

[52] The attribution of this formula to Young is due to Walsh (1901; 536) (1932; 657).

[53] Fisher's 1922 book is famous for developing the value ratio decomposition approach to index number theory but his introductory chapters took the share weighted average point of view: "An index number of prices, then shows the *average percentage change* of prices from one point of time to another." Irving Fisher (1922; 3). Fisher went on to note the importance of economic weighting: "The preceding calculation treats all the commodities as equally important; consequently, the average was called 'simple'. If one commodity is more important than another, we may treat the more important as though it were two or three commodities, thus giving it two or three times as much 'weight' as the other commodity." Irving Fisher (1922; 6). Walsh (1901; 430-431) considered both approaches: "We can either (1) draw some average of the total money values of the classes during an epoch of years, and with weighting so determined employ the geometric average of the price variations [ratios]; or (2) draw some average of the mass quantities of the classes during the epoch, and apply to them Scrope's method." Scrope's method is the same as using the Lowe index. Walsh (1901; 88-90) consistently stressed the importance of weighting price ratios by their economic importance (rather than using equally weighted averages of price relatives).

Statistical agencies sometimes regard the Young index defined above as an approximation to the Laspeyres price index $P_L(p^0,p^t,q^0)$. Hence, it is of interest to see how the two indexes compare. Defining the long term monthly price relatives going from month 0 to t as $r_i \equiv p_i^t/p_i^0$ and using definitions (32) and (44) leads to the following formula:

$$\begin{aligned}(45)\ P_Y(p^0,p^t,s^b) - P_L(p^0,p^t,q^0) &= \Sigma_{i=1}^N s_i^b(p_i^t/p_i^0) - \Sigma_{i=1}^N s_i^0(p_i^t/p_i^0)\\ &= \Sigma_{i=1}^N [s_i^b - s_i^0]r_i\\ &= \Sigma_{i=1}^N [s_i^b - s_i^0][r_i - r^*]\end{aligned}$$

where $r_i \equiv p_i^t/p_i^0$ for i = 1,...,N and $r^* \equiv \Sigma_{i=1}^N s_i^0(p_i^t/p_i^0)$. The last equality follows from the line above since $\Sigma_{i=1}^N [s_i^b - s_i^0]r^* = [1 - 1]r^* = 0$. Thus the Young index $P_Y(p^0,p^t,s^b)$ is equal to the Laspeyres index $P_L(p^0,p^t,q^0)$ plus the *covariance* between the difference in the annual shares pertaining to year b and the month 0 shares, $s_i^b - s_i^0$, and the deviations of the relative prices from their mean, $r_i - r^*$.

It is no longer possible to guess at what the likely sign of the covariance term is. The question is no longer whether the *quantity* demanded goes down as the price of commodity i goes up (the answer to this question is usually yes) but the new question is: does the *share* of expenditure go down as the price of commodity i goes up? The answer to this question depends on the elasticity of demand for the product. However, let us provisionally assume both that there are long run trends in commodity prices and that if the trend in prices for commodity i is above the mean, then the expenditure share for the commodity trends *down* (and vice versa). Thus we are assuming high elasticities or very strong substitution effects. Assuming also that the base year b is prior to month 0, then under these conditions, suppose that there is a long term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With the assumed very elastic consumer substitution responses, s_i will tend to decrease relatively over time and since s_i^b is assumed to be prior to s_i^0, s_i^0 is expected to be less than s_i^b or $s_i^b - s_i^0$ will likely be positive. Thus, the covariance is likely to be *positive* under these circumstances. *Hence with long run trends in prices and very elastic responses of consumers to price changes, the Young index is likely to be greater than the corresponding Laspeyres index.*

Assume that there are long run trends in commodity prices. If the trend in price for commodity i is above the mean, then suppose that the expenditure share for the commodity trends *up* (and vice versa). Thus we are assuming low elasticities or very weak substitution effects. Assume also that the base year b is prior to month 0 and suppose that there is a long term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With the assumed very inelastic consumer substitution responses, s_i will tend to increase relatively over time and since s_i^b is assumed to be prior to s_i^0, it will be the case that s_i^0 is greater than s_i^b or $s_i^b - s_i^0$ is negative. Thus, the covariance is likely to be *negative* under these circumstances. *Hence with long run trends in prices and very inelastic responses of consumers to price changes, the Young index is likely to be less than the corresponding Laspeyres index.*

The previous two paragraphs indicate that, a priori, it is not known what the likely difference between the Young index and the corresponding Laspeyres index will be. If elasticities of substitution are close to one, then the two sets of expenditure shares, s_i^b and s_i^0, will be close to each other and the difference between the two indexes will be close to zero. However, if monthly expenditure shares have strong seasonal components (or if there are missing products for some months for whatever reason), then the annual shares s_i^b could differ substantially from the monthly shares s_i^0.

It is useful to have a formula for updating the previous month's Young price index using just month over month price relatives. The Young index for month t+1, $P_Y(p^0,p^{t+1},s^b)$, can be written in terms of the Young index for month t, $P_Y(p^0,p^t,s^b)$ and an updating factor as follows:

$$(46)\ P_Y(p^0,p^{t+1},s^b) \equiv \Sigma_{i=1}^N s_i^b(p_i^{t+1}/p_i^0)$$
$$= P_Y(p^0,p^t,s^b)[\Sigma_{i=1}^N s_i^b(p_i^{t+1}/p_i^0)/\Sigma_{n=1}^N s_n^b(p_n^t/p_n^0)]$$
$$= P_Y(p^0,p^t,s^b)[\Sigma_{i=1}^N p_i^bq_i^b(p_i^t/p_i^0)(p_i^{t+1}/p_i^t)/\Sigma_{n=1}^N p_n^bq_n^b(p_n^t/p_n^0)]$$
$$= P_Y(p^0,p^t,s^b)[\Sigma_{i=1}^N s_i^{b0t}(p_i^{t+1}/p_i^t)]$$

where the hybrid weights s_i^{b0t} are defined as follows:

$$(47)\ s_i^{b0t} \equiv p_i^bq_i^b(p_i^t/p_i^0)/\Sigma_{n=1}^N p_n^bq_n^b(p_n^t/p_n^0) = s_i^b(p_i^t/p_i^0)/\Sigma_{n=1}^N s_n^b(p_n^t/p_n^0)\ ; \qquad i = 1,...,N.$$

Thus the hybrid weights s_i^{b0t} can be obtained from the base year weights s_i^b by updating them; i.e., by multiplying them by the price relatives, (or indexes at higher levels of aggregation), p_i^t/p_i^0. Thus the required updating factor, going from month t to month t+1, is the chain link index, $\sum_{i=1}^N s_i^{b0t}\ (p_i^{t+1}/p_i^t)$, which uses the hybrid share weights s_i^{b0t} defined by (47). Note that we require the period t prices, p_i^t, to be positive in order to ensure that the link factor is well defined.

Even if the Young index provides a close approximation to the corresponding Laspeyres index, it is difficult to recommend the use of the Young index as a final estimate of the change in prices going from period 0 to t, just as it was difficult to recommend the use of the Laspeyres index as the *final* estimate of inflation going from period 0 to t. Recall that the problem with the Laspeyres index was its lack of symmetry in the treatment of the two periods under consideration; i.e., using the justification for the Laspeyres index as a good fixed basket index, there was an identical justification for the use of the Paasche index as an equally good fixed basket index to compare prices in periods 0 and t. The Young index suffers from a similar lack of symmetry with respect to the treatment of the base period. The problem can be explained as follows. The Young index, $P_Y(p^0,p^t,s^b)$ defined by (44) calculates the price change between months 0 and t treating month 0 as the base. But there is no particular reason to necessarily treat month 0 as the base month other than convention. Hence, if we treat month t as the base and use the same formula to measure the price change from month t back to month 0, the index $P_Y(p^t,p^0,s^b) = \sum_{i=1}^N s_i^b(p_i^0/p_i^t)$ would be appropriate. This estimate of price change can then be made comparable to the original Young index by taking its reciprocal, leading to the following *rebased Young index*[54], $P_Y^*(p^0,p^t,s^b)$, defined as

$$(48)\ P_Y^*(p^0,p^t,s^b) \equiv 1/\Sigma_{i=1}^N s_i^b(p_i^0/p_i^t) = [\Sigma_{i=1}^N s_i^b(p_i^t/p_i^0)^{-1}]^{-1}.$$

Thus the rebased Young index, $P_Y^*(p^0,p^t,s^b)$, that uses the current month as the base period is a *share weighted harmonic mean* of the price relatives going from month 0 to month t, whereas the original Young index, $P_Y(p^0,p^t,s^b)$, is a *share weighted arithmetic mean* of the same price relatives.

Fisher argued as follows that an index number formula should give the same answer no matter which period is chosen as the base:

"Either one of the two times may be taken as the 'base'. Will it make a difference which is chosen? Certainly, it *ought* not and our Test 1 demands that it shall not. More fully expressed, the test is that the formula for calculating an index number should be such that it will give the same ratio between one point of comparison and the other point, *no matter which of the two is taken as the base*." Irving Fisher (1922; 64).

[54] Using Fisher's (1922; 118) terminology, $P_Y^*(p^0,p^t,s^b) \equiv 1/[P_Y(p^t,p^0,s^b)]$ is the *time antithesis* of the original Young index, $P_Y(p^0,p^t,s^b)$.

The problem with the Young index is that not only does it not coincide with its rebased counterpart, there is a definite inequality between the two indexes, namely:

(49) $P_Y^*(p^0,p^t,s^b) \le P_Y(p^0,p^t,s^b)$

with a strict inequality provided that the period t price vector p^t is not proportional to the period 0 price vector p^0.[55] Thus a statistical agency that uses the direct Young index $P_Y(p^0,p^t,s^b)$ will generally show a higher inflation rate than a statistical agency that uses the same raw data but uses the rebased Young index, $P_Y^*(p^0,p^t,s^b)$.

The inequality (49) does not tell us by how much the Young index will exceed its rebased time antithesis. However in Appendix 3, it is shown that to the accuracy of a certain second order Taylor series approximation, the following relationship holds between the direct Young index and its time antithesis:

(50) $P_Y(p^0,p^t,s^b) = P_Y^*(p^0,p^t,s^b) + P_Y(p^0,p^t,s^b)\mathrm{Var}(e)$

where Var(e) is defined as

(51) $\mathrm{Var}(e) \equiv \Sigma_{n=1}^N s_n^b[e_n - e^*]^2$.

The deviations e_n are defined by $1+e_n = r_n/r^*$ for n = 1,…,N where the r_n and their weighted mean r^* are defined as follows:

(52) $r_n \equiv p_n^t/p_n^0$; n = 1,...,N;
(53) $r^* \equiv \Sigma_{n=1}^N s_n^b r_n = P_Y(p^0,p^t,s^b)$.

The weighted mean of the e_n is defined as e^*:

(54) $e^* \equiv \Sigma_{n=1}^N s_n^b e_n$

which turns out to equal 0. *Hence the more dispersion there is in the price relatives p_n^t/p_n^0, to the accuracy of a second order approximation, the more the direct Young index will exceed its counterpart that uses month t as the initial base period rather than month 0.*

Given two a priori equally plausible index number formulae that give different answers, such as the Young index and its time antithesis, Fisher (1922; 136) generally suggested taking the geometric average of the two indexes[56] and a benefit of this averaging is that the resulting

[55] These inequalities follow from the fact that a harmonic mean of M positive numbers is always equal to or less than the corresponding arithmetic mean; see Walsh (1901;517) or Fisher (1922; 383-384). This inequality is a special case of Schlömilch's (1858) Inequality; see Hardy, Littlewood and Polyá (1934; 26). Walsh (1901; 330-332) explicitly noted the inequality (49) and also noted that the corresponding geometric average would fall between the harmonic and arithmetic averages. Walsh (1901; 432) computed some numerical examples of the Young index and found big differences between it and his "best" indexes, even using weights that were representative for the periods being compared. Recall that the Lowe index becomes the Walsh index when geometric mean quantity weights are chosen and so the Lowe index can perform well when representative weights are used. This is not necessarily the case for the Young index, even using representative weights. Walsh (1901; 433) summed up his numerical experiments with the Young index as follows: "In fact, Young's method, in every form, has been found to be bad."
[56] "We now come to a third use of these tests, namely, to 'rectify' formulae, i.e., to derive from any given formula which does not satisfy a test another formula which does satisfy it; …. This is easily done by 'crossing', that is, by averaging antitheses. If a given formula fails to satisfy Test 1 [the time reversal test], its time antithesis will also fail to satisfy it; but the two will fail, as it were, in opposite ways, so that a cross between them (obtained by *geometrical* averaging) will give the golden mean

formula will satisfy the time reversal test. Thus rather than using *either* the base period 0 Young index, $P_Y(p^0,p^t,s^b)$, *or* the current period t Young index, $P_Y^*(p^0,p^t,s^b)$, which is always below the base period 0 Young index if there is any dispersion in relative prices, it seems preferable to use the following index, which is the *geometric average* of the two alternatively based Young indexes:[57]

(55) $P_Y^{**}(p^0,p^t,s^b) \equiv [P_Y(p^0,p^t,s^b)P_Y^*(p^0,p^t,s^b)]^{1/2}$.

If the base year shares s_i^b happen to coincide with both the month 0 and month t shares, s_i^0 and s_i^t respectively, it can be seen that the time rectified Young index $P_Y^{**}(p^0,p^t,s^b)$ defined by (55) will coincide with the Fisher ideal price index between months 0 and t, $P_F(p^0,p^t,q^0,q^t)$.[58] Note also that the index P_Y^{**} defined by (55) can be produced on a timely basis by a statistical agency since it does not depend on quantity information for months 0 and t. However, this point illustrates the problem with using out of date base year shares (or annual quantities) as weights for monthly prices: the base year shares may not be representative for the actual expenditure shares (or quantities) for month 0 and the subsequent months. Thus in general, the use of the Fisher or Walsh indexes is recommended over the use of indexes that rely on annual baskets of a prior year. However, this recommendation is tempered by the fact that the statistical agency may not be able to obtain information on current period quantities or expenditures in a timely fashion and thus it may be necessary to use indexes that do not depend on the availability of current information on expenditures or quantities.

8. Fixed Base Versus Chained Indexes

In this section[59], the merits of using the chain system for constructing price indexes in the time series context versus using the fixed base system are discussed.

The chain system[60] measures the change in prices going from one period to a subsequent period using a bilateral index number formula involving the prices and quantities pertaining to the two adjacent periods. These one period rates of change (the links in the chain) are then cumulated to yield the relative levels of prices over the entire period under consideration. Thus if the bilateral price index is P, the chain system generates the following pattern of price levels for the first three periods:[61]

(56) $1, P(p^0,p^1,q^0,q^1), P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)$.

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which does satisfy.” Irving Fisher (1922; 136). Actually the basic idea behind Fisher’s rectification procedure was suggested by Walsh, who was a discussant for Fisher (1921) when Fisher gave a preview of his 1922 book: “We merely have to take any index number, find its antithesis in the way prescribed by Professor Fisher, and then draw the geometric mean between the two.” Correa Moylan Walsh (1921b; 542).

[57] This index is a base year weighted counterpart to an equally weighted index proposed by Carruthers, Sellwood and Ward (1980; 25) and Dalén (1992; 140) in the context of elementary index number formulae. See Chapter 6 for further discussion of this unweighted index.

[58] However, if there are systematic trends in shares, then the s_i^b will not coincide with s_i^0 and s_i^t and it is likely that the rectified Young index will differ from the Fisher index since the base year shares will not in general be representative for the shares for months 0 and t.

[59] This section is largely based on the work of Hill (1988) (1993; 385-390).

[60] The chain principle was introduced independently into the economics literature by Lehr (1885; 45-46) and Marshall (1887; 373). Both authors observed that the chain system would mitigate the difficulties due to the introduction of new commodities into the economy, a point also mentioned by Hill (1993; 388). Fisher (1911; 203) introduced the term “chain system”.

[61] Let the value of transactions in period t be $V^t \equiv \Sigma_{n=1}^N p_n^t q_n^t$ for t = 0,1,2. Then the period t quantity aggregates that correspond to the price levels defined by (56) are equal to the following expressions: $Q^0 \equiv V^0$; $Q^1 \equiv V^1/P(p^0,p^1,q^0,q^1)$ and $Q^2 \equiv V^2/P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)$.

On the other hand, the fixed base system of price levels using the same bilateral index number formula P simply computes the level of prices in period t relative to the base period 0 as $P(p^0,p^t,q^0,q^t)$. Thus the fixed base pattern of price levels for periods 0,1 and 2 is:[62]

(57) 1, $P(p^0,p^1,q^0,q^1)$, $P(p^0,p^2,q^0,q^2)$.

Note that in both the chain system and the fixed base system of price levels defined by (56) and (57) above, the base period price level is set equal to 1. The usual practice in statistical agencies is to set the base period price level equal to 100. If this is done, then it is necessary to multiply each of the numbers in (56) and (57) by 100.

Because of the difficulties involved in obtaining current period information on quantities (or equivalently, on expenditures), many statistical agencies loosely base their Consumer Price Index on the use of the Laspeyres formula (5) and the fixed base system. Therefore, it is of some interest to look at some of the possible problems associated with the use of fixed base Laspeyres indexes.

The main problem with the use of fixed base Laspeyres indexes is that the period 0 fixed basket of commodities that is being priced in period t can often be quite different from the period t basket. Thus if there are systematic *trends* in at least some of the prices and quantities[63] in the index basket, the fixed base Laspeyres price index $P_L(p^0,p^t,q^0,q^t)$ can be quite different from the corresponding fixed base Paasche price index, $P_P(p^0,p^t,q^0,q^t)$.[64] This means that both indexes are likely to be an inadequate representation of the movement in average prices over the time period under consideration.

The fixed base Laspeyres quantity index cannot be used forever: eventually, the base period quantities q^0 are so far removed from the current period quantities q^t that the base must be changed. Chaining is merely the limiting case where the base is changed each period.[65]

A main advantage of the chain system is that under normal conditions, chaining will reduce the spread between the Paasche and Laspeyres indexes.[66] These two indexes each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration and it could be expected that a single point estimate of the aggregate price change should lie between these two estimates. Thus the use of either a

[62] The period t quantity aggregates that correspond to the price levels defined by (57) are equal to the following expressions: $Q^0 \equiv V^0$; $Q^1 \equiv V^1/P(p^0,p^1,q^0,q^1)$ and $Q^2 \equiv V^2/P(p^0,p^2,q^0,q^2)$.

[63] Examples of rapidly downward trending prices and upward trending quantities are computers, electronic equipment of all types, internet access and (quality adjusted) telecommunication charges.

[64] Note that $P_L(p^0,p^t,q^0,q^t)$ will equal $P_P(p^0,p^t,q^0,q^t)$ if *either* the two quantity vectors q^0 and q^t are proportional *or* the two price vectors p^0 and p^t are proportional. Thus in order to obtain a difference between the Paasche and Laspeyres indexes, nonproportional movements in *both* prices and quantities are required.

[65] Regular seasonal fluctuations can cause monthly or quarterly data to "bounce" using the term due to Szulc (1983) and chaining bouncing data can lead to a considerable amount of index "drift"; i.e., if after 12 months, prices and quantities return to their levels of a year earlier, then a chained monthly index will usually not return to unity. Hence, the use of chained indexes for "noisy" monthly or quarterly data is not recommended. The chain drift problem will be studied in more detail in Chapter 7.

[66] See Diewert (1978; 895) and Hill (1988) (1993; 387-388). Another main advantage of using chained indexes is that chaining will in general increase the number of matched prices in situations where there is a considerable amount of product turnover.

chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth".[67]

Hill (1993; 388), drawing on the earlier research of Szulc (1983) and Hill (1988; 136-137), noted that it is not appropriate to use the chain system when prices oscillate (or "bounce" to use Szulc's (1983; 548) term). This phenomenon can occur in the context of regular seasonal fluctuations or in the context of price wars or highly discounted sale prices. However, in the context of roughly monotonically changing prices and quantities, Hill (1993; 389) recommended the use of chained symmetrically weighted indexes. The Fisher and Walsh indexes are examples of symmetrically weighted indexes.

It is possible to be a bit more precise under what conditions one should chain or not chain. Basically, one should chain if the prices and quantities pertaining to adjacent periods are *more similar* than the prices and quantities of more distant periods, since this strategy will lead to a narrowing of the spread between the Paasche and Laspeyres indexes at each link.[68] Of course, one needs a measure of how similar are the prices and quantities pertaining to two periods. The similarity measures could be *relative* ones or *absolute* ones. In the case of absolute comparisons, two vectors of the same dimension are similar if they are identical and dissimilar otherwise. In the case of relative comparisons, two vectors are similar if they are proportional and dissimilar if they are nonproportional.[69] Once a similarity measure has been defined, the prices and quantities of each period can be compared to each other using this measure and a "tree" or path that links all of the observations can be constructed where the most similar observations are compared with each other using a bilateral index number formula.[70] Hill (1995) defined the price structures between the two countries to be more dissimilar the bigger is the spread between P_L and P_P; i.e., the bigger is max $\{P_L/P_P, P_P/P_L\}$. The problem with this measure of dissimilarity in the price structures of the two countries is that it could be the case that $P_L = P_P$ (so that the Hill measure would register a maximal

[67] However, if the underlying data are very volatile, then chaining may not reduce the spread between the Paasche and Laspeyres indexes. In this case, the methods based on multilateral index number theory should be used; see Chapter 7 below.

[68] Walsh, in discussing whether fixed base or chained index numbers should be constructed, took for granted that the precision of all reasonable bilateral index number formulae would improve, provided that the two periods or situations being compared were more similar and hence, for this reason, favored the use of chained indexes: "The question is really, in which of the two courses [fixed base or chained index numbers] are we likely to gain greater exactness in the comparisons actually made? Here the probability seems to incline in favor of the second course; for the conditions are likely to be less diverse between two contiguous periods than between two periods say fifty years apart." Correa Moylan Walsh (1901; 206). Walsh (1921a; 84-85) later reiterated his preference for chained index numbers. Fisher also made use of the idea that the chain system would usually make bilateral comparisons between price and quantity data that was more similar and hence the resulting comparisons would be more accurate: "The index numbers for 1909 and 1910 (each calculated in terms of 1867-1877) are compared with each other. But direct comparison between 1909 and 1910 would give a different and more valuable result. To use a common base is like comparing the relative heights of two men by measuring the height of each above the floor, instead of putting them back to back and directly measuring the difference of level between the tops of their heads." Irving Fisher (1911; 204). "It seems, therefore, advisable to compare each year with the next, or, in other words, to make each year the base year for the next. Such a procedure has been recommended by Marshall, Edgeworth and Flux. It largely meets the difficulty of non-uniform changes in the Q's, for any inequalities for successive years are relatively small." Irving Fisher (1911; 423-424).

[69] Diewert (2009) took an axiomatic approach to defining various indexes of absolute and relative dissimilarity. Measures of relative price similarity or dissimilarity will be discussed in Chapter 7.

[70] Fisher (1922; 271-276) hinted at the possibility of using spatial linking; i.e., of linking countries that are similar in structure. However, the modern literature has grown due to the pioneering efforts of Robert Hill (1995) (2009). Hill (1995) used the spread between the Paasche and Laspeyres price indexes as an indicator of similarity and showed that this criterion gives the same results as a criterion that looks at the spread between the Paasche and Laspeyres quantity indexes.

degree of similarity) but p^0 could be very different from p^t. Thus there is a need for a more systematic study of similarity (or dissimilarity) measures in order to pick the "best" one that could be used as an input into Hill's (1999a) (1999b) (2001) (2009) spanning tree algorithm for linking observations.

The method of linking observations explained in the previous paragraph based on the similarity of the price and quantity structures of any two observations may not be practical in a statistical agency context since the addition of a new period may lead to a reordering of the previous links. However, as will be seen in chapter 7, it is possible to come up with a similarity linking method that does not involve changing index values for prior periods.

Some index number theorists have objected to the chain principle on the grounds that it has no counterpart in the spatial context:

"They [chain indexes] only apply to intertemporal comparisons, and in contrast to direct indices they are not applicable to cases in which no natural order or sequence exists. Thus the idea of a chain index for example has no counterpart in interregional or international price comparisons, because countries cannot be sequenced in a 'logical' or 'natural' way (there is no k+1 nor k−1country to be compared with country k)." Peter von der Lippe (2001; 12).[71]

This is of course correct but the approach of Robert Hill does lead to a "natural" set of spatial links. Applying the same approach to the time series context will lead to a set of links between periods that may not be month to month but it will in many cases justify year over year linking of the data pertaining to the same month. This problem will be addressed in Chapters 7 and 9.

It is of some interest to determine if there are index number formulae that give the same answer when either the fixed base or chain system is used. Comparing the sequence of chain indexes defined by (56) above to the corresponding fixed base indexes, it can be seen that we will obtain the same answer in all three periods if the index number formula P satisfies the following functional equation for all price and quantity vectors:

(58) $P(p^0,p^2,q^0,q^2) = P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)$.

If an index number formula P satisfies (58), then P satisfies the *circularity test*.[72]

If it is assumed that the index number formula P satisfies certain properties or tests in addition to the circularity test above[73], then Funke, Hacker and Voeller (1979) showed that P must have the following functional form due originally to Konüs and Byushgens[74] (1926; 163-166):[75]

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[71] It should be noted that von der Lippe (2001; 56-58) was a vigorous critic of all index number tests based on symmetry in the time series context, although he was willing to accept symmetry in the context of making international comparisons. "But there are good reasons *not* to insist on such criteria in the *intertemporal* case. When no symmetry exists between 0 and t, there is no point in interchanging 0 and t." Peter von der Lippe (2001; 58).

[72] The test name is due to Fisher (1922; 413) and the concept was originally due to Westergaard (1890; 218-219).

[73] The additional tests are: (i) positivity and continuity of $P(p^0,p^1,q^0,q^1)$ for all strictly positive price and quantity vectors p^0,p^1,q^0,q^1; (ii) the identity test; (iii) the commensurability test; (iv) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree one in the components of p^1 and (v) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree zero in the components of q^1. These tests will be explained in Chapter 3.

[74] Konüs and Byushgens showed that the index defined by (59) is exact for Cobb-Douglas (1928) preferences; see also Pollak (1983; 119-120). The concept of an exact index number formula will be explained in Chapter 5.

[75] This result can be derived using results in Eichhorn (1978; 167-168) and Vogt and Barta (1997; 47). A simple proof can be found in Balk (1995). This result vindicates Irving Fisher's (1922; 274) intuition

(59) $P_{KB}(p^0,p^1,q^0,q^1) \equiv \Pi_{n=1}^N (p_n^1/p_n^0)^{\alpha_n}$

where the N constants α_n satisfy the following restrictions:

(60) $\Sigma_{n=1}^N \alpha_n = 1$ and $\alpha_n > 0$ for n = 1,...,N.

Thus under very weak regularity conditions, the only price index satisfying the circularity test (and the additional tests listed above in a footnote) is a weighted geometric average of all the individual price ratios, the weights being constant through time.[76]

An interesting special case of the family of indexes defined by (59) occurs when the weights α_i are all equal. In this case, P_{KB} reduces to the Jevons (1865) index:

(61) $P_J(p^0,p^1) \equiv \Pi_{n=1}^N (p_n^1/p_n^0)^{1/N}$.

The problem with the indexes defined by Konüs and Byushgens and Jevons is that the individual price ratios, p_n^1/p_n^0, have weights (either α_n or 1/N) that are *independent* of the economic importance of commodity n in the two periods under consideration. Put another way, these price weights are independent of the quantities of commodity n consumed or the expenditures on commodity n during the two periods. Hence, these indexes are not really suitable for use by statistical agencies at higher levels of aggregation when expenditure share or quantity information is available.

The above results indicate that it is not useful to ask that the price index P satisfy the circularity test *exactly*. However, it is of some interest to find index number formulae that satisfy the circularity test to some degree of *approximation*, since the use of such an index number formula will lead to measures of aggregate price change that are more or less the same no matter whether we use the chain or fixed base systems. Irving Fisher (1922; 284) found that deviations from circularity using his data set and the Fisher ideal price index P_F defined by (12) above were quite small. This relatively high degree of correspondence between fixed base and chain indexes has been found to hold for other symmetrically weighted formulae like the Walsh index P_W defined by (19) above.[77] Thus in most time series applications of index number theory where the base year in fixed base indexes is changed every 5 years or so, it will not matter very much whether the statistical agency uses a fixed base price index or a chain index, provided that a symmetrically weighted formula is used.[78] This of course depends on the length of the time series considered and the degree of variation

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who asserted that "the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*…" Fisher (1922; 275) went on to assert: "But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915, we need, theoretically at least, another set of weights. … Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another."

[76] This result will be discussed in more detail in chapter 3 below.

[77] See for example Diewert (1978; 894). Walsh (1901; 424 and 429) found that his 3 preferred formulae all approximated each other very well as did the Fisher ideal for his artificial data set.

[78] More specifically, most superlative indexes (which are symmetrically weighted) will usually satisfy the circularity test to a high degree of approximation in the time series context using aggregated data. See chapter 5 for the definition of a superlative index. It is worth stressing that fixed base Paasche and Laspeyres indexes are very likely to diverge considerably over a 5 year period if computers (or any other commodity that has price and quantity trends that are quite different from the trends in the other commodities) are included in the value aggregate under consideration. See Chapters 7 and 11 for some empirical evidence on the divergence between the Laspeyres and Paasche indexes.

in the prices and quantities as we go from period to period. The more prices and quantities are subject to large fluctuations (rather than smooth trends), the less will be the correspondence.[79]

It is possible to give a theoretical explanation for the approximate satisfaction of the circularity test for symmetrically weighted index number formulae. Another symmetrically weighted formula is the Törnqvist index P_T.[80] The natural logarithm of this index is defined as follows:

(62) $\ln P_T(p^0,p^1,q^0,q^1) \equiv \Sigma_{n=1}^N \frac{1}{2}(s_n^0 + s_n^1)\ln(p_n^1/p_n^0)$

where the period t expenditure shares s_n^t are defined by (7) above. Alterman, Diewert and Feenstra (1999; 61) showed that if the logarithmic price ratios $\ln(p_n^t/p_n^{t-1})$ trended linearly with time t and the expenditure shares s_n^t also trended linearly with time, then the Törnqvist index P_T will satisfy the circularity test exactly.[81] Since many economic time series on prices and quantities satisfy these assumptions approximately, then under these conditions, the Törnqvist index P_T will satisfy the circularity test approximately. As will be seen in chapter 7, the Törnqvist index generally closely approximates the symmetrically weighted Fisher and Walsh indexes, so that for many economic time series (with smooth trends), all three of these symmetrically weighted indexes will satisfy the circularity test to a high enough degree of approximation that it will not matter whether we use the fixed base or chain principle.[82]

Walsh (1901; 401) (1921a; 98) (1921b; 540) introduced the following useful variant of the circularity test:

(63) $1 = P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)...P(p^{T-1},p^T,q^{T-1},q^T)P(p^T,p^0,q^T,q^0)$.

The motivation for this test is the following one. Use the bilateral index formula $P(p^0,p^1,q^0,q^1)$ to calculate the change in prices going from period 0 to 1, use the same formula evaluated at the data corresponding to periods 1 and 2, $P(p^1,p^2,q^1,q^2)$, to calculate the change in prices going from period 1 to 2, … , use $P(p^{T-1},p^T,q^{T-1},q^T)$ to calculate the change in prices going from period T−1 to T, introduce an artificial period T+1 that has exactly the price and quantity of the initial period 0 and use $P(p^T,p^0,q^T,q^0)$ to calculate the change in prices going from period T to 0. Finally, multiply all of these indexes together and since we end up where we started, then the product of all of these indexes should ideally be one. Diewert (1993a; 40) called this test a *multiperiod identity test*.[83] Note that if T = 2 (so that the number of periods is 3 in total), then Walsh's test reduces to Fisher's (1921; 534) (1922; 64) *time reversal test*.[84]

Walsh (1901; 423-433) showed how his circularity test could be used in order to evaluate how "good" any bilateral index number formula was. What he did was invent artificial price and

[79] Again, see Szulc (1983) and Hill (1988). This topic will be studied in more detail in Chapters 7 and 11.

[80] This formula was implicitly introduced in Törnqvist (1936) and explicitly defined in Törnqvist and Törnqvist (1937).

[81] This result will be proven in Chapter 7. This exactness result can be extended to cover the case when there are monthly proportional variations in prices and the expenditure shares have constant seasonal effects in addition to linear trends; see Alterman, Diewert and Feenstra (1999; 65).

[82] However, if the smooth trends assumption is violated to a considerable degree or if there are a substantial number of new and disappearing products, then this result will not hold as will be seen in Chapter 7. If prices and quantities are subject to big fluctuations, then it will be necessary to move to a multilateral index; see chapter 7. Note that with new and disappearing products, fixed base indexes can only be used if the base is changed frequently.

[83] Walsh (1921a; 98) called his test the *circular test* but since Fisher also used this term to describe his transitivity test defined earlier by (58), it seems best to stick to Fisher's terminology since it is well established in the literature.

[84] Walsh (1921b; 540-541) noted that the time reversal test was a special case of his circularity test.

quantity data for 5 periods and he added a sixth period that had the data of the first period. He then evaluated the right hand side of (63) for various formula, $P(p^0,p^1,q^0,q^1)$, and determined how far from unity the results were. His "best" formulae had products that were close to one.[85]

This same framework is often used to evaluate the efficacy of chained indexes versus their direct counterparts. Thus if the right hand side of (63) turns out to be different than unity, the chained indexes are said to suffer from "chain drift". If a formula does suffer from chain drift, it is sometimes recommended that fixed base indexes be used in place of chained ones. However, this advice, if accepted would *always* lead to the adoption of fixed base indexes, provided that the bilateral index formula satisfies the identity test, $P(p^0,p^0,q^0,q^0) = 1$. But at the first level of aggregation, there will be tremendous product turnover in most economies. Under these conditions, the adoption of a fixed base index would soon lead to a lack of matching of the products and the resulting fixed base indexes would lose their relevance. Thus it is not recommended that Walsh's circularity test be used to decide whether fixed base or chained indexes should be calculated. However, it is fair to use Walsh's circularity test as he originally used it i.e., as an approximate method for deciding how "good" a particular index number formula is. In order to decide whether to chain or use fixed base indexes, one should decide on the basis of how similar the observations being compared are and choose the method that will best link up the most similar observations.

one should decide on the basis of how similar are the observations being compared and choose the method that will best link up the most similar observations. The question of when to chain and when not to will be discussed in more detail in Chapter 7.

9. Two Stage Aggregation versus Single Stage Aggregation

Does a Laspeyres or Paasche or Fisher index that is constructed in two stages equal the corresponding index that is constructed in a single stage? This question is addressed in the present section. In practice, it is a big advantage to be consistent in aggregation because consistency in aggregation allows the production of an index to be decentralized.

Suppose that the price and quantity data for period t, p^t and q^t, can be written in terms of M subvectors as follows:

$$(64)\ p^t = [p^{t1},p^{t2},...,p^{tM}]\ ;\ q^t = [q^{t1},q^{t2},...,q^{tM}]\ ;\ t = 0,1$$

where the dimensionality of the subvectors p^{tm} and q^{tm} is N(m) for m = 1,2,…,M with the sum of the dimensions N(m) equal to N. These subvectors correspond to the price and quantity data for subcomponents of an overall consumer price index for period t. For the first stage of aggregation, construct subindexes for each of these components going from period 0 to 1. For the base period, set the aggregate price level for each of these subcomponents, say P_m^0 for m = 1,2,…M, equal to 1 and set the corresponding base period subcomponent quantities, say Q_m^0 for m = 1,2,…,M, equal to the base period value of consumption for that subcomponent for m = 1,2,…,M:

$$(65)\ P_m^0 \equiv 1\ ;\ Q_m^0 \equiv \Sigma_{i=1}^{N(m)}\, p_i^{0m}q_i^{0m}\ ; \qquad m = 1,...,M.$$

Now use the *Laspeyres formula* in order to construct a period 1 price for each subcomponent, say P_m^1 for m = 1,2,…,M, of the consumer price index. Since the dimensionality of the subcomponent vectors, p^{tm} and q^{tm}, differs from the dimensionality of the complete period t vectors of prices and quantities, p^t and q^t, it is necessary to use different symbols for these

[85] This is essentially a variant of the methodology that Fisher (1922; 284) used to check how well various formulae corresponded to his version of the circularity test.

subcomponent Laspeyres indexes, say P_L^m for m = 1,2,…M. Thus the period 1 subcomponent prices are defined as follows:

$$(66)\ P_m^1 \equiv P_L^m(p^{0m},p^{1m},q^{0m},q^{1m}) \equiv \Sigma_{i=1}^{N(m)}\ p_i^{1m}q_i^{0m}/\Sigma_{i=1}^{N(m)}\ p_i^{0m}q_i^{0m}\ ; \qquad m = 1,...,M.$$

Once the period 1 prices for the M subindexes have been defined by (66), then the corresponding subcomponent period 1 quantities Q_m^1 for m = 1,2,…,M can be defined by deflating the period 1 subcomponent values $\sum_{i=1}^{N(m)} p_i^{1m}\, q_i^{1m}$ by the period 1 price levels, P_m^1:

$$(67)\ Q_m^1 \equiv \Sigma_{i=1}^{N(m)}\ p_i^{1m}q_i^{1m}/P_m^1\ ; \qquad m = 1,...,M.$$

Now define the period 0 and 1 *subcomponent price level vectors* P^0 and P^1 as follows:

$$(68)\ P^0 \equiv [P_1^0,P_2^0,...,P_M^0] \equiv 1_M\ ;\ P^1 \equiv [P_1^1,P_2^1,...,P_M^1]$$

where 1_M denotes a vector of ones of dimension M and the components of P^1 are defined by (67). The period 0 and 1 subcomponent quantity vectors Q^0 and Q^1 are defined as follows:

$$(69)\ Q^0 \equiv [Q_1^0,Q_2^0,\ldots,Q_M^0]\ ;\ Q^1 \equiv [Q_1^1,Q_2^1,\ldots,Q_M^1]$$

where the components of Q^0 are defined by definitions (66) and the components of Q^1 are defined by definitions (67). The price and quantity vectors in (68) and (69) represent the results of the first stage aggregation. Now use these vectors as inputs into the second stage aggregation problem; i.e., apply the Laspeyres price index formula using the information in (68) and (69) as inputs into the index number formula. Since the price and quantity vectors that are inputs into this second stage aggregation problem have dimension M instead of the single stage formula which utilized vectors of dimension N, a different symbol is required for the new Laspeyres index which we choose to be P_L^*. Thus the Laspeyres price index computed in two stages is denoted as $P_L^*(P^0,P^1,Q^0,Q^1)$. This index is defined as follows:

$$\begin{aligned}(70)\ P_L^*(P^0,P^1,Q^0,Q^1) &\equiv \Sigma_{m=1}^M\ P_m^1Q_m^0/\Sigma_{m=1}^M\ P_m^0Q_m^0 \\ &= \Sigma_{m=1}^M\ P_m^1[\Sigma_{i=1}^{N(m)}\ p_i^{0m}q_i^{0m}]/\Sigma_{m=1}^M\ [\Sigma_{i=1}^{N(m)}\ p_i^{0m}q_i^{0m}] && \text{using (65)} \\ &= \Sigma_{m=1}^M\ [\Sigma_{i=1}^{N(m)}\ p_i^{1m}q_i^{0m}/\Sigma_{i=1}^{N(m)}\ p_i^{0m}q_i^{0m}][\Sigma_{i=1}^{N(m)}\ p_i^{0m}q_i^{0m}]/\Sigma_{m=1}^M\ \Sigma_{i=1}^{N(m)}\ p_i^{0m}q_i^{0m} && \text{using (66)} \\ &= \Sigma_{m=1}^M\ \Sigma_{i=1}^{N(m)}\ p_i^{1m}q_i^{0m}/\Sigma_{m=1}^M\ \Sigma_{i=1}^{N(m)}\ p_i^{0m}q_i^{0m} \\ &\equiv P_L(p^0,p^1,q^0,q^1)\end{aligned}$$

where $P_L(p^0,p^1,q^0,q^1)$ is the overall Laspeyres price index calculated in a single stage. Thus *the two stage Laspeyres index exactly equals the single stage Laspeyres index*:[86]

$$(71)\ P_L^*(P^0,P^1,Q^0,Q^1) = P_L(p^0,p^1,q^0,q^1).$$

Recall that (26) established that the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, was equal to the ratio of two Laspeyres indexes, $P_L(p^b,p^t,q^b)/P_L(p^b,p^0,q^b)$. Thus the two stage aggregation result (71) for the Laspeyres formula implies that the Lowe index is also consistent in aggregation.[87]

Does the same two stage aggregation result hold for the Paasche index? The single stage Paasche index is defined as:

$$(72)\ P_P(p^0,p^1,q^0,q^1) \equiv \Sigma_{m=1}^M\ \Sigma_{i=1}^{N(m)}\ p_i^{1m}q_i^{1m}/\Sigma_{m=1}^M\ \Sigma_{i=1}^{N(m)}\ p_i^{0m}q_i^{1m}.$$

[86] Balk (1996; 362) (2008; 106-107) established this two stage consistency in aggregation result for both the Laspeyres and Paasche indexes. Blackorby and Primont (1980; 88) established the result for the Laspeyres index.
[87] This result was established in Eurostat (2018; 173).

The Paasche subaggregate price and quantity levels for period 0 are still defined by equations (65). However, the period 1 subcomponent Paasche price levels are defined as follows:

$$(73)\ P_m^1 \equiv P_P^m(p^{0m},p^{1m},q^{0m},q^{1m}) \equiv \Sigma_{i=1}^{N(m)} p_i^{1m}q_i^{1m}/\Sigma_{i=1}^{N(m)} p_i^{0m}q_i^{1m}\ ; \qquad m = 1,\ldots,M.$$

Using definitions (73) for the period 1 price levels P_m^1, the Paasche period 1 subaggregate quantity levels are defined by definitions (67). The Paasche price index computed in two stages is denoted as $P_P^*(P^0,P^1,Q^0,Q^1)$ and is defined as follows:

$$\begin{aligned}
(74)\ P_P^*(P^0,P^1,Q^0,Q^1) &\equiv \Sigma_{m=1}^M P_m^1Q_m^1/\Sigma_{m=1}^M P_m^0Q_m^1 \\
&= \Sigma_{m=1}^M P_m^1[\Sigma_{i=1}^{N(m)} p_i^{1m}q_i^{1m}/P_m^1]/\Sigma_{m=1}^M P_m^0[\Sigma_{i=1}^{N(m)} p_i^{1m}q_i^{1m}/P_m^1] && \text{using (67)} \\
&= \Sigma_{m=1}^M [\Sigma_{i=1}^{N(m)} p_i^{1m}q_i^{1m}]/\Sigma_{m=1}^M [\Sigma_{i=1}^{N(m)} p_i^{1m}q_i^{1m}/P_m^1] && \text{using (65)} \\
&= \Sigma_{m=1}^M [\Sigma_{i=1}^{N(m)} p_i^{1m}q_i^{1m}]/\Sigma_{m=1}^M [\Sigma_{i=1}^{N(m)} p_i^{1m}q_i^{1m}/(\Sigma_{i=1}^{N(m)} p_i^{1m}q_i^{1m}/\Sigma_{i=1}^{N(m)} p_i^{0m}q_i^{1m})] && \text{using (73)} \\
&= \Sigma_{m=1}^M [\Sigma_{i=1}^{N(m)} p_i^{1m}q_i^{1m}]/\Sigma_{m=1}^M [\Sigma_{i=1}^{N(m)} p_i^{0m}q_i^{1m}] \\
&= P_P(p^0,p^1,q^0,q^1) && \text{using (72).}
\end{aligned}$$

Thus *the two stage Paasche index exactly equals the single stage Paasche index*.[88]

Definitions (65)-(69) can be used to construct first stage subaggregates for any index number formula except that in definitions (66), replace $P_m^1 \equiv P_L^m(p^{0m},p^{1m},q^{0m},q^{1m})$ or $P_m^1 \equiv P_P^m(p^{0m},p^{1m},q^{0m},q^{1m})$ by $P_m^1 \equiv P^m(p^{0m},p^{1m},q^{0m},q^{1m})$, where $P^m(p^{0m},p^{1m},q^{0m},q^{1m})$ can represent any bilateral index number formula.

Suppose that the Fisher or Törnqvist formula is used at each stage of the aggregation; i.e., in equations (66), suppose that the Laspeyres formula $P_L^m(p^{0m},p^{1m},q^{0m},q^{1m})$ is replaced by the Fisher formula $P_F^m(p^{0m},p^{1m},q^{0m},q^{1m})$ (or by the Törnqvist formula $P_T^m(p^{0m},p^{1m},q^{0m},q^{1m})$) and in equation (70), $P_L^*(P^0,P^1,Q^0,Q^1)$ is replaced by $P_F^*(P^0,P^1,Q^0,Q^1)$ (or by $P_T^*(P^0,P^1,Q^0,Q^1)$) and $P_L(p^0,p^1,q^0,q^1)$ is replaced by $P_F(p^0,p^1,q^0,q^1)$ (or by $P_T(p^0,p^1,q^0,q^1)$). Then the two stage aggregation equality does not hold for these index number formulae. It can be shown that, in general:

$$(75)\ P_F^*(P^0,P^1,Q^0,Q^1) \neq P_F(p^0,p^1,q^0,q^1) \text{ and } P_T^*(P^0,P^1,Q^0,Q^1) \neq P_T(p^0,p^1,q^0,q^1).$$

However, even though the Fisher and Törnqvist formulae are not *exactly* consistent in aggregation, it can be shown that these formulae are *approximately* consistent in aggregation. More specifically, it can be shown that the two stage Fisher formula P_F^* and the single stage Fisher formula P_F in (75), both regarded as functions of the 4N variables in the vectors p^0,p^1,q^0,q^1, approximate each other to the second order around a point where the two price vectors are equal (so that $p^0 = p^1$) and where the two quantity vectors are equal (so that $q^0 = q^1$) and a similar result holds for the two stage and single stage Törnqvist indexes in (75).[89] Thus for normal time series data, single stage and two stage Fisher and Törnqvist indexes will usually be numerically very close.[90]

Appendix 1. The Relationship between the Paasche and Laspeyres Indexes

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88 For additional results on consistency on aggregation over three or more stages of aggregation, see Appendix 5 below. For further materials on the problem of consistency in aggregation, see the references in Blackorby and Primont (1980), Diewert (1978) (1980) and Balk (1996).

89 See Diewert (1978; 889). In fact, these derivative equalities are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$.

90 For an empirical comparison of the four indexes, see Diewert (1978; 894-895). For the Canadian consumer data considered there, the chained two stage Fisher in 1971 was 2.3228 and the corresponding chained two stage Törnqvist was 2.3230, the same values as for the corresponding single stage indexes. Additional empirical results will be exhibited in subsequent chapters.

Recall the notation used in section 2 above. Define the nth relative price or price relative r_n and the nth quantity relative t_n as follows:

(A1.1) $r_n \equiv p_n^1/p_n^0$; $t_n \equiv q_n^1/q_n^0$; $n = 1,...,N$.

Using formula (8) above for the *Laspeyres price index* P_L and definitions (A1.1), we have:

(A1.2) $P_L = \Sigma_{n=1}^N r_n s_n^0 \equiv r^*$;

i.e., we define the "average" price relative r^* as the base period expenditure share weighted average of the individual price relatives, r_i .

The *Laspeyres quantity index*, $Q_L(q^0,q^1,p^0)$, that compares quantities in month 1, q^1, to the corresponding quantities in month 0, q^0, using the prices of month 0, p^0, as weights can be defined as a weighted average of the quantity ratios t_n as follows:

(A1.3) $Q_L(q^0,q^1,p^0) = \sum_{n=1}^N s_n^0 t_n \equiv t^*$.

Before we compare the Paasche and Laspeyres price indexes, we need to undertake a preliminary computation using the above definitions of r_n and t_n. Define the *weighted covariance* between the r_n and t_n as follows:

$$\begin{aligned}
\text{(A1.4) } Cov(r,t,s^0) &\equiv \sum_{n=1}^N (r_n - r^*)(t_n - t^*)s_n^0 \\
&= \sum_{n=1}^N r_n t_n s_n^0 - \sum_{n=1}^N r_n t^* s_n^0 - \sum_{n=1}^N r^* t_n s_n^0 + \sum_{n=1}^N r^* t^* s_n^0 \\
&= \sum_{n=1}^N r_n t_n s_n^0 - t^*\sum_{n=1}^N r_n s_n^0 - r^*\sum_{n=1}^N t_n s_n^0 + r^* t^*\sum_{n=1}^N s_n^0 \\
&= \sum_{n=1}^N r_n t_n s_n^0 - t^*\sum_{n=1}^N r_n s_n^0 - r^*\sum_{n=1}^N t_n s_n^0 + r^* t^* && \text{using } \textstyle\sum_{n=1}^N s_n^0 = 1 \\
&= \sum_{n=1}^N r_n t_n s_n^0 - t^* r^* - r^* t^* + r^* t^* && \text{using (A1.2) and (A1.3)} \\
&= \sum_{n=1}^N r_n t_n s_n^0 - t^* r^*.
\end{aligned}$$

Rearranging (A1.4) leads to the following *covariance identity*:[91]

(A1.5) $\sum_{n=1}^N r_n t_n s_n^0 = \sum_{n=1}^N (r_n - r^*)(t_n - t^*)s_n^0 + r^* t^*$.

Using formula (6) for the Paasche price index P_P, we have:

$$\begin{aligned}
\text{(A1.6) } P_P &\equiv \Sigma_{n=1}^N p_n^1 q_n^1/\Sigma_{i=1}^N p_i^0 q_i^1 \\
&= \Sigma_{n=1}^N r_n t_n p_n^0 q_n^0/\Sigma_{i=1}^N t_i p_i^0 q_i^0 && \text{using definitions (A1.1)} \\
&= \Sigma_{n=1}^N r_n t_n s_n^0/\Sigma_{i=1}^N t_i s_i^0 \\
&= \sum_{n=1}^N r_n t_n s_n^0/t^* && \text{using definition (A1.3)} \\
&= [\{\sum_{n=1}^N (r_n - r^*)(t_n - t^*)s_n^0\} + r^* t^*]/t^* && \text{using (A1.5)} \\
&= [\sum_{n=1}^N (r_n - r^*)(t_n - t^*)s_n^0/t^*] + r^* \\
&= [\sum_{n=1}^N (r_n - r^*)(t_n - t^*)s_n^0/Q_L(q^0,q^1,p^0)] + P_L(p^0,p^1,q^0)
\end{aligned}$$

where the last equality follows using definitions (A1.2) and (A1.3). Taking the difference between P_P and P_L and using (A1.6) yields:

(A1.7) $P_P - P_L = \sum_{n=1}^N (r_n - r^*)(t_n - t^*)s_n^0/Q_L(q^0,q^1,p^0)$.

Thus the difference between the Paasche and Laspeyres price indexes is equal to the covariance between the price ratios, $r_n = p_n^1/p_n^0$, and the corresponding quantity ratios, $t_n =$

[91] The analysis in this appendix is due to Bortkiewicz (1923; 374-375).

q_n^1/q_n^0, divided by the (positive) Laspeyres quantity index, $Q_L(q^0,q^1,p^0)$. If this covariance is negative, which is the usual case in the consumer context, then P_P will be less than P_L.

Appendix 2. The Relationship between the Lowe and Laspeyres Indexes

We shall use the same notation for the long term monthly price relatives $r_n \equiv p_n^t/p_n^0$ that was used in Appendix 1. However, we shall change the definition of the t_n in order to relate the base year annual quantities q_n^b to the base month quantities q_n^0:

$$\text{(A2.1)} \quad t_n \equiv q_n^b/q_n^0 \; ; \qquad n = 1,\ldots,N.$$

We also define a new *Laspeyres quantity index* $Q_L(q^0,q^b,p^0)$, which compares the base year quantity vector q^b to the base month quantity vector q^0, using the price weights of the base month p^0, as follows:

$$\begin{aligned}
\text{(A2.2)} \quad Q_L(q^0,q^b,p^0) &\equiv \textstyle\sum_{n=1}^N p_n^0q_n^b/\sum_{i=1}^N p_i^0q_i^0 \\
&= \textstyle\sum_{n=1}^N p_n^0q_n^0(q_n^b/q_n^0)/\sum_{i=1}^N p_i^0q_i^0 \\
&= \textstyle\sum_{n=1}^N s_n^0\,(q_n^b/q_n^0) && \text{using definitions (7)} \\
&= \textstyle\sum_{n=1}^N s_n^0t_n && \text{using definitions (A2.1)} \\
&\equiv t^*.
\end{aligned}$$

Using definition (26) in the main text, the *Lowe index* comparing the prices in month t to those of month 0, using the quantity weights of the base year b, is equal to:

$$\begin{aligned}
\text{(A2.3)} \quad P_{Lo}(p^0,p^t,q^b) &\equiv \textstyle\sum_{n=1}^N p_n^tq_n^b \,/\, \sum_{n=1}^N p_n^0q_n^b \\
&= \textstyle\sum_{n=1}^N p_n^t\,t_nq_n^0 \,/\, \sum_{n=1}^N p_n^0\,t_nq_n^0 && \text{using definitions (A2.1)} \\
&= \textstyle\sum_{n=1}^N r_np_n^0\,t_nq_n^0 \,/\, \sum_{n=1}^N p_n^0\,t_nq_n^0 && \text{using definitions (A1.1)} \\
&= \textstyle\sum_{n=1}^N r_nt_n\,s_n^0 \,/\, \sum_{n=1}^N t_n\,s_n^0 && \text{using definitions (7)} \\
&= \textstyle\sum_{n=1}^N r_nt_n\,s_n^0/t^* && \text{using (A2.2)} \\
&= [\{\textstyle\sum_{n=1}^N (r_n - r^*)(t_n - t^*)s_n^0\} + r^*t^*]/t^* && \text{using the identity (A1.5)} \\
&= [\textstyle\sum_{n=1}^N (r_n - r^*)(t_n - t^*)s_n^0/t^*] + r^* \\
&= [\textstyle\sum_{n=1}^N (r_n - r^*)(t_n - t^*)s_n^0/t^*] + P_L(p^0,p^t,q^0) && \text{using definition (A1.2)} \\
&= [\mathrm{Cov}(r,t,s^0)/Q_L(q^0,q^b,p^0)] + P_L(p^0,p^t,q^0)
\end{aligned}$$

where the last equality follows using definitions (A1.4) and (A2.2). Subtracting the Laspeyres price index relating the prices of month t to those of month 0, $P_L(p^0,p^t,q^0)$, from both sides of (A2.3) leads to the following relationship of this monthly Laspeyres price index to its Lowe counterpart:

$$\begin{aligned}
\text{(A2.4)} \quad P_{Lo}(p^0,p^t,q^b) - P_L(p^0,p^t,q^0) &= \textstyle\sum_{n=1}^N (r_n - r^*)(t_n - t^*)s_n^0/Q_L(q^0,q^b,p^0) \\
&= \mathrm{Cov}(r,t,s^0)/Q_L(q^0,q^b,p^0).
\end{aligned}$$

Appendix 3. The Relationship between the Young Index and its Time Antithesis

Recall that the direct Young index, $P_Y(p^0,p^t,s^b)$, was defined by (44) and its time antithesis, $P_Y^*(p^0,p^t,s^b)$, was defined by (48). Define the nth relative price between months 0 and t as

$$\text{(A3.1)} \quad r_n \equiv p_n^t/p_n^0 \; ; \qquad n = 1,\ldots,N$$

and define the weighted average (using the base year weights s_i^b) of the r_n as

$$\text{(A3.2)} \quad r^* \equiv \textstyle\sum_{n=1}^N s_n^br_n$$

which turns out to equal the direct Young index, $P_Y(p^0,p^t,s^b)$. Define the deviation e_n of r_n from their weighted average r^* using the following equations:

$$\text{(A3.3)}\quad r_n = r^*(1+e_n) \; ; \qquad n = 1,...,N.$$

If equations (A3.3) are substituted into equation (A3.2), the following equation is obtained:

$$\begin{aligned}\text{(A3.4)}\quad r^* &= \Sigma_{n=1}^{N} s_n^b r^*(1+e_n)\\ &= r^* + r^*\Sigma_{n=1}^{N} s_n^b e_n\end{aligned}$$

since $\Sigma_{n=1}^{N} s_n^b = 1$. Thus

$$\text{(A3.5)}\quad e^* \equiv \Sigma_{n=1}^{N} s_n^b e_n = 0.$$

Thus the weighted mean e^* of the deviations e_n equals 0.

The direct Young index, $P_Y(p^0,p^t,s^b)$, and its time antithesis, $P_Y^*(p^0,p^t,s^b)$, can be written as functions of r*, the annual share weights s_n^b and the deviations of the price relatives e_n from their weighted mean as follows:

$$\text{(A3.6)}\quad P_Y(p^0,p^t,s^b) = r^* \; ;$$
$$\text{(A3.7)}\quad P_Y^*(p^0,p^t,s^b) = [\Sigma_{n=1}^{N} s_n^b\{r^*(1 + e_n)\}^{-1}]^{-1} = r^*[\Sigma_{n=1}^{N} s_n^b(1 + e_n)^{-1}]^{-1}.$$

Now regard $P_Y^*(p^0,p^t,s^b)$ as a function of the vector of deviations, $e \equiv [e_1,\ldots,e_N]$, say $P_Y^*(e)$. The second order Taylor series approximation to $P_Y^*(e)$ around the point $e = 0_N$ is given by the following expression:[92]

$$\begin{aligned}\text{(A3.8)}\quad P_Y^*(e) &\approx r^* + r^*\Sigma_{n=1}^{N} s_n^b e_n + r^*\Sigma_{n=1}^{N}\Sigma_{i=1}^{N} s_n^b s_i^b e_n e_i - r^*\Sigma_{n=1}^{N} s_n^b[e_n]^2\\ &= r^* + r^*[0] + r^*\Sigma_{n=1}^{N}[\Sigma_{i=1}^{N} s_n^b e_n]s_i^b e_i - r^*\Sigma_{n=1}^{N} s_n^b[e_n - e^*]^2 && \text{using (A3.5)}\\ &= r^* + r^*\Sigma_{n=1}^{N}[0]s_i^b e_i - r^*\Sigma_{n=1}^{N} s_n^b[e_n - e^*]^2 && \text{using (A3.5)}\\ &= r^* - r^*\mathrm{Var}(e)\end{aligned}$$

where the weighted sample variance of the vector e of price deviations is defined as

$$\text{(A3.9)}\quad \mathrm{Var}(e) \equiv \Sigma_{n=1}^{N} s_n^b[e_n - e^*]^2.$$

Using $P_Y(p^0,p^t,s^b) = r^*$, rearranging (A3.8) gives us the following approximate relationship between the direct Young index $P_Y(p^0,p^t,s^b)$ and its time antithesis $P_Y^*(p^0,p^t,s^b)$, to the accuracy of a second order Taylor series approximation about a price point where the month t price vector is proportional to the month 0 price vector:

$$\text{(A3.10)}\quad P_Y(p^0,p^t,s^b) \approx P_Y^*(p^0,p^t,s^b) + P_Y(p^0,p^t,s^b)\mathrm{Var}(e).$$

Thus to the accuracy of a second order approximation, the direct Young index will *exceed* its time antithesis by a term equal to the direct Young index times the weighted variance of the deviations of the price relatives from their weighted mean. Thus the bigger is the dispersion in relative prices, the more the direct Young index will exceed its time antithesis.

Appendix 4. The Relationship between the Lowe Index and the Young Index[93]

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[92] This type of second order approximation is due to Dalén (1992; 143) for the case $r^* = 1$ and to Diewert (1995; 29) for the case of a general r^*.

[93] This Appendix benefited from comments by Paul Armknecht.

This Chapter has indicated that the Laspeyres, Paasche and Fisher indexes are preferred target indexes because they weight prices by the most relevant quantity vectors for making overall price comparisons between two periods; i.e., they use the quantity vectors that are equal or proportional to actual consumption for the two periods in the comparison. However, often national statistical offices cannot collect current period expenditure or quantity information and so their options are limited to a choice between the Lowe and the Young index. This choice is based on the fact that they have limited resources to conduct a household expenditure survey and in some instances, there can be a 5 to 10 year time lapse between the survey periods. The question to be addressed here is: which of these two indexes is the preferred option under these circumstances?

Recall that the *Young index* between periods 0 and t, $P_Y(p^0,p^t,s^b)$, was defined by (44) where p^0 and p^t are the price vectors for periods 0 and t and s^b is the vector of expenditure share weights for a previous period (usually a year prior to month 0). For convenience, we repeat this definition here:

$$\text{(A4.1)}\quad P_Y(p^0,p^t,s^b) \equiv \Sigma_{n=1}^N s_n^b(p_n^t/p_n^0).$$

The Young index between the base period b for the weights and the base period 0 for the monthly prices is defined as follows:

$$\text{(A4.2)}\quad P_Y(p^b,p^0,s^b) \equiv \Sigma_{n=1}^N s_n^b(p_n^0/p_n^b).$$

Using definition (26) in the main text, the *Lowe index* comparing the prices in month t to those of month 0, using the quantity weights q^b of the base year b, is equal to:

$$\begin{aligned}
\text{(A4.3)}\quad P_{Lo}(p^0,p^t,q^b) &\equiv \textstyle\sum_{n=1}^N p_n^t q_n^b / \sum_{n=1}^N p_n^0 q_n^b \\
&= \textstyle\sum_{n=1}^N (p_n^t/p_n^0)p_n^0 q_n^b / \sum_{n=1}^N p_n^0 q_n^b \\
&= \textstyle\sum_{n=1}^N (p_n^t/p_n^0)p_n^b q_n^b (p_n^0/p_n^b) / \sum_{n=1}^N p_n^b q_n^b (p_n^0/p_n^b) \\
&= \textstyle\sum_{n=1}^N s_n^b (p_n^t/p_n^0)(p_n^0/p_n^b) / \sum_{n=1}^N s_n^b (p_n^0/p_n^b) && \text{using } s_n^b \equiv p_n^b q_n^b / p^b\cdot q^b \\
&= \textstyle\sum_{n=1}^N s_n^b (p_n^t/p_n^0)(p_n^0/p_n^b) / P_Y(p^b,p^0,s^b) && \text{using definition (A4.2)} \\
&= \textstyle\sum_{n=1}^N s_n^b r_n t_n / P_Y(p^b,p^0,s^b) && \text{defining } r_n \equiv p_n^t/p_n^0;\ t_n \equiv p_n^0/p_n^b \\
&= [\textstyle\sum_{n=1}^N s_n^b (r_n - r^*)(t_n - t^*) + r^* t^*] / P_Y(p^b,p^0,s^b) && \text{using the identity (A1.5)} \\
&= [\mathrm{Cov}(r,t,s^b) + P_Y(p^0,p^t,s^b) P_Y(p^b,p^0,s^b)] / P_Y(p^b,p^0,s^b) \\
&= [\mathrm{Cov}(r,t,s^b) / P_Y(p^b,p^0,s^b)] + P_Y(p^0,p^t,s^b)
\end{aligned}$$

since $r^* \equiv \Sigma_{n=1}^N s_n^b r_n = \Sigma_{n=1}^N s_n^b(p_n^t/p_n^0) = P_Y(p^0,p^t,s^b)$ is the Young index going from period 0 to t and $t^* \equiv \Sigma_{n=1}^N s_n^b t_n = \Sigma_{n=1}^N s_n^b(p_n^0/p_n^b) = P_Y(p^b,p^0,s^b)$ is the Young index going from period b to 0. The *weighted covariance* between the vectors of relative prices r and t is defined as:

$$\begin{aligned}
\text{(A4.4)}\quad \mathrm{Cov}(r,t,s^b) &\equiv \textstyle\sum_{n=1}^N s_n^b (r_n - r^*)(t_n - t^*) \\
&= \textstyle\sum_{n=1}^N s_n^b [(p_n^t/p_n^0) - r^*][(p_n^0/p_n^b) - t^*] \\
&= \textstyle\sum_{n=1}^N s_n^b [(p_n^t/p_n^0) - P_Y(p^0,p^t,s^b)][(p_n^0/p_n^b) - P_Y(p^b,p^0,s^b)].
\end{aligned}$$

If there are *diverging long run trends in prices*, we would expect $\mathrm{Cov}(r,t,s^b)$ to be positive; i.e., if product n has an increasing price (relative to other products) over the entire period running from period 0 to t, then $(p_n^t/p_n^0) - r^*$ and $(p_n^0/p_n^b) - t^*$ will both be positive; if product n has an decreasing price (relative to other products) over the entire period, then $(p_n^t/p_n^0) - r^*$ and $(p_n^0/p_n^b) - t^*$ will both be negative. Thus the covariance will be positive in either case. Under these conditions, the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, will exceed the corresponding Young index, $P_Y(p^0,p^t,s^b)$, using (A4.3). Since both the Lowe and Young index will both tend to be above our preferred target index (the Fisher index), the national statistical office would come closer to the target index by using the Young index over the corresponding Lowe index.

Appendix 5: Three Stage Aggregation

Suppose that we have price and quantity data for two periods that is classified by three distinct categories. For example, commodities may be classified by type of product or service at the first level of aggregation, by the type of outlet or household at the second level of aggregation and by their location or region at the third level of aggregation. We suppose that the first classification has N categories, the second has M categories and the third has K categories. Denote the period t price, quantity and value transacted for the category indexed by k, m and n by $p_{kmn}{}^t$, $q_{kmn}{}^t$ and $v_{kmn}{}^t \equiv p_{kmn}{}^t q_{kmn}{}^t$ respectively for t = 1,t, k = 1,...,K; m = 1,...,M and n = 1,...,N.[94] Below we will show that the Laspeyres (1871) and Paasche (1874) indexes are consistent in aggregation if they are constructed in a three stage aggregation procedure. As was seen in section 9 above, this consistency in aggregation property for the Laspeyres and Paasche indexes is well known if there are two stages of aggregation but it does not see to be well known for three or more stages of aggregation. In this Appendix, we extend the results to show that the Lowe (1823) and Young (1812) indexes are also consistent in aggregation over two or three stages of aggregation.

Conditional on k and m (choices of the last two categories), we can calculate the aggregate value of transactions over the third category for period t, $v_{km}{}^t$, as follows:

(A5.1) $v_{km}{}^t \equiv \Sigma_{n=1}{}^N v_{kmn}{}^t > 0$; $\qquad$ t = 0,1; k = 1,...,K; m = 1,...,M.

The *overall Laspeyres price index* that compares the prices of period 0 to period 1 is defined as follows:

(A5.2) $P_L{}^1 \equiv \Sigma_{k=1}{}^K \Sigma_{m=1}{}^M \Sigma_{n=1}{}^N p_{kmn}{}^1 q_{kmn}{}^0 / \Sigma_{k=1}{}^K \Sigma_{m=1}{}^M \Sigma_{n=1}{}^N p_{kmn}{}^0 q_{kmn}{}^0$;
$\qquad = \Sigma_{k=1}{}^K \Sigma_{m=1}{}^M \Sigma_{n=1}{}^N p_{kmn}{}^1 q_{kmn}{}^0 / \Sigma_{k=1}{}^K \Sigma_{m=1}{}^M v_{km}{}^0$

where we have used definitions (A5.1) to derive the second line of (A5.2).

The same data will be used to aggregate in three stages; first aggregate over the n category, then in the second stage, aggregate over the m category and in the third stage, aggregate over the k category. Thus in the first stage of aggregation, a family of KM Laspeyres indexes will be constructed where we condition on categories k and m and construct a *conditional Laspeyres index* for period t, $P_{Lkm}{}^1$, that aggregates over the last category. Thus construct the following period t *first stage of aggregation Laspeyres price indexes*:

(A5.3) $P_{Lkm}{}^t \equiv \Sigma_{n=1}{}^N p_{kmn}{}^t q_{kmn}{}^0 / \Sigma_{n=1}{}^N p_{kmn}{}^0 q_{kmn}{}^0$; $\qquad$ t = 0,1; k = 1,...,K; m = 1,...,M
$\qquad = \Sigma_{n=1}{}^N p_{kmn}{}^t q_{kmn}{}^0 / v_{km}{}^0$

where the second line follows using definitions (A5.1). Using definitions (A5.3) when t = 0, we see that the following equations hold:

(A5.4) $P_{Lkm}{}^0 = 1$; $\qquad$ k = 1,...,K; m = 1,...,M.

[94] It is not necessary to assume that all prices and quantities be positive. However, we do require that each $v_{km}{}^t$ be positive for all k, m and t; see definitions 1 below. At the first stage of aggregation, it is likely that many commodities will not be transacted in both periods under consideration. In this case, prices and quantities for the missing products can be set equal to 0. However, if a commodity is transacted in one period but not the other, then there can be a problem. In general, bilateral price indexes are not meaningful (or well defined) unless there are positive matching prices in the two periods being compared. Thus suppose $p_{kmn}{}^1 > 0$, $q_{kmn}{}^1 > 0$ and $q_{kmn}{}^t = 0$. In order to obtain a meaningful price index that compares prices in period t to prices in period 1, it will be necessary to either artificially set $q_{kmn}{}^1$ equal to 0 or to provide an artificial positive imputed price for $p_{kmn}{}^t$.

Define the period t quantity or volume index $Q_{km}{}^t$ that pairs up with the period t price index $P_{Lkm}{}^t$ defined by (A5.3) as the period t transactions value over n (conditional on choosing categories k and m), $v_{km}{}^t$, divided by $P_{Lkm}{}^t$:

$$\text{(A5.5)}\ Q_{km}{}^t \equiv v_{km}{}^t/P_{Lkm}{}^t\ ; \qquad t = 0,1;\ k = 1,\ldots,K;\ m = 1,\ldots,M$$
$$= v_{km}{}^t/[\Sigma_{n=1}{}^N\ p_{kmn}{}^t q_{kmn}{}^0/v_{km}{}^0]$$

where the second line in (A5.5) follows using (A5.3). Using definitions (A5.1) and (A5.5), it can be seen that

$$\text{(A5.6)}\ Q_{km}{}^0 = v_{km}{}^0\ ; \qquad k = 1,\ldots,K;\ m = 1,\ldots,M.$$

For our second stage of the three stage aggregation procedure, we will aggregate over the second category using the Laspeyres price and quantity indexes, $P_{Lkm}{}^t$ and $Q_{km}{}^t$ defined by (A5.3) and (A6.6), as our basic building blocks. Thus define the *conditional on k Laspeyres price index* for period t, $P_{Lk}{}^t$, as follows:

$$\text{(A5.7)}\ P_{Lk}{}^t \equiv \Sigma_{m=1}{}^M\ P_{Lkm}{}^t Q_{km}{}^0/\Sigma_{m=1}{}^M\ P_{Lkm}{}^0 Q_{km}{}^0\ ; \qquad t = 0,1;\ k = 1,\ldots,K$$
$$= \Sigma_{m=1}{}^M\ [\Sigma_{n=1}{}^N\ p_{kmn}{}^t q_{kmn}{}^0/v_{km}{}^0]v_{km}{}^0/\Sigma_{m=1}{}^M\ v_{km}{}^0 \qquad \text{using (A5.3) and (A5.6)}$$
$$= [\Sigma_{m=1}{}^M\ \Sigma_{n=1}{}^N\ p_{kmn}{}^t q_{kmn}{}^0]/\Sigma_{m=1}{}^M\ v_{km}{}^0\ .$$ [95]

Using (A5.1) and (A5.7) when t is set equal to 0, we find that the following equalities hold:

$$\text{(A5.8)}\ P_{Lk}{}^0 = 1\ ; \qquad k = 1,\ldots,K.$$

The Laspeyres price index $P_{Lk}{}^t$ defined by (A5.7) applies to the conditional on k expenditures $\Sigma_{m=1}{}^M\ v_{km}{}^t = \Sigma_{m=1}{}^M\ [\Sigma_{n=1}{}^N\ \ v_{kmn}{}^t]$. Thus we define the companion quantity or volume index that matches up with $P_{Lk}{}^t$ defined by (7) as follows:

$$\text{(A5.9)}\ Q_k{}^t \equiv \Sigma_{m=1}{}^M\ v_{km}{}^t/P_{Lk}{}^t\ ; \qquad t = 0,1;\ k = 1,\ldots,K$$
$$= \Sigma_{m=1}{}^M\ v_{km}{}^t/[\Sigma_{m=1}{}^M\ \Sigma_{n=1}{}^N\ p_{kmn}{}^t q_{kmn}{}^0/\Sigma_{m=1}{}^M\ v_{km}{}^0]$$

where the second line in (A5.9) follows using definitions (A5.7). When t = 1, it can be seen that definitions (A5.1) and (A5.9) imply the following equations:

$$\text{(A5.10)}\ Q_k{}^0 = \Sigma_{m=1}{}^M\ v_{km}{}^0\ ; \qquad k = 1,\ldots,K.$$

Our third and final stage of aggregation is to use the prices and quantities defined by (A5.7)-(A5.10) for t = 0,1 to form a Laspeyres index which aggregates over the k classification. This is the final *three stages of aggregation Laspeyres price index* $P_L{}^{1*}$ defined as follows:

$$\text{(A5.11)}\ P_L{}^{1*} \equiv \Sigma_{k=1}{}^K\ P_{Lk}{}^1 Q_k{}^0/\Sigma_{k=1}{}^K\ P_{Lk}{}^0 Q_k{}^0\ ;$$
$$= \Sigma_{k=1}{}^K\ [\Sigma_{m=1}{}^M\ \Sigma_{n=1}{}^N\ p_{kmn}{}^1 q_{kmn}{}^0/\Sigma_{i=1}{}^M\ v_{ki}{}^0][\Sigma_{j=1}{}^M\ v_{kj}{}^0]/\Sigma_{k=1}{}^K[\Sigma_{m=1}{}^M\ v_{km}{}^0]$$
$$\text{using (A5.7) and (A5.10)}$$
$$= \Sigma_{k=1}{}^K\ \Sigma_{m=1}{}^M\ \Sigma_{n=1}{}^N\ p_{kmn}{}^1 q_{kmn}{}^0/\Sigma_{k=1}{}^K\ \Sigma_{m=1}{}^M\ v_{km}{}^0$$
$$= P_L{}^1 \qquad \text{using definition (A5.2).}$$

Thus the Laspeyres price index constructed in three stages is equal to the corresponding single stage Laspeyres price index. The same method of proof can be used to show that the Laspeyres index constructed in four or more stages of aggregation is equal to the single stage Laspeyres index.

[95] If K = 1, then it can be verified that (A5.7) establishes the consistency in aggregation of the Laspeyres price index over two stages of aggregation.

The above proof can be modified to show that the single stage Paasche index is equal to its counterpart Paasche index constructed in two or three stages.

We now consider the consistency in aggregation properties of the Lowe (1823) index. The situation is a bit more complex than the framework that was described above in that *three* periods are involved in a comparison of prices between two periods. Thus let $q_{kmn}{}^{b}$ be the quantity transacted in the quantity base period b for the commodity category indexed by k,m and n. The *Lowe index* that compares the prices of period 1, $p_{kmn}{}^{1}$, with the prices of period 0, $p_{kmn}{}^{0}$, is $P_{Lo}{}^{1}$ defined as follows:

$$\begin{aligned}(A5.12)\ P_{Lo}{}^{1} &\equiv \Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/\Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{0}q_{kmn}{}^{b} ;\\ &= \Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/\Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} v_{kmn}{}^{0b}\end{aligned}$$

where the *hybrid expenditure weights* using the prices of period 0 and the quantities of period b for commodity category indexed by k,m and n are defined as follows:

$$(A5.13)\ v_{kmn}{}^{0b} \equiv p_{kmn}{}^{0}q_{kmn}{}^{b} ; \qquad k = 1,...,K;\ m = 1,...,M;\ n = 1,...,N.$$

For each k and m, define the *period 0 hybrid conditional on k and m total expenditure* on commodities indexed by n as follows:[96]

$$(A5.14)\ v_{km}{}^{0b} \equiv \Sigma_{n=1}{}^{N} v_{kmn}{}^{0b} ; \qquad k = 1,...,K;\ m = 1,...,M.$$

Substituting (A5.14) into definition (A5.12), we see that the Lowe index for period 1 relative to period 0 can be written as follows;

$$(A5.15)\ P_{Lo}{}^{1} = \Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/\Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} v_{km}{}^{0b} .$$

The above data will be used to aggregate in three stages; first aggregate over the n category, then in the second stage, aggregate over the m category and in the third stage, aggregate over the k category. Thus in the first stage of aggregation, a family of KM Lowe indexes will be constructed where we condition on categories k and m and construct a *conditional Lowe index* for period t, $P_{Lokm}{}^{t}$, that aggregates over the n category. We now compare the prices of period 1 to the prices of period 0 using the Lowe formula. Thus construct the following period 1 *first stage of aggregation Lowe price indexes*:

$$\begin{aligned}(A5.16)\ P_{Lokm}{}^{1b} &\equiv \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/\Sigma_{n=1}{}^{N} p_{kmn}{}^{0}q_{kmn}{}^{b} ; \qquad k = 1,...,K;\ m = 1,...,M\\ &= \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/v_{km}{}^{0b}\end{aligned}$$

where the second line follows using definitions (A5.13) and (A5.14). The above conditional Lowe indexes can act as period 1 Lowe conditional price levels. The corresponding period 0 Lowe conditional price levels are defined as follows:

$$(A5.17)\ P_{Lokm}{}^{0b} = \Sigma_{n=1}{}^{N} p_{kmn}{}^{0}q_{kmn}{}^{b}/\Sigma_{n=1}{}^{N} p_{kmn}{}^{0}q_{kmn}{}^{b} = 1 ; \qquad k = 1,...,K;\ m = 1,...,M.$$

It is not obvious how to define the subaggregate quantity $Q_{km}{}^{0b}$ that should match up with the Lowe subaggregate price index for period 0, $P_{Lokm}{}^{0b}$. In order to achieve consistency in aggregation for the Lowe index, we will set the subaggregate hybrid value for period 0, $v_{km}{}^{0b}$ equal to subaggregate price $P_{Lokm}{}^{0b}$ times subaggregate quantity $Q_{km}{}^{0b}$. Thus we have the following definitions:

[96] We assume that $v_{km}{}^{0b} > 0$ for k = 1,...,K; m = 1,...,M.

$$(A5.18)\ Q_{km}{}^{0b} \equiv v_{km}{}^{0b}/P_{Lokm}{}^{0b}\ ; \qquad k = 1,...,K;\ m = 1,...,M$$
$$= v_{km}{}^{0b}$$

where the second line in (A5.18) follows using (A5.17).

For our second stage of the three stage aggregation procedure, we aggregate over the second category using the Lowe price and quantity subindexes, $P_{Lokm}{}^{0b}$, $P_{Lokm}{}^{1b}$ and $Q_{km}{}^{1b}$ defined by (A5.16)-(A5.18) as our basic building blocks. Thus define the *conditional on k Lowe price index* for period 1 relative to period 0, $P_{Lok}{}^{1b}$, as follows:

$$(A5.19)\ P_{Lok}{}^{1b} \equiv \Sigma_{m=1}{}^{M} P_{Lokm}{}^{1b}Q_{km}{}^{0b}/\Sigma_{m=1}{}^{M} P_{Lokm}{}^{0b}Q_{km}{}^{0b}\ ; \qquad k = 1,...,K$$
$$= \Sigma_{m=1}{}^{M} [\Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/v_{km}{}^{0b}]v_{km}{}^{0b}/\Sigma_{m=1}{}^{M} v_{km}{}^{0b} \qquad \text{using (A5.16) and (A5.18)}$$
$$= \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/\Sigma_{m=1}{}^{M} v_{km}{}^{0b}\ .$$[97]

We treat the $P_{Lok}{}^{1b}$ as period 1 conditional on k Lowe price levels. The corresponding period 0 Lowe conditional on k price levels are defined as follows:

$$(A5.20)\ P_{Lok}{}^{0b} \equiv \Sigma_{m=1}{}^{M} P_{Lokm}{}^{0b}Q_{km}{}^{0b}/\Sigma_{m=1}{}^{M} P_{Lokm}{}^{0b}Q_{km}{}^{0b} = 1\ ; \qquad k = 1,...,K.$$

Total hybrid expenditures on category k goods and services for period 0, $v_k{}^{0b}$,[98] are defined as follows:

$$(A5.21)\ v_k{}^{0b} \equiv \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} v_{kmn}{}^{0b}$$
$$= \Sigma_{m=1}{}^{M}\ v_{km}{}^{0b} \qquad \text{using definitions (A5.14).}$$

Define the period 0 *Lowe quantity subaggregate for category k*, $Q_k{}^{0b}$, as period 0 hybrid expenditures on the category, $v_k{}^{0b}$, deflated by the subaggregate Lowe price index for category k in period 0, $P_{Lok}{}^{0b}$:

$$(A5.22)\ Q_k{}^{0b} \equiv v_k{}^{0b}/P_{Lokm}{}^{0b}\ ; \qquad k = 1,...,K;$$
$$= v_k{}^{0b} \qquad \text{using (A5.20)}$$
$$= \Sigma_{m=1}{}^{M} v_{km}{}^{0b} \qquad \text{using (A5.21).}$$

Our third and final stage of aggregation is to use the prices and quantities defined by (A5.19), (A5.20) and (A5.22) to form a Lowe index which aggregates over the k classification. This is the *three stages of aggregation Lowe price index* $P_{Lo}{}^{1*}$ defined as follows:

$$(A5.23)\ P_{Lo}{}^{1*} \equiv \Sigma_{k=1}{}^{K} P_{Lok}{}^{1b}Q_k{}^{0b}/\Sigma_{k=1}{}^{K} P_{Lok}{}^{0b}Q_k{}^{0b}\ ;$$
$$= \Sigma_{k=1}{}^{K} [\Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/\Sigma_{j=1}{}^{M} v_{kj}{}^{0b}][\Sigma_{m=1}{}^{M}\ v_{km}{}^{0b}]/[\Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M}\ v_{km}{}^{0b}]$$
$$\text{using (A5.19) and (A5.22)}$$
$$= \Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/[\Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M}\ v_{km}{}^{0b}] \qquad \text{cancelling terms}$$
$$= \Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} p_{kmn}{}^{1}q_{kmn}{}^{b}/[\Sigma_{k=1}{}^{K} \Sigma_{m=1}{}^{M} \Sigma_{n=1}{}^{N} v_{kmn}{}^{0b}] \qquad \text{using (A5.14)}$$
$$= P_{Lo}{}^{1} \qquad \text{using (A5.12).}$$

Thus the Lowe price index constructed in three stages is equal to the corresponding single stage Lowe price index. The same method of proof can be used to show that the Lowe index constructed in four or more stages of aggregation is equal to the single stage Lowe index.

[97] If K = 1, then it can be verified that (A5.19) establishes the consistency in aggregation of the Lowe price index over two stages of aggregation.

[98] This is the cost of purchasing the base period basket of commodities that are in category k using the prices of period 0 and the quantities of the base period b.

We turn to the Young index and it consistency in aggregation properties. Let p_{kmn}^{b}, q_{kmn}^{b} and $v_{kmn}^{b} \equiv p_{knm}^{b}q_{kmn}^{b}$ be the price, quantity and transaction value for commodity class indexed by k, m and n for the base period b for k = 1,...,K, m = 1,...,M and n = 1,...,N. As usual, p_{kmn}^{t} is the price of the commodity that is indexed by categories k, m and n in period t for t = 0,1. The base period expenditure share for commodity k,m and n, s_{kmn}^{b}, is defined as follows:

$$\text{(A5.24)}\quad s_{kmn}^{b} \equiv v_{kmn}^{b}/\Sigma_{r=1}^{K}\, \Sigma_{s=1}^{M}\, \Sigma_{t=1}^{N}\, v_{rst}^{b}\ ; \qquad k = 1,...,K;\ m = 1,...,M;\ n = 1,...,N.$$

The period 1 Young index P_Y^{1} that compares the prices of period 1 with the prices of period 0 is the following share weighted average of the price ratios p_{kmn}^{1}/p_{kmn}^{1}: [99]

$$\text{(A5.25)}\quad P_Y^{1} \equiv \Sigma_{k=1}^{K}\, \Sigma_{m=1}^{M}\, \Sigma_{n=1}^{N}\, s_{kmn}^{b}[p_{kmn}^{1}/p_{kmn}^{0}]\ .$$

We will compare the above single stage Young index with a corresponding Young index that aggregates the price ratios in three stages. For the first stage of aggregation, we need to define the following *conditional shares* that condition on k and m and aggregate over n, s_{km}^{b}:

$$\text{(A5.26)}\quad s_{km}^{b} \equiv \Sigma_{n=1}^{N}\, s_{kmn}^{b}\ ; \qquad k = 1,...,K;\ m = 1,...,M.$$

The *first stage conditional Young indexes*, P_{Ymn}^{1}, that compare the prices of the commodities in the class of commodities indexed by k and m for period 1 relative to period 0 are defined as follows:

$$\text{(A5.27)}\quad P_{Ykm}^{1} \equiv \Sigma_{n=1}^{N}\, s_{kmn}^{b}[p_{kmn}^{1}/p_{kmn}^{0}]/s_{km}^{b}\ ; \qquad k = 1,...,K;\ m = 1,...,M.$$

The above conditional Young indexes can act as period 1 Young conditional price levels. The corresponding period 0 Young conditional price levels are defined as follows:

$$\text{(A5.28)}\quad P_{Ykm}^{1} \equiv \Sigma_{n=1}^{N}\, s_{kmn}^{b}[p_{kmn}^{0}/p_{kmn}^{0}]/s_{km}^{b} = 1\ ; \qquad k = 1,...,K;\ m = 1,...,M.$$

The second stage of aggregation uses the prices defined by (A5.27) and (A5.28) and the shares defined by (A5.26). Thus define the *second stage conditional Young indexes*, P_{Yk}^{t}, that condition on expenditures in the k category and compare the aggregate prices defined by (A5.27) for period 1 to their counterparts in period 0: [100]

$$\begin{aligned}
\text{(A5.29)}\quad P_{Yk}^{1} &\equiv \Sigma_{m=1}^{M}\, s_{km}^{b}[P_{Ykm}^{1}/P_{Ykm}^{0}]/\Sigma_{m=1}^{M}\, s_{km}^{b}\ ; && k = 1,...,K \\
&= \Sigma_{m=1}^{M}\, s_{km}^{b}\{\Sigma_{n=1}^{N}\, s_{kmn}^{b}[p_{kmn}^{1}/p_{kmn}^{0}]/s_{km}^{b}\}/\Sigma_{m=1}^{M}\, s_{km}^{b} && \text{using (A5.27) and (A5.28)} \\
&= \Sigma_{m=1}^{M}\, \Sigma_{n=1}^{N}\, s_{kmn}^{b}(p_{kmn}^{1}/p_{kmn}^{0})/\Sigma_{m=1}^{M}\, s_{km}^{b} && \text{cancelling terms} \\
&= \Sigma_{m=1}^{M}\, \Sigma_{n=1}^{N}\, s_{kmn}^{b}(p_{kmn}^{1}/p_{kmn}^{0})/\Sigma_{m=1}^{M}\, \Sigma_{n=1}^{N}\, s_{kmn}^{b} && \text{using (A5.26).}
\end{aligned}$$

The above conditional on k Young indexes can act as period 1 Young conditional on k price levels. The corresponding period 0 Young conditional on k price levels are defined as follows:

$$\text{(A5.30)}\quad P_{Yk}^{0} \equiv \Sigma_{m=1}^{M}\, s_{km}^{b}[P_{Ykm}^{0}/P_{Ykm}^{0}]/\Sigma_{m=1}^{M}\, s_{km}^{b} = 1\ ; \qquad k = 1,...,K.$$

Finally, in order to implement the third stage of aggregation, we define aggregate *shares that condition on expenditure category k*, s_k^{b}:

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[99] In order for this index to be well defined, we require all period 1 prices to be positive. If a product is present in just one of the two periods under consideration, then it is necessary to exclude that product from the index or, alternatively, to generate an imputed price for the product for the period where it is missing.

[100] If K = 1, (A5.29) shows that the two stage Young index is equal to its single stage counterpart.

(A5.31) $s_k{}^b \equiv \Sigma_{m=1}{}^M s_{km}{}^b$; $\qquad k = 1,...,K$
$\qquad = \Sigma_{m=1}{}^M \Sigma_{n=1}{}^N s_{kmn}{}^b$

where the second line in (A5.31) follows using definitions (A5.26). Note that the above shares sum to one:

(A5.32) $\Sigma_{k=1}{}^K s_k{}^b = \Sigma_{k=1}{}^K \Sigma_{m=1}{}^M \Sigma_{n=1}{}^N s_{kmn}{}^b = 1.$

The final stage of aggregation is to aggregate over the k classification. The three stage Young index that compares the prices of period 1 to period 0 is $P_Y{}^{1*}$ defined as follows:

(A5.33) $P_Y{}^{1*} \equiv \Sigma_{k=1}{}^K s_k{}^b[P_{Yk}{}^1/P_{Yk}{}^0]$
$\qquad = \Sigma_{k=1}{}^K s_k{}^b P_{Yk}{}^1$ $\qquad$ using (A5.30)
$\qquad = \Sigma_{k=1}{}^K s_k{}^b [\Sigma_{m=1}{}^M \Sigma_{n=1}{}^N s_{kmn}{}^b(p_{kmn}{}^1/p_{kmn}{}^0)/\Sigma_{m=1}{}^M s_{km}{}^b]$ $\qquad$ using (A5.29)
$\qquad = \Sigma_{k=1}{}^K s_k{}^b [\Sigma_{m=1}{}^M \Sigma_{n=1}{}^N s_{kmn}{}^b(p_{kmn}{}^1/p_{kmn}{}^0)/s_k{}^b]$ $\qquad$ using (A5.31)
$\qquad = \Sigma_{k=1}{}^K \Sigma_{m=1}{}^M \Sigma_{n=1}{}^N s_{kmn}{}^b(p_{kmn}{}^1/p_{kmn}{}^0)$ $\qquad$ cancelling terms
$\qquad = P_Y{}^1$ $\qquad$ using (A5.25).

Thus the Young price index constructed in three stages is equal to the corresponding single stage Young price index. The same method of proof can be used to show that the Young index constructed in four or more stages of aggregation is equal to the single stage Young index.

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